

STOCHASTIC INTEGRALS AND BROWNIAN MOTION ON ABSTRACT NILPOTENT LIE GROUPS

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ABSTRACT. We construct a class of iterated stochastic integrals with respect to Brownian motion on an abstract Wiener space which allows for the definition of Brownian motions on a general class of infinite-dimensional nilpotent Lie groups based on abstract Wiener spaces. We then prove that a Cameron–Martin type quasi-invariance result holds for the associated heat kernel measures in the non-degenerate case, and give estimates on the associated Radon–Nikodym derivative. We also prove that a log Sobolev estimate holds in this setting.

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1. INTRODUCTION

The construction of diffusions on infinite-dimensional manifolds and the study of the regularity properties of their induced measures have been a topic of great interest for at least the past 50 years; see for example [13, 26, 27, 19, 20, 3, 2, 30], although many other references exist. The purpose of the present paper is to construct diffusions on a general class of infinite-dimensional nilpotent Lie groups, and to show that the associated heat kernel measures are quasi-invariant under appropriate translations. We demonstrate that this class of groups is quite rich. We focus here on the elliptic setting, but comment that, as nilpotent groups are

2010 *Mathematics Subject Classification*. Primary 60J65 28D05; Secondary 58J65 22E65.

Key words and phrases. Heat kernel measure, infinite-dimensional Lie group, quasi-invariance, logarithmic Sobolev inequality.

* This research was supported in part by NSF Grants DMS-0907293 and DMS-1255574.

standard first models for studying hypoellipticity, examples of infinite-dimensional versions of such spaces are important for the more general study of hypoellipticity in infinite dimensions. This is an area of great interest, and is of particular relevance in the study of stochastic PDEs and their applications [1, 5, 8, 22, 29, 31]. The present paper studies heat kernel measures for elliptic diffusions on these spaces, which is a necessary precursor to understanding the degenerate case.

1.1. Main results. Let $(\mathfrak{g}, \mathfrak{g}_{CM}, \mu)$ denote an abstract Wiener space, where \mathfrak{g} is a Banach space equipped with a centered non-degenerate Gaussian measure μ and associated Cameron–Martin Hilbert space \mathfrak{g}_{CM} . We will assume that \mathfrak{g}_{CM} additionally carries a nilpotent Lie algebra structure $[\cdot, \cdot] : \mathfrak{g}_{CM} \times \mathfrak{g}_{CM} \rightarrow \mathfrak{g}_{CM}$, and we will further assume that this Lie bracket is Hilbert-Schmidt. We will call such a space an *abstract nilpotent Lie algebra*. Via the standard Baker-Campbell-Hausdorff-Dynkin formula, we may then equip \mathfrak{g}_{CM} with an explicit group operation under which \mathfrak{g}_{CM} becomes an infinite-dimensional group. When thought of as a group, we will denote this space by G_{CM} . We equip G_{CM} with the left-invariant Riemannian metric which agrees with the inner product on $\mathfrak{g}_{CM} \cong T_e G_{CM}$, and we denote the Riemannian distance by d . Note that, despite the use of the notation \mathfrak{g} , we do not assume that the Lie bracket structure extends to \mathfrak{g} , and so this space is not necessarily a Lie algebra or a Lie group. Still, when \mathfrak{g} is playing a role typically played by the group, we will denote \mathfrak{g} by G .

If \mathfrak{g} were a Lie algebra (and thus carried an associated group structure), we could construct a Brownian motion on G as the solution to the Stratonovich stochastic differential equation

$$\delta g_t = g_t \delta B_t := L_{g_t*} \delta B_t, \text{ with } g_0 = \mathbf{e} = (0, 0),$$

where L_x is left translation by $x \in G$ and $\{B_t\}_{t \geq 0}$ is a standard Brownian motion on \mathfrak{g} (as a Banach space) with $\text{Law}(B_1) = \mu$. In finite dimensions, the solution to this stochastic differential equation may be obtained explicitly as a formula involving the Lie bracket. In particular, for $t > 0$ and $n \in \mathbb{N}$, let $\Delta_n(t)$ denote the simplex in \mathbb{R}^n given by

$$\{s = (s_1, \dots, s_n) \in \mathbb{R}^n : 0 < s_1 < s_2 < \dots < s_n < t\}.$$

Let \mathcal{S}_n denote the permutation group on $(1, \dots, n)$, and, for each $\sigma \in \mathcal{S}_n$, let $e(\sigma)$ denote the number of “errors” in the ordering $(\sigma(1), \sigma(2), \dots, \sigma(n))$, that is, $e(\sigma) = \#\{j < n : \sigma(j) > \sigma(j+1)\}$. Then the Brownian motion on G could be written as

$$(1.1) \quad g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in \mathcal{S}_n} \left((-1)^{e(\sigma)} / n^2 \binom{n-1}{e(\sigma)} \right) \int_{\Delta_n(t)} [[\dots [\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \dots], \delta B_{s_{\sigma(n)}}],$$

where this sum is finite under the assumed nilpotence. An obstacle to the development of a general theory of stochastic differential equations on infinite-dimensional Banach spaces is the lack of smoothness of the norm in a general Banach space which is necessary to define a stochastic integral on it. Still, in Section 2, we prove a general result to define a class of iterated stochastic integrals with respect to Brownian motion on the Banach space \mathfrak{g} that includes the expression above. Additionally, we show that one may make sense of the above expression when the Lie bracket on \mathfrak{g}_{CM} does not necessarily extend to \mathfrak{g} . Thus we are able to define a

“group Brownian motion” $\{g_t\}_{t \geq 0}$ on G via (1.1). We let $\nu_t := \text{Law}(g_t)$ be the heat kernel measure on G .

In particular, the integrals above are defined as a limit of stochastic integrals on finite-dimensional subgroups G_π of G_{CM} . We show that these G_π are nice in the sense that they approximate G_{CM} and that there exists a uniform lower bound on their Ricci curvatures.

Using these results, we are able to prove the following main theorem.

Theorem 1.1. *For $h \in G_{CM}$, let $L_h, R_h : G_{CM} \rightarrow G_{CM}$ denote left and right translation by h , respectively. Then L_h and R_h define measurable transformations on G , and for all $T > 0$, $\nu_t \circ L_h^{-1}$ and $\nu_t \circ R_h^{-1}$ are absolutely continuous with respect to ν_t . Let*

$$J_t^l(h, \cdot) := \frac{d(\nu_t \circ L_h^{-1})}{d\nu_t} \quad \text{and} \quad J_t^r(h, \cdot) := \frac{d(\nu_t \circ R_h^{-1})}{d\nu_t}$$

be the Radon-Nikodym derivatives, k be the uniform lower bound on the Ricci curvatures of the finite-dimensional approximation groups G_π and

$$c(t) := \frac{t}{e^t - 1}, \quad \text{for all } t \in \mathbb{R},$$

with the convention that $c(0) = 1$. Then, for all $p \in [1, \infty)$, $J_t^l(h, \cdot), J_t^r(h, \cdot) \in L^p(\nu_t)$ and both satisfy the estimate

$$\|J_t^*(h, \cdot)\|_{L^p(\nu_t)} \leq \exp\left(\frac{c(kt)(p-1)}{2t} d(\mathbf{e}, h)^2\right),$$

where $*$ = l or $*$ = r .

The fact that one may define a measurable left or right action on G by an element of G_{CM} is discussed in Section 3. The lower bound on the Ricci curvature is proved in Proposition 3.10.

1.2. Discussion. The present paper builds on the previous work in [14] and [32], significantly generalizing these previous works in several ways. In particular, the paper [32] considered analogous results for “semi-infinite Lie groups”, which are infinite-dimensional nilpotent Lie groups constructed as extensions of finite-dimensional nilpotent Lie groups \mathfrak{v} by an infinite-dimensional abstract Wiener space (see Example 3.3). At several points in the analysis there, the fact that $\dim(\mathfrak{v}) < \infty$ was used in a critical way. In particular, it was used to show that the stochastic integrals defining the Brownian motion on G as in equation (1.1) were well-defined. In the present paper, we have removed this restriction, as well as removing the “stratified” structure implicit in the construction as a Lie group extension.

Again, we note that, despite the use of the notation \mathfrak{g} , it is not assumed that the Lie bracket structure on \mathfrak{g}_{CM} extends to \mathfrak{g} , and so \mathfrak{g} itself is not necessarily a Lie algebra or Lie group. In [14] and [32], it was assumed that the Lie bracket was a continuous map defined on \mathfrak{g} . However, it turns out that the group construction on \mathfrak{g}_{CM} is the only necessary structure for the subsequent analysis. As is usual for the infinite-dimensional setting, while the heat kernel measure is itself supported on the larger space \mathfrak{g} , its critical analysis depends more on the structure of \mathfrak{g}_{CM} . Still, as was originally done in [14] and then in [32], one may instead define an abstract nilpotent Lie algebra starting with a continuous nilpotent bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. For example, in the event that $\mathfrak{g} = W \times \mathfrak{v}$ where \mathfrak{v} is a finite-dimensional Lie

algebra and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{v}$, it is well-known that this implies that the restriction of the bracket to $\mathfrak{g}_{CM} = H \times \mathfrak{v}$ is Hilbert-Schmidt. (For any continuous bilinear $\omega : W \times W \rightarrow K$ where K is a Hilbert space, one has that $\|\omega\|_{(H^{\otimes 2})^* \otimes K} < \infty$; this follows for example from Corollary 4.4 of [28].) More generally, in order for the subsequent theory to make sense, one would naturally need to require that \mathfrak{g}_{CM} be a Lie subalgebra of \mathfrak{g} , that is, for the restriction of the Lie bracket to \mathfrak{g}_{CM} to preserve \mathfrak{g}_{CM} . As the proofs in the sequel rely strongly on the bracket being Hilbert-Schmidt, it would be then necessary to add the Hilbert-Schmidt hypothesis as it does not follow immediately if one only assumes a continuous bracket on \mathfrak{g} which preserves \mathfrak{g}_{CM} .

Additionally, the spaces studied in the present paper are well-designed for the study of infinite-dimensional hypoelliptic heat kernel measures, and there has already been progress on proving quasi-invariance and stronger smoothness properties for these measures in the simplest case of a step two Lie algebra with finite-dimensional center; see [7] and [15]. More generally, the paper [35] explores related interesting lines of inquiry for heat kernel measures on infinite-dimensional groups, largely in the context of groups of maps from manifolds to Lie groups.

Acknowledgements. The author thanks Bruce Driver and Nathaniel Eldredge for helpful conversations during the writing of this paper.

2. ITERATED ITÔ INTEGRALS

Recall the standard construction of abstract Wiener spaces. Suppose that W is a real separable Banach space and \mathcal{B}_W is the Borel σ -algebra on W .

Definition 2.1. A measure μ on (W, \mathcal{B}_W) is called a (mean zero, non-degenerate) *Gaussian measure* provided that its characteristic functional is given by

$$\hat{\mu}(u) := \int_W e^{iu(x)} d\mu(x) = e^{-\frac{1}{2}q(u,u)}, \quad \text{for all } u \in W^*,$$

for $q = q_\mu : W^* \times W^* \rightarrow \mathbb{R}$ a symmetric, positive definite quadratic form. That is, q is a real inner product on W^* .

Theorem 2.2. *Let μ be a Gaussian measure on W . For $w \in W$, let*

$$\|w\|_H := \sup_{u \in W^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u,u)}},$$

and define the Cameron–Martin subspace $H \subset W$ by

$$H := \{h \in W : \|h\|_H < \infty\}.$$

Then H is a dense subspace of W , and there exists a unique inner product $\langle \cdot, \cdot \rangle_H$ on H such that $\|h\|_H^2 = \langle h, h \rangle_H$ for all $h \in H$, and H is a separable Hilbert space with respect to this inner product. For any $h \in H$, $\|h\|_W \leq C\|h\|_H$ for some $C < \infty$.

Alternatively, given W a real separable Banach space and H a real separable Hilbert space continuously embedded in W as a dense subspace, then for each $w^* \in W^*$ there exists a unique $h_{w^*} \in H$ such that $\langle h, w^* \rangle = \langle h, h_{w^*} \rangle_H$ for all $h \in H$. Then $W^* \ni w^* \mapsto h_{w^*} \in H$ is continuous, linear, and one-to-one with a dense range

$$(2.1) \quad H_* := \{h_{w^*} : w^* \in W^*\},$$

and $W^* \ni w^* \mapsto h_{w^*} \in W$ is continuous. A Gaussian measure on W is a Borel probability measure μ such that, for each $w^* \in W^*$, the random variable $w \mapsto \langle w, w^* \rangle$ under μ is a centered Gaussian with variance $\|h_{w^*}\|_H^2$.

Suppose that $P : H \rightarrow H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{h_j\}_{j=1}^m$ be an orthonormal basis for PH . Then we may extend P to a (unique) continuous operator from $W \rightarrow H$ (still denoted by P) by letting

$$(2.2) \quad Pw := \sum_{j=1}^m \langle w, h_j \rangle_H h_j$$

for all $w \in W$.

Notation 2.3. Let $\text{Proj}(W)$ denote the collection of finite rank projections on W such that $PW \subset H_*$ and $P|_H : H \rightarrow H$ is an orthogonal projection, that is, P has the form given in equation (2.2).

Let $\{B_t\}_{t \geq 0}$ be a Brownian motion on W with variance determined by

$$\mathbb{E}[\langle B_s, h \rangle_H \langle B_t, k \rangle_H] = \langle h, k \rangle_H \min(s, t),$$

for all $s, t \geq 0$ and $h, k \in H_*$, where H_* is as in (2.1). Note that for any $P \in \text{Proj}(W)$, PB is a Brownian motion on PH . In the rest of this section, we will verify the existence of martingales defined as certain iterated stochastic integrals with respect to B_t .

The following is Proposition 4.1 of [32]. Note that again this was stated in the context where $H = \mathfrak{g}_{CM}$ was a ‘‘semi-infinite Lie algebra’’, but a brief inspection of the proof shows that this is a general statement about stochastic integrals on Hilbert spaces.

Proposition 2.4. Let $\{P_m\}_{m=1}^\infty \subset \text{Proj}(W)$ such that $P_m|_H \uparrow I_H$. Then, for $\xi \in L^2(\Delta_n(t), H^{\otimes n})$ a continuous mapping, let

$$\begin{aligned} J_n^m(\xi)_t &:= \int_{\Delta_n(t)} \langle P_m^{\otimes n} \xi(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n} \rangle_{H^{\otimes n}} \\ &= \int_{\Delta_n(t)} \langle \xi(s), dP_m B_{s_1} \otimes \cdots \otimes dP_m B_{s_n} \rangle_{H^{\otimes n}}. \end{aligned}$$

Then $\{J_n^m(\xi)_t\}_{t \geq 0}$ is a continuous L^2 -martingale, and there exists a continuous L^2 -martingale $\{J_n(\xi)_t\}_{t \geq 0}$ such that

$$(2.3) \quad \lim_{m \rightarrow \infty} \mathbb{E} \left[\sup_{\tau \leq t} |J_n^m(\xi)_\tau - J_n(\xi)_\tau|^2 \right] = 0$$

and

$$(2.4) \quad \mathbb{E}|J_n(\xi)_t|^2 \leq \|\xi\|_{L^2(\Delta_n(t), H^{\otimes n})}^2$$

for all $t < \infty$. The process $J_n(\xi)$ is well-defined independent of the choice of increasing orthogonal projections $\{P_m\}_{m=1}^\infty$ into H_* , and so will be denoted by

$$J_n(\xi)_t = \int_{\Delta_n(t)} \langle \xi(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n} \rangle_{H^{\otimes n}}.$$

Now we may use this result to define stochastic integrals taking values in another Hilbert space K .

Proposition 2.5. *Let K be a Hilbert space and $F \in L^2(\Delta_n(t), (H^{\otimes n})^* \otimes K)$ be a continuous map. That is, $F : \Delta_n(t) \times H^{\otimes n} \rightarrow K$ is a map continuous in s and linear on $H^{\otimes n}$ such that*

$$\int_{\Delta_n(t)} \|F(s)\|_2^2 ds = \int_{\Delta_n(t)} \sum_{j_1, \dots, j_n=1}^{\infty} \|F(s)(h_{j_1} \otimes \dots \otimes h_{j_n})\|_K^2 ds < \infty.$$

Then

$$J_n^m(F)_t := \int_{\Delta_n(t)} F(dP_m B_{s_1} \otimes \dots \otimes dP_m B_{s_n})$$

is a continuous K -valued L^2 -martingale, and there exists a continuous K -valued L^2 -martingale $\{J_n(F)_t\}_{t \geq 0}$ such that

$$(2.5) \quad \lim_{m \rightarrow \infty} \mathbb{E} \left[\sup_{\tau \leq t} \|J_n^m(F)_\tau - J_n(F)_\tau\|_K^2 \right] = 0,$$

for all $t < \infty$. The martingale $J_n(F)$ is well-defined independent of the choice of orthogonal projections, and thus will be denoted by

$$J_n(F)_t = \int_{\Delta_n(t)} F(dB_{s_1} \otimes \dots \otimes dB_{s_n}).$$

Proof. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of K . Since $\langle F(s)(\cdot), e_j \rangle$ is linear on $H^{\otimes n}$, for each s there exists $\xi_j(s) \in H^{\otimes n}$ such that

$$(2.6) \quad \langle \xi_j(s), k_1 \otimes \dots \otimes k_n \rangle = \langle F(s)(k_1 \otimes \dots \otimes k_n), e_j \rangle.$$

If $\xi_j : \Delta_n(t) \rightarrow H^{\otimes n}$ is defined by equation (2.6), then clearly $\xi_j \in L^2(\Delta_n(t), H^{\otimes n})$ and in particular

$$\|F\|_{L^2(\Delta_n(t) \times H^{\otimes n}, K)}^2 = \sum_{j=1}^{\infty} \|\xi_j\|_{L^2(\Delta_n(t), H^{\otimes n})}^2 < \infty.$$

Thus, for $J_n(\xi_j)$ as defined in Proposition 2.4,

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^{\infty} |J_n(\xi_j)_t|^2 \right] &\leq \frac{t^n}{n!} \mathbb{E} \left[\int_{\Delta_n(t)} \sum_{j=1}^{\infty} |\langle \xi_j(s), dB_{s_1} \otimes \dots \otimes dB_{s_n} \rangle_{H^{\otimes n}}|^2 \right] \\ &= \frac{t^n}{n!} \sum_{j=1}^{\infty} \|\xi_j\|_{L^2(\Delta_n(t), H^{\otimes n})}^2 < \infty, \end{aligned}$$

and so we may write

$$\begin{aligned} \sum_{j=1}^{\infty} J_n(\xi_j)_t e_j &= \sum_{j=1}^{\infty} \int_{\Delta_n(t)} \langle \xi_j(s), dB_{s_1} \otimes \dots \otimes dB_{s_n} \rangle_{H^{\otimes n}} e_j \\ &= \int_{\Delta_n(t)} \sum_{j=1}^{\infty} \langle F(s)(dB_{s_1} \otimes \dots \otimes dB_{s_n}), e_j \rangle_K e_j \\ &= \int_{\Delta_n(t)} F(s)(dB_{s_1} \otimes \dots \otimes dB_{s_n}). \end{aligned}$$

Thus, taking $J_n(F)_t := \sum_{j=1}^{\infty} J_n(\xi_j)_t e_j$, we also have that

$$\mathbb{E} \|J_n(F)_t - J_n^m(F)_t\|_K^2 = \mathbb{E} \left[\sum_{j=1}^{\infty} |J_n(\xi_j)_t - J_n^m(\xi_j)_t|^2 \right] \rightarrow 0$$

as $m \rightarrow \infty$ by (2.3) and dominated convergence since

$$\mathbb{E} |J_n(\xi_j)_t - J_n^m(\xi_j)_t|^2 \leq 4 \|\xi_j\|_{L^2(\Delta_n(t), H^{\otimes n})}^2$$

by (2.4). Then equation (2.5) holds by Doob's maximal inequality. \square

Note that the preceding results then imply that one may define the above stochastic integrals with respect to *any* increasing sequence of orthogonal projections – that is, we need not require that the projections extend continuously to W .

Proposition 2.6. *Let V be an arbitrary finite-dimensional subspace of H , and let $\pi : H \rightarrow V$ denote orthogonal projection onto V . Then for any Hilbert space K and $F \in L^2(\Delta_n(t), (H^{\otimes n})^* \otimes K)$ a continuous map, the stochastic integral*

$$J_n^\pi(F)_t := \int_{\Delta_n(t)} F(d\pi B_{s_1} \otimes \cdots \otimes d\pi B_{s_n})$$

is well-defined, and $\{J_n^\pi(F)_t\}_{t \geq 0}$ is a continuous K -valued L^2 -martingale. Moreover, if V_m is an increasing sequence of finite-dimensional subspaces of H such that the corresponding orthogonal projections $\pi_m \uparrow I_H$, then

$$\lim_{m \rightarrow \infty} \mathbb{E} \left[\sup_{\tau \leq t} \|J_n^{\pi_m}(F)_\tau - J_n(F)_\tau\|^2 \right] = 0,$$

where $J_n(F)$ is as defined in Proposition 2.5.

Proof. First consider the case that $K = \mathbb{R}$, and thus $F(s) = \langle \xi(s), \cdot \rangle$ for a continuous $\xi \in L^2(\Delta_n(t), H^{\otimes n})$. Since $\pi^{\otimes n} \xi \in L^2(\Delta_n(t), H^{\otimes n})$, the definition of $J_n^\pi(\xi) = J_n(\pi^{\otimes n} \xi)$ follows from Proposition 2.4. Moreover, by equation (2.4),

$$\begin{aligned} \mathbb{E} |J_n^{\pi_m}(\xi)_t - J_n(\xi)_t|^2 &= \mathbb{E} |J_n(\pi_m^{\otimes n} \xi)_t - J_n(\xi)_t|^2 = \mathbb{E} |J_n(\pi_m^{\otimes n} \xi - \xi)_t|^2 \\ &\leq \|\pi_m^{\otimes n} \xi - \xi\|_{L^2(\Delta_n(t), H^{\otimes n})}^2 \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. Now the proof for general F follows just as in Proposition 2.5. \square

3. ABSTRACT NILPOTENT LIE ALGEBRAS AND GROUPS

Definition 3.1. Let $(\mathfrak{g}, \mathfrak{g}_{CM}, \mu)$ be an abstract Wiener space such that \mathfrak{g}_{CM} is equipped with a nilpotent Hilbert-Schmidt Lie bracket. Then we will call $(\mathfrak{g}, \mathfrak{g}_{CM}, \mu)$ an *abstract nilpotent Lie algebra*.

The Baker-Campbell-Hausdorff-Dynkin formula implies that

$$\log(e^A e^B) = A + B + \sum_{k=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} a_{n,m}^k \text{ad}_A^{n_1} \text{ad}_B^{m_1} \cdots \text{ad}_A^{n_k} \text{ad}_B^{m_k} A,$$

for all $A, B \in \mathfrak{g}_{CM}$, where

$$a_{n,m}^k := \frac{(-1)^k}{(k+1)m!n!(|n|+1)},$$

$\mathcal{I}_k := \{(n, m) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k : n_i + m_i > 0 \text{ for all } 1 \leq i \leq k\}$, and for each multi-index $n \in \mathbb{Z}_+^k$,

$$n! = n_1! \cdots n_k! \quad \text{and} \quad |n| = n_1 + \cdots + n_k;$$

see, for example, [18]. If \mathfrak{g}_{CM} is nilpotent of step r , then

$$\text{ad}_A^{n_1} \text{ad}_B^{m_1} \cdots \text{ad}_A^{n_k} \text{ad}_B^{m_k} A = 0 \quad \text{if } |n| + |m| \geq r.$$

for $A, B \in \mathfrak{g}_{CM}$. Since \mathfrak{g}_{CM} is simply connected and nilpotent, the exponential map is a global diffeomorphism (see, for example, Theorems 3.6.2 of [38] or 1.2.1 of [12]). In particular, we may view \mathfrak{g}_{CM} as both a Lie algebra and Lie group, and one may verify that

$$(3.1) \quad g \cdot h = g + h + \sum_{k=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} d_{n,m}^k \text{ad}_g^{n_1} \text{ad}_h^{m_1} \cdots \text{ad}_g^{n_k} \text{ad}_h^{m_k} g$$

defines a group structure on \mathfrak{g}_{CM} . Note that $g^{-1} = -g$ and the identity $\mathbf{e} = (0, 0)$.

Definition 3.2. When we wish to emphasize the group structure on \mathfrak{g}_{CM} , we will denote \mathfrak{g}_{CM} by G_{CM} .

Lemma 3.3. *The Banach space topology on \mathfrak{g}_{CM} makes G_{CM} into a topological group.*

Proof. Since \mathfrak{g}_{CM} is a topological vector space, $g \mapsto g^{-1} = -g$ and $(g_1, g_2) \mapsto g_1 + g_2$ are continuous by definition. The map $(g_1, g_2) \mapsto [g_1, g_2]$ is continuous in the \mathfrak{g}_{CM} topology by the boundedness of the Lie bracket. It then follows from (3.1) that $(g_1, g_2) \mapsto g_1 \cdot g_2$ is continuous as well. \square

3.1. Measurable group actions on G . As discussed in the introduction, given a Hilbert-Schmidt Lie bracket on \mathfrak{g}_{CM} and a subsequently defined group operation on G_{CM} , one may define a measurable action on G by left or right multiplication by an element of G_{CM} .

In particular, let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of \mathfrak{g}_{CM} . For now, fix n and consider the mapping $\mathfrak{g}_{CM} \rightarrow \mathfrak{g}_{CM}^*$ given by $h \mapsto \langle \text{ad}_h \cdot, e_n \rangle$. Then this is a continuous linear map on \mathfrak{g}_{CM} and in the usual way we may make the identification of $\mathfrak{g}_{CM}^* \cong \mathfrak{g}_{CM}$ so that we define the operator $A_n : \mathfrak{g}_{CM} \rightarrow \mathfrak{g}_{CM}$ given by

$$\langle A_n h, k \rangle = \langle \text{ad}_h k, e_n \rangle;$$

in particular, $A_n h = \text{ad}_h^* e_n$. Note that, for any $h, k \in \mathfrak{g}_{CM}$

$$\langle A_n^* h, k \rangle = \langle A_n k, h \rangle = \langle \text{ad}_k^* e_n, h \rangle = \langle e_n, \text{ad}_k h \rangle = -\langle e_n, \text{ad}_h k \rangle = -\langle \text{ad}_h^* e_n, k \rangle$$

and thus $A_n^* = -A_n$. Now fix $h \in G_{CM} = \mathfrak{g}_{CM}$. Then for $\text{ad}_h : \mathfrak{g}_{CM} \rightarrow \mathfrak{g}_{CM}$ we may write

$$\text{ad}_h k = \sum_n \langle \text{ad}_h k, e_n \rangle e_n = \sum_n \langle A_n h, k \rangle e_n.$$

Since each $\langle A_n h, \cdot \rangle \in \mathfrak{g}_{CM}^*$ has a measurable linear extension to G such that $\|\langle A_n h, \cdot \rangle\|_{L^2(\mu)} = \|A_n h\|_{\mathfrak{g}_{CM}}$ (see, for example, Theorem 2.10.11 of [10]), we may extend ad_h to a measurable linear transformation from $G = \mathfrak{g}$ to $G_{CM} = \mathfrak{g}_{CM}$ (still denoted by ad_h) given by

$$\text{ad}_h g := \sum_n \langle A_n h, g \rangle e_n.$$

Note that here we are using the fact that

$$\begin{aligned} \sum_n \|\langle A_n h, \cdot \rangle\|_{L^2(\mu)}^2 &= \sum_n \|A_n h\|_H^2 = \sum_{n,m} \langle A_n h, e_m \rangle^2 \\ &= \sum_{n,m} \langle \text{ad}_h e_m, e_n \rangle^2 \leq \|h\|^2 \|\cdot\|_{HS}^2 \end{aligned}$$

which implies that

$$\sum_n \langle A_n h, g \rangle^2 < \infty \quad g\text{-a.s.}$$

Similarly, we may define

$$\text{ad}_g h := -\text{ad}_h g = -\sum_n \langle A_n h, g \rangle e_n.$$

In a similar way, note that we may write, for $h, k \in G_{CM}$ and $m < r$,

$$\begin{aligned} \text{ad}_h^m k &= \sum_{\ell_1} \cdots \sum_{\ell_m} \left(\prod_{b=1}^{m-1} \langle A_{\ell_b} h, e_{\ell_{b+1}} \rangle \right) \langle A_{\ell_m} h, k \rangle e_{\ell_1} \\ &= (-1)^m \sum_{\ell_1} \cdots \sum_{\ell_m} \left(\prod_{b=1}^{m-1} \langle A_{\ell_b} e_{\ell_{b+1}}, h \rangle \right) \langle A_{\ell_m} k, h \rangle e_{\ell_1}. \end{aligned}$$

and thus for $h, k, x \in G_{CM}$ and $n + m < r$

$$\begin{aligned} \text{ad}_k^n \text{ad}_h^m x &= (-1)^n \sum_{j_1} \cdots \sum_{j_n} \sum_{\ell_1} \cdots \sum_{\ell_m} \left(\prod_{a=1}^{n-1} \langle A_{j_a} e_{j_{a+1}}, k \rangle \right) \langle A_{j_n} e_{\ell_1}, k \rangle \\ &\quad \left(\prod_{b=1}^{m-1} \langle A_{\ell_b} h, e_{\ell_{b+1}} \rangle \right) \langle A_{\ell_m} h, x \rangle e_{j_1}. \end{aligned}$$

More generally for $|n| + |m| < r$

$$\begin{aligned} &\text{ad}_k^{n_1} \text{ad}_h^{m_1} \cdots \text{ad}_k^{n_s} \text{ad}_h^{m_s} k \\ &= (-1)^{|n|} \sum_{j_1^1} \cdots \sum_{\ell_{m_s}^s} \prod_{c=1}^s \left\{ \left(\prod_{a_c=1}^{n_c-1} \langle A_{j_{a_c}^c} e_{j_{a_c+1}^c}, k \rangle \right) \langle A_{j_{n_c}^c} e_{\ell_1^c}, k \rangle \right. \\ &\quad \left. \times \left(\prod_{b_c=1}^{m_c-1} \langle A_{\ell_{b_c}^c} h, e_{\ell_{b_c+1}^c} \rangle \right) \right\} \left\{ \prod_{c=1}^{s-1} \langle A_{\ell_{m_c}^c} h, e_{j_1^{c+1}} \rangle \right\} \langle A_{\ell_{m_s}^s} h, k \rangle e_{j_1^1} \end{aligned}$$

Thus, for $h \in G_{CM}$ and $|n| + |m| < r$, we may define a measurable action on G given by

$$\begin{aligned} G \ni g &\mapsto \text{ad}_g^{n_1} \text{ad}_h^{m_1} \cdots \text{ad}_g^{n_s} \text{ad}_h^{m_s} g \\ &:= (-1)^{|n|} \sum_{j_1^1} \cdots \sum_{\ell_{m_s}^s} \prod_{c=1}^s \left\{ \left(\prod_{a_c=1}^{n_c-1} \langle A_{j_{a_c}^c} e_{j_{a_c+1}^c}, g \rangle \right) \langle A_{j_{n_c}^c} e_{\ell_1^c}, g \rangle \right. \\ &\quad \left. \times \left(\prod_{b_c=1}^{m_c-1} \langle A_{\ell_{b_c}^c} h, e_{\ell_{b_c+1}^c} \rangle \right) \right\} \left\{ \prod_{c=1}^{s-1} \langle A_{\ell_{m_c}^c} h, e_{j_1^{c+1}} \rangle \right\} \langle A_{\ell_{m_s}^s} h, k \rangle e_{j_1^1} \in G_{CM}. \end{aligned}$$

(For $|n| + |m| \geq r$, we define this mapping to be 0, which is certainly measurable.)
Again, we are using that

$$\sum_{j_1^1} \left\| \sum_{j_2^1} \cdots \sum_{\ell_{m_s}^s} \prod_{c=1}^s \left\{ \left(\prod_{a_c=1}^{n_c-1} \langle A_{j_{a_c}^c} e_{j_{a_c+1}^c}, \cdot \rangle \right) \langle A_{j_{n_c}^c} e_{\ell_1^c}, \cdot \rangle \right. \right. \\ \left. \left. \times \left(\prod_{b_c=1}^{m_c-1} \langle A_{\ell_{b_c}^c} h, e_{\ell_{b_c+1}^c} \rangle \right) \right\} \left\{ \prod_{c=1}^{s-1} \langle A_{\ell_{m_c}^c} h, e_{j_1^{c+1}} \rangle \right\} \langle A_{\ell_{m_s}^s} h, k \rangle e_{j_1^1} \right\|_{L^2(\mu)}^2 < \infty.$$

This holds by straightforward but tedious computations — in fact, iterative applications of Cauchy-Schwarz combined with the fact that $\sum_{n,m} \|A_n e_m\|^2 = \|[\cdot, \cdot]\|_{HS}^2 < \infty$. Thus,

$$\sum_{j_1^1} \left(\sum_{j_2^1} \cdots \sum_{\ell_{m_s}^s} \prod_{c=1}^s \left\{ \left(\prod_{a_c=1}^{n_c-1} \langle A_{j_{a_c}^c} e_{j_{a_c+1}^c}, g \rangle \right) \langle A_{j_{n_c}^c} e_{\ell_1^c}, g \rangle \right. \right. \\ \left. \left. \times \left(\prod_{b_c=1}^{m_c-1} \langle A_{\ell_{b_c}^c} h, e_{\ell_{b_c+1}^c} \rangle \right) \right\} \left\{ \prod_{c=1}^{s-1} \langle A_{\ell_{m_c}^c} h, e_{j_1^{c+1}} \rangle \right\} \langle A_{\ell_{m_s}^s} h, k \rangle e_{j_1^1} \right)^2 < \infty,$$

g -a.s. and $\text{ad}_g^{n_1} \text{ad}_h^{m_1} \cdots \text{ad}_g^{n_s} \text{ad}_h^{m_s} g$ as given above is defined a.s. Thus, we have the following result.

Proposition 3.4. *For $h \in G_{CM}$, the mapping $G \ni g \mapsto g \cdot h \in G$ defined analogously to (3.1) is a measurable right group action by G_{CM} on G , and similarly for the left action $g \mapsto h \cdot g$.*

3.2. Examples of abstract nilpotent Lie algebras.

Example 3.1 (Free nilpotent Lie algebras). Starting with an abstract Wiener space (W, H, μ) , one may construct in the standard way the abstract free nilpotent Lie algebra of step r with generators $\{h_i\}_{i=1}^\infty$ an orthonormal basis of H . See for example Section 0 of [21].

Example 3.2 (Heisenberg-like algebras). Let (W_i, H_i, μ_i) for $i = 1, 2$ be abstract Wiener spaces. Then for any $\omega : H_1 \times H_1 \rightarrow H_2$ a Hilbert-Schmidt map, we may define a Lie bracket on $\mathfrak{g}_{CM} = H_1 \times H_2$ by

$$[(h_1, h_2), (h'_1, h'_2)] := (0, \omega(h_1, h'_1)),$$

and $\mathfrak{g} = W_1 \times W_2$ may be thought of as an abstract Heisenberg-like algebra as in [14].

These abstract Heisenberg-like algebras are central extensions of one abstract Wiener space by another abstract Wiener space. The next example generalizes this construction.

Example 3.3 (Extensions of Lie algebras). Let \mathfrak{v} and \mathfrak{h} be Lie algebras, and let $\text{Der}(\mathfrak{v})$ denote the set of derivations on \mathfrak{v} ; that is, $\text{Der}(\mathfrak{v})$ consists of all linear maps $\rho : \mathfrak{v} \rightarrow \mathfrak{v}$ satisfying Leibniz's rule:

$$\rho([X, Y]_{\mathfrak{v}}) = [\rho(X), Y]_{\mathfrak{v}} + [X, \rho(Y)]_{\mathfrak{v}}.$$

Now suppose there are a linear mapping $\alpha : \mathfrak{h} \rightarrow \text{Der}(\mathfrak{v})$ and a skew-symmetric bilinear mapping $\omega : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{v}$, satisfying, for all $X, Y, Z \in \mathfrak{h}$,

$$(B1) \quad [\alpha_X, \alpha_Y] - \alpha_{[X, Y]_{\mathfrak{h}}} = \text{ad}_{\omega(X, Y)}$$

and

$$(B2) \quad \sum_{\text{cyclic}} (\alpha_X \omega(Y, Z) - \omega([X, Y]_{\mathfrak{h}}, Z)) = 0.$$

Then, one may verify that, for $X_1 + V_1, X_2 + V_2 \in \mathfrak{h} \oplus \mathfrak{v}$,

$$[X_1 + V_1, X_2 + V_2]_{\mathfrak{g}} := [X_1, X_2]_{\mathfrak{h}} + \omega(X_1, X_2) + \alpha_{X_1} V_2 - \alpha_{X_2} V_1 + [V_1, V_2]_{\mathfrak{v}}$$

defines a Lie bracket on $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{v}$, and we say \mathfrak{g} is an extension of \mathfrak{h} over \mathfrak{v} . That is, \mathfrak{g} is the Lie algebra with ideal \mathfrak{v} and quotient algebra $\mathfrak{g}/\mathfrak{v} = \mathfrak{h}$. The associated exact sequence is

$$0 \rightarrow \mathfrak{v} \xrightarrow{\iota_1} \mathfrak{g} \xrightarrow{\pi_2} \mathfrak{h} \rightarrow 0,$$

where ι_1 is inclusion and π_2 is projection. In fact, these are the only extensions of \mathfrak{h} over \mathfrak{v} (see, for example, [4]).

Now suppose that (W, H, μ) is a real abstract Wiener space, and $(\mathfrak{v}, \mathfrak{v}_{CM}, \mu^0)$ is an abstract nilpotent Lie algebra. Motivated by the previous discussion, we may consider H as an abelian Lie algebra and construct extensions of H over \mathfrak{v}_{CM} . In this case, we need a linear mapping $\alpha : H \rightarrow \text{Der}(\mathfrak{v}_{CM})$ and a skew-symmetric bilinear mapping $\omega : H \times H \rightarrow \mathfrak{v}_{CM}$, such that ω and $\alpha : H \times \mathfrak{v}_{CM} \rightarrow \mathfrak{v}_{CM}$ are both Hilbert-Schmidt and together ω and α satisfy (B1) and (B2), which in this setting become

$$[\alpha_X, \alpha_Y] = \text{ad}_{\omega(X, Y)}$$

and

$$\alpha_X \omega(Y, Z) + \alpha_Y \omega(Z, X) + \alpha_Z \omega(X, Y) = 0,$$

for all $X, Y, Z \in H$. Then we may define a Lie algebra structure on $\mathfrak{g}_{CM} = H \oplus \mathfrak{v}_{CM}$ via the Lie bracket

$$[(X_1, V_1), (X_2, V_2)]_{\mathfrak{g}_{CM}} := (0, \omega(X_1, X_2) + \alpha_{X_1} V_2 - \alpha_{X_2} V_1).$$

Again, these are in fact the only extensions of H over \mathfrak{v}_{CM} .

Combining Examples 3.2 and 3.3 shows that these constructions are iterative, in that one may construct new abstract nilpotent Lie algebras as Lie algebra extensions of another abstract nilpotent Lie algebra. The next example builds on the previous one to give one precise way to construct some Lie algebra extensions.

Example 3.4. Let $\beta : H \rightarrow \mathfrak{v}_{CM}$ be any Hilbert-Schmidt map, and for $X, Y \in H$ and $V \in \mathfrak{v}_{CM}$ define

$$\begin{aligned} \omega(X, Y) &:= [\beta(X), \beta(Y)]_{\mathfrak{v}_{CM}} \\ \alpha_X V &:= \text{ad}_{\beta(X)} V = [\beta(X), V]_{\mathfrak{v}_{CM}}. \end{aligned}$$

Then the Lie bracket on \mathfrak{g}_{CM} is given by

$$\begin{aligned} [(X, V), (Y, U)] \\ = (0, [\beta(X), \beta(Y)]_{\mathfrak{v}_{CM}} + [\beta(X), U]_{\mathfrak{v}_{CM}} - [\beta(Y), V]_{\mathfrak{v}_{CM}} + [V, U]_{\mathfrak{v}_{CM}}). \end{aligned}$$

Note that, if \mathfrak{v}_{CM} is nilpotent of step r , then \mathfrak{g}_{CM} will automatically be nilpotent of step r .

The examples above demonstrate that the space of abstract nilpotent Lie algebras is quite rich, and there are many natural examples with a straightforward construction. This significantly improves the results of [32], which studied heat kernel measures on nilpotent extensions of abstract Wiener spaces over *finite-dimensional*

nilpotent Lie algebras. For example, the restriction $\dim(\mathfrak{v}_{CM}) < \infty$ trivializes Example 3.4. We elaborate in the following remark.

Remark 3.5. Returning to Example 3.4, since β is linear and continuous, we have the decomposition

$$H = \text{Nul}(\beta) \oplus \text{Nul}(\beta)^\perp,$$

where $\dim(\text{Nul}(\beta)^\perp) \leq \dim(\mathfrak{v}_{CM})$. Thus, for $X, Y \in H$ we may write $X = X_1 + X_2, Y = Y_1 + Y_2 \in \text{Nul}(\beta) \oplus \text{Nul}(\beta)^\perp$ and

$$[(X_1 + X_2, 0), (Y_1 + Y_2, 0)] = [\beta(X_1 + X_2), \beta(Y_1 + Y_2)] = [\beta(X_2), \beta(Y_2)],$$

and ω is a map on $\text{Nul}(\beta)^\perp \times \text{Nul}(\beta)^\perp$. Thus, $[\text{Nul}(\beta), \text{Nul}(\beta)] = \{0\}$ and similarly $[\text{Nul}(\beta), \mathfrak{v}] = \{0\}$. So

$$\mathfrak{g}_{CM} = H \oplus \mathfrak{v}_{CM} = \text{Nul}(\beta) \oplus \text{Nul}(\beta)^\perp \oplus \mathfrak{v}_{CM},$$

and, in particular, when $\dim(\mathfrak{v}_{CM}) < \infty$, \mathfrak{g}_{CM} is just an extension of the finite-dimensional vector space $\text{Nul}(\beta)^\perp$ by the finite-dimensional Lie algebra \mathfrak{v}_{CM} , and the construction is not truly infinite-dimensional.

3.3. Properties of \mathfrak{g}_{CM} . This section collects some results for topological and geometric properties of \mathfrak{g}_{CM} that we'll require for the sequel.

Proposition 3.6. *For all $m \geq 2$, $[[[\cdot, \cdot], \dots], \cdot] : \mathfrak{g}_{CM}^{\otimes m} \rightarrow \mathfrak{g}_{CM}$ is Hilbert-Schmidt.*

Proof. For $m = 2$, this follows from the definition of G . Now assume the statement holds for all $m = \ell$, and consider $m = \ell + 1$. Writing $[[h_{i_1}, h_{i_2}], \dots, h_{i_{\ell+1}}] \in \mathfrak{g}_{CM}$ in terms of the orthonormal basis $\{e_j\}_{j=1}^\infty$ and using Hölder's inequality gives

$$\begin{aligned} \|[[\cdot, \cdot], \dots], \cdot] \|_2^2 &= \| [[[\cdot, \cdot], \dots], \cdot] \|_{(\mathfrak{g}_{CM}^*)^{\otimes \ell+1} \otimes \mathfrak{g}_{CM}} \\ &= \sum_{i_1, \dots, i_{\ell+1}=1}^{\infty} \| [[[h_{i_1}, h_{i_2}], \dots, h_{i_\ell}], h_{i_{\ell+1}}] \|^2 \\ &= \sum_{i_1, \dots, i_{\ell+1}=1}^{\infty} \left\| \sum_{j=1}^{\infty} [e_j, h_{i_{\ell+1}}] \langle e_j, [[h_{i_1}, h_{i_2}], \dots, h_{i_\ell}] \rangle \right\|^2 \\ &\leq \sum_{i_1, \dots, i_{\ell+1}=1}^{\infty} \left(\sum_{j=1}^{\infty} \| [e_j, h_{i_{\ell+1}}] \|^2 \right) \left(\sum_{j=1}^{\infty} | \langle e_j, [[h_{i_1}, h_{i_2}], \dots, h_{i_\ell}] \rangle |^2 \right) \\ &= \left(\sum_{i_{\ell+1}=1}^{\infty} \sum_{j=1}^{\infty} \| [e_j, h_{i_{\ell+1}}] \|^2 \right) \left(\sum_{i_1, \dots, i_\ell=1}^{\infty} \| [[h_{i_1}, h_{i_2}], \dots, h_{i_\ell}] \|^2 \right) \\ &\leq \| [\cdot, \cdot] \|_{(\mathfrak{g}_{CM}^*)^{\otimes 2} \otimes \mathfrak{g}_{CM}}^2 \| [[[\cdot, \cdot], \dots], \cdot] \|_{(\mathfrak{g}_{CM}^*)^{\otimes \ell} \otimes \mathfrak{g}_{CM}}^2 \end{aligned}$$

and the last line is finite by the induction hypothesis. \square

Next we will recall that the flat and geometric topologies on \mathfrak{g}_{CM} are equivalent. First we set the following notation.

Notation 3.7. For $g \in G_{CM}$, let $L_g : G_{CM} \rightarrow G_{CM}$ and $R_g : G_{CM} \rightarrow G_{CM}$ denote left and right multiplication by g , respectively. As G_{CM} is a vector space, to each $g \in G_{CM}$ we can associate the tangent space $T_g G_{CM}$ to G_{CM} at g , which

is naturally isomorphic to G_{CM} . For $f : G_{CM} \rightarrow \mathbb{R}$ a Frechét smooth function and $v, x \in G_{CM}$ and $h \in \mathfrak{g}_{CM}$, let

$$f'(x)h := \partial_h f(x) = \left. \frac{d}{dt} \right|_0 f(x + th),$$

and let $v_x \in T_x G_{CM}$ denote the tangent vector satisfying $v_x f = f'(x)v$. If $\sigma(t)$ is any smooth curve in G_{CM} such that $\sigma(0) = x$ and $\dot{\sigma}(0) = v$ (for example, $\sigma(t) = x + tv$), then

$$L_{g*} v_x = \left. \frac{d}{dt} \right|_0 g \cdot \sigma(t).$$

Notation 3.8. Let $C^1([0, 1], G_{CM})$ denote the collection of C^1 -paths $g : [0, 1] \rightarrow G_{CM}$. The length of g is defined as

$$\ell_{CM}(g) := \int_0^1 \|L_{g^{-1}(s)*} g'(s)\|_{\mathfrak{g}_{CM}} ds.$$

The Riemannian distance between $x, y \in G_{CM}$ is then defined as

$$d(x, y) := \inf \{ \ell_{CM}(g) : g \in C_h^1([0, 1], G_{CM}) \text{ such that } g(0) = x \text{ and } g(1) = y \}.$$

The following proposition was proved as Corollary 4.13 in [32]. This was proved under the conditions that \mathfrak{g}_{CM} was a “semi-infinite Lie algebra”, that is, under the assumption that \mathfrak{g}_{CM} was a nilpotent Lie algebra extension (as in Example 3.3) of a finite-dimensional nilpotent Lie algebra \mathfrak{v} by an abstract Wiener space. However, a cursory inspection of the proofs there will show that they only depended on the fact that the Lie bracket was Hilbert-Schmidt, and not on the “stratified” structure of $\mathfrak{g}_{CM} = H \times \mathfrak{v}$ or the fact that the image of the Lie bracket was a finite-dimensional subspace.

Proposition 3.9. *The topology on G_{CM} induced by d is equivalent to the Hilbert topology induced by $\|\cdot\|_{\mathfrak{g}_{CM}}$.*

We may find a uniform lower bound on the Ricci curvature of all finite-dimensional subgroups of G_{CM} . Finite-dimensional subgroups of G_{CM} may be obtained by taking the Lie algebra generated by any finite-dimensional subspace of \mathfrak{g}_{CM} (which is again necessarily finite-dimensional by the nilpotence of the bracket) and endowing it with the standard group operation via the Baker-Campbell-Hausdorff-Dynkin formula. An analogue of the following proposition was proved as Proposition 3.23 and Corollary 3.24 in [32]. It follows directly from the form of the Ricci curvature on nilpotent groups (endowed with a left invariant metric) and the assumption that the Lie bracket is Hilbert-Schmidt. This proof is essentially the same as in [32], but it is quite brief and so is included for completeness.

Proposition 3.10. *Let*

$$k := -\frac{1}{2} \sup \left\{ \|[\cdot, X]\|_{\mathfrak{g}_{CM}^* \otimes \mathfrak{g}_{CM}}^2 : \|X\|_{\mathfrak{g}_{CM}} = 1 \right\}.$$

Then $k > -\infty$ and k is the largest constant such that

$$\langle \text{Ric}^\pi X, X \rangle_{\mathfrak{g}_\pi} \geq k \|X\|_{\mathfrak{g}_\pi}^2, \quad \text{for all } X \in \mathfrak{g}_\pi,$$

holds uniformly for all \mathfrak{g}_π finite-dimensional Lie subalgebras of \mathfrak{g}_{CM} .

Proof. For \mathfrak{g} any nilpotent Lie algebra with orthonormal basis Γ ,

$$\langle \text{Ric } X, X \rangle = \frac{1}{4} \sum_{Y \in \Gamma} \|\text{ad}_Y^* X\|^2 - \frac{1}{2} \sum_{Y \in \Gamma} \|\text{ad}_Y X\|^2 \geq -\frac{1}{2} \sum_{Y \in \Gamma} \|[Y, X]\|^2$$

for all $X \in \mathfrak{g}$. Thus, for \mathfrak{g}_π any finite-dimensional Lie algebra

$$\langle \text{Ric}^\pi X, X \rangle_{\mathfrak{g}_\pi} \geq k_\pi \|X\|_{\mathfrak{g}_\pi}^2, \quad \text{for all } X \in \mathfrak{g}_\pi,$$

where

$$(3.2) \quad k_\pi := -\frac{1}{2} \sup \left\{ \|[\cdot, X]\|_{\mathfrak{g}_\pi^* \otimes \mathfrak{g}_\pi}^2 : \|X\|_{\mathfrak{g}_\pi} = 1 \right\} \geq -\frac{1}{2} \|[\cdot, \cdot]\|_2^2 > -\infty.$$

Taking the infimum of k_π over all \mathfrak{g}_π completes the proof. \square

4. BROWNIAN MOTION ON G

Suppose that B_t is a smooth curve in \mathfrak{g}_{CM} with $B_0 = 0$, and consider the differential equation

$$\dot{g}_t = g_t \dot{B}_t := L_{g_t^*} \dot{B}_t, \quad \text{with } g_0 = \mathbf{e}.$$

The solution g_t may be written as follows (see [36]): For $t > 0$, let $\Delta_n(t)$ denote the simplex in \mathbb{R}^n given by

$$\{s = (s_1, \dots, s_n) \in \mathbb{R}^n : 0 < s_1 < s_2 < \dots < s_n < t\}.$$

Let \mathcal{S}_n denote the permutation group on $(1, \dots, n)$, and, for each $\sigma \in \mathcal{S}_n$, let $e(\sigma)$ denote the number of ‘‘errors’’ in the ordering $(\sigma(1), \sigma(2), \dots, \sigma(n))$, that is, $e(\sigma) = \#\{j < n : \sigma(j) > \sigma(j+1)\}$. Then

$$(4.1) \quad g_t = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} \left((-1)^{e(\sigma)} / n^2 \binom{n-1}{e(\sigma)} \right) \times \int_{\Delta_n(t)} [\dots [\dot{B}_{s_{\sigma(1)}}, \dot{B}_{s_{\sigma(2)}}], \dots, \dot{B}_{s_{\sigma(n)}}] ds,$$

where the $n = 1$ term is understood to be $\int_0^t dB_s = B_t$. Using this as our motivation, we first explore stochastic integral analogues of equation (4.1) where the smooth curve B is replaced by Brownian motion on \mathfrak{g} .

4.1. Brownian motion and finite-dimensional approximations. We now return to the setting of an abstract Wiener space $(\mathfrak{g}, \mathfrak{g}_{CM}, \mu)$ endowed with a nilpotent Hilbert-Schmidt Lie bracket on \mathfrak{g}_{CM} . Again, let B_t denote Brownian motion on \mathfrak{g} . By equation (4.1), the solution to the Stratonovich stochastic differential equation

$$\delta g_t = L_{g_t^*} \delta B_t, \quad \text{with } g_0 = \mathbf{e},$$

should be given by

$$(4.2) \quad g_t = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} c_n^\sigma \int_{\Delta_n(t)} [[\dots [\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \dots], \delta B_{s_{\sigma(n)}}],$$

for coefficients c_n^σ determined by equation (4.1).

To understand the integrals in (4.2), consider the following heuristic computation. Let $\{M_n(t)\}_{t \geq 0}$ denote the process in $\mathfrak{g}^{\otimes n}$ defined by

$$M_n(t) := \int_{\Delta_n(t)} \delta B_{s_1} \otimes \dots \otimes \delta B_{s_n}.$$

By repeatedly applying the definition of the Stratonovich integral, the iterated Stratonovich integral $M_n(t)$ may be realized as a linear combination of iterated Itô integrals:

$$M_n(t) = \sum_{m=\lceil n/2 \rceil}^n \frac{1}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} I_t^n(\alpha),$$

where

$$\mathcal{J}_n^m := \left\{ (\alpha_1, \dots, \alpha_m) \in \{1, 2\}^m : \sum_{i=1}^m \alpha_i = n \right\},$$

and, for $\alpha \in \mathcal{J}_n^m$, $I_t^n(\alpha)$ is the iterated Itô integral

$$I_t^n(\alpha) = \int_{\Delta_m(t)} dX_{s_1}^1 \otimes \cdots \otimes dX_{s_m}^m$$

with

$$dX_s^i = \begin{cases} dB_s & \text{if } \alpha_i = 1 \\ \sum_{j=1}^{\infty} h_j \otimes h_j ds & \text{if } \alpha_i = 2 \end{cases};$$

compare with Proposition 1 of [9]. This change from multiple Stratonovich integrals to multiple Itô integrals may also be recognized as a specific case of the Hu-Meyer formulas [23, 24], but we will compute more explicitly to verify that our integrals are well-defined.

Define $F_1 : \mathfrak{g}_{CM} \rightarrow \mathfrak{g}_{CM}$ by $F_1(k) = k$, and for $n \in \{2, \dots, r\}$ define $F_n : \mathfrak{g}_{CM}^{\otimes n} \rightarrow \mathfrak{g}_{CM}$ by

$$(4.3) \quad F_n(k_1 \otimes \cdots \otimes k_n) := [[[\cdots [k_1, k_2], k_3], \cdots], k_n].$$

For each fixed n and $\sigma \in \mathcal{S}_n$, define $F_n^\sigma : \mathfrak{g}_{CM}^{\otimes n} \rightarrow \mathfrak{g}_{CM}$ by

$$(4.4) \quad \begin{aligned} F_n^\sigma(k_1 \otimes \cdots \otimes k_n) &:= F_n(k_{\sigma(1)} \otimes \cdots \otimes k_{\sigma(n)}) \\ &= [[\cdots [k_{\sigma(1)}, k_{\sigma(2)}], \cdots], k_{\sigma(n)}]. \end{aligned}$$

Then we may write

$$g_t = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} c_n^\sigma F_n^\sigma(M_n(t)) = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} \sum_{m=\lceil n/2 \rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} F_n^\sigma(I_t^n(\alpha)),$$

presuming we can make sense of the integrals $F_n^\sigma(I_t^n(\alpha))$.

For each α , let $p_\alpha = \#\{i : \alpha_i = 1\}$ and $q_\alpha = \#\{i : \alpha_i = 2\}$ (so that $p_\alpha + q_\alpha = m$ when $\alpha \in \mathcal{J}_n^m$), and let

$$\mathcal{J}_n := \bigcup_{m=\lceil n/2 \rceil}^n \mathcal{J}_n^m.$$

Then, for each $\sigma \in \mathcal{S}_n$ and $\alpha \in \mathcal{J}_n$,

$$F_n^\sigma(I_t^n(\alpha)) = \int_{\Delta_{p_\alpha}(t)} f_\alpha(s, t) \hat{F}_n^{\sigma, \alpha}(dB_{s_1} \otimes \cdots \otimes dB_{s_{p_\alpha}}),$$

where $\hat{F}_n^{\sigma, \alpha}$ and f_α are as follows.

The map $\hat{F}_n^{\sigma, \alpha} : \mathfrak{g}^{\otimes p_\alpha} \rightarrow \mathfrak{g}$ is defined by

$$(4.5) \quad \begin{aligned} &\hat{F}_n^{\sigma, \alpha}(k_1 \otimes \cdots \otimes k_{p_\alpha}) \\ &:= \sum_{j_1, \dots, j_{q_\alpha}=1}^{\infty} F_n^{\sigma'}(k_1 \otimes \cdots \otimes k_{p_\alpha} \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots \otimes h_{j_{q_\alpha}} \otimes h_{j_{q_\alpha}}), \end{aligned}$$

for $\{h_j\}_{j=1}^\infty$ an orthonormal basis of \mathfrak{g}_{CM} and $\sigma' = \sigma'(\alpha) \in \mathcal{S}_n$ given by $\sigma' = \sigma \circ \tau^{-1}$, for any $\tau \in \mathcal{S}_n$ such that

$$\begin{aligned} & \tau(dX_{s_1}^1 \otimes \cdots \otimes dX_{s_m}^m) \\ &= \sum_{j_1, \dots, j_{q_\alpha}=1}^\infty dB_{s_1} \otimes \cdots \otimes dB_{s_{p_\alpha}} \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots \otimes h_{j_{q_\alpha}} \otimes h_{j_{q_\alpha}} ds_1 \cdots ds_{q_\alpha}. \end{aligned}$$

To define f_α , first consider the polynomial of order q_α , in the variables s_i with i such that $\alpha_i = 1$ and in the variable t , given by evaluating the integral

$$(4.6) \quad f'_\alpha((s_i : \alpha_i = 1), t) = \int_{\Delta'_{q_\alpha}(t)} \prod_{i: \alpha_i=2} ds_i,$$

where $\Delta'_{q_\alpha}(t) = \{s_{i-1} < s_i < s_{i+1} : \alpha_i = 2\}$ with $s_0 = 0$ and $s_{m+1} = t$. Then f_α is f'_α with the variables reindexed by the bijection $\{i : \alpha_i = 1\} \rightarrow \{1, \dots, p_\alpha\}$ that maintains the natural ordering of these sets. (For example, for $\alpha = (1, 2, 1, 2) \in \mathcal{J}_6^4$,

$$f'_\alpha(s_1, s_3, t) = \int_{\{s_1 < s_2 < s_3, s_3 < s_4 < t\}} ds_2 ds_4 = (t - s_3)(s_3 - s_1),$$

so that $f_\alpha(s_1, s_2, t) = (t - s_2)(s_2 - s_1)$.)

This explicit realization of f_α is not critical to the sequel. It is really only necessary to know that f_α is a polynomial of order q_α in $s = (s_1, \dots, s_{p_\alpha})$ and t , and thus may be written as

$$f_\alpha(s, t) = \sum_{a=0}^{q_\alpha} b_\alpha^a t^a \tilde{f}_{\alpha,a}(s),$$

for some coefficients $b_\alpha^a \in \mathbb{R}$ and polynomials $\tilde{f}_{\alpha,a}$ of degree $q_\alpha - a$ in s . Now, if $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt on $\mathfrak{g}_{CM}^{\otimes p_\alpha}$, then

$$\int_{\Delta_{p_\alpha}(t)} \left\| \tilde{f}_{\alpha,a}(s) \hat{F}_n^{\sigma,\alpha} \right\|_2^2 ds = \left\| \tilde{f}_{\alpha,a} \right\|_{L^2(\Delta_{p_\alpha}(t))}^2 \left\| \hat{F}_n^{\sigma,\alpha} \right\|_2^2 < \infty,$$

and

$$(4.7) \quad F_n^\sigma(I_t^n(\alpha)) = \sum_{a=0}^{q_\alpha} b_\alpha^a t^a J_n(\tilde{f}_{\alpha,a} \hat{F}_n^{\sigma,\alpha})_t$$

may be understood in the sense of the limit integrals in Proposition 2.5. (In particular, if $\alpha_m = 1$, then $f_\alpha = f_\alpha(s)$ does not depend on t , and Proposition 2.5 implies that $F_n^\sigma(I_t^n(\alpha))$ is a \mathfrak{v} -valued L^2 -martingale.)

The above computations show that, if for all n , $\sigma \in \mathcal{S}_n$, and $\alpha \in \mathcal{J}_n$, $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt, then we may rewrite (4.2) as

$$g_t = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} \sum_{m=\lceil n/2 \rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} \sum_{a=0}^{q_\alpha} b_\alpha^a t^a J_n(\tilde{f}_{\alpha,a} \hat{F}_n^{\sigma,\alpha})_t,$$

where J_n is as defined in Proposition 2.5. The next proposition shows that $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt as desired, and thus g_t in (4.2) is well-defined.

Proposition 4.1. *Let $n \in \{2, \dots, r\}$, $\sigma \in \mathcal{S}_n$, and $\alpha \in \mathcal{J}_n$. Then $\hat{F}_n^{\sigma,\alpha} : \mathfrak{g}_{CM}^{\otimes p_\alpha} \rightarrow \mathfrak{g}_{CM}$ is Hilbert-Schmidt.*

Proof. For the whole of this proof, all sums will be taken over an orthonormal basis of \mathfrak{g}_{CM} .

Now, for F_n and F_n^σ as defined in equations (4.3) and (4.4), we may write

$$F_n^\sigma(k_1 \otimes \cdots \otimes k_{p_\alpha} \otimes h_1 \otimes h_1 \otimes \cdots \otimes h_{q_\alpha} \otimes h_{q_\alpha}) = F_n(A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)})$$

where

$$A_b := \begin{cases} k_b & \text{if } b = 1, \dots, p_\alpha \\ h_{\lceil (b-p_\alpha)/2 \rceil} & \text{if } b = p_\alpha + 1, \dots, n \end{cases}.$$

If $q_\alpha = 0$, then $p_\alpha = n$, each $A_{\sigma(j)} = k_i$ for some $i = 1, \dots, n$, and

$$\begin{aligned} \|\hat{F}_n^{\sigma, \alpha}\|_2^2 &= \sum_{k_1, \dots, k_n} \|F_n^\sigma(k_1 \otimes \cdots \otimes k_n)\|_{\mathfrak{g}_{CM}}^2 \\ &= \sum_{k_1, \dots, k_n} \|F_n(k_1 \otimes \cdots \otimes k_n)\|_{\mathfrak{g}_{CM}}^2 = \|\llbracket \llbracket \llbracket \cdots [\cdot, \cdot], \cdots \cdot \rrbracket \rrbracket\|_{(\mathfrak{g}_{CM}^*)^{\otimes n} \otimes \mathfrak{g}_{CM}}^2 \end{aligned}$$

which is finite by Proposition 3.6.

Now, if $q_\alpha = 1$, let $N = N(\sigma)$ denote the second j such that $\sigma(j) \in \{n-1, n\}$; that is,

$$\begin{aligned} F_n^\sigma(k_1 \otimes \cdots \otimes k_{n-1} \otimes h_1 \otimes h_1) \\ &= F_n(A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(N-1)} \otimes h_1 \otimes A_{\sigma(N+1)} \otimes \cdots \otimes A_{\sigma(n)}) \\ &= \llbracket \llbracket \llbracket \llbracket \cdots [A_{\sigma(1)}, A_{\sigma(2)}], \cdots \rrbracket, A_{\sigma(N-1)}, h_1, A_{\sigma(N+1)}, \cdots \rrbracket, A_{\sigma(n)} \rrbracket \end{aligned}$$

where exactly one of $A_{\sigma(1)}, \dots, A_{\sigma(N-1)}$ is h_1 and, for all $j > N$, $A_{\sigma(j)} = k_i$ for some

$$i \in I := I(\sigma) := \{\sigma(j) : j = N+1, \dots, n\} \subseteq \{1, \dots, p_\alpha\} = \{1, \dots, n-2\}.$$

Thus, writing

$$\mathcal{A}(h_1, k_i : i \in I^c) := \llbracket \llbracket \llbracket \cdots [A_{\sigma(1)}, A_{\sigma(2)}], \cdots \rrbracket, A_{\sigma(N-1)} \rrbracket,$$

we have that

$$\begin{aligned} F_n^\sigma(k_1 \otimes \cdots \otimes k_{n-1} \otimes h_1 \otimes h_1) \\ &= \sum_{e_1} \langle \mathcal{A}(h_1, k_i : i \in I^c), e_1 \rangle_{\mathfrak{g}_{CM}} \llbracket \llbracket \llbracket \cdots [e_1, h_1], A_{\sigma(N+1)}, \cdots \rrbracket, A_{\sigma(n)} \rrbracket, \end{aligned}$$

and so

$$\begin{aligned} \|\hat{F}_n^{\sigma, \alpha}\|_2^2 &= \sum_{k_1, \dots, k_{n-1}} \left\| \sum_{h_1} F_n^\sigma(k_1 \otimes \cdots \otimes k_{n-1} \otimes h_1 \otimes h_1) \right\|_{\mathfrak{g}_{CM}}^2 \\ &\leq \sum_{k_1, \dots, k_{n-1}} \left(\sum_{h_1, e_1} |\langle \mathcal{A}(h_1, k_i : i \in I^c), e_1 \rangle_{\mathfrak{g}_{CM}}|^2 \right) \\ &\quad \times \left(\sum_{h_1, e_1} \|\llbracket \llbracket \llbracket \cdots [e_1, h_1], A_{\sigma(N+1)}, \cdots \rrbracket, A_{\sigma(n)} \rrbracket\|_{\mathfrak{g}_{CM}}^2 \right) \\ &= \left(\sum_{k_i : i \in I^c, h_1, e_1} |\langle \mathcal{A}(h_1, k_i : i \in I^c), e_1 \rangle_{\mathfrak{g}_{CM}}|^2 \right) \end{aligned}$$

$$\begin{aligned} & \times \left(\sum_{k_i: i \in I, h_1, e_1} \left\| \left[\dots \left[e_1, h_1 \right], A_{\sigma(N+1)}, \dots \right], A_{\sigma(n)} \right\|_{\mathfrak{g}_{CM}}^2 \right) \\ & \leq \left\| \left[\dots \left[\cdot, \cdot \right], \dots \right], \cdot \right\|_{(\mathfrak{g}_{CM}^*)^{\otimes N-1} \otimes \mathfrak{g}_{CM}}^2 \left\| \left[\dots \left[\cdot, \cdot \right], \dots \right], \cdot \right\|_{(\mathfrak{g}_{CM}^*)^{\otimes n-N+1} \otimes \mathfrak{g}_{CM}}^2. \end{aligned}$$

Now more generally when $q_\alpha \geq 2$, we may similarly “separate” the pairs of h_j ’s as above. More precisely, define

$$\Phi(b) := \Phi_\alpha(b) := \begin{cases} b & \text{if } b = 1, \dots, p_\alpha \\ \lceil \frac{b-p_\alpha}{2} \rceil + p_\alpha & \text{if } b = p_\alpha + 1, \dots, n \end{cases}.$$

Let $N_0 = 1$, and set

$$\begin{aligned} \Omega_1^j &:= \{\Phi(\sigma(\ell)) : \ell = N_0, \dots, j-1\} \quad \text{and} \quad N_1 := \min\{j > N_0 : \Phi(\sigma(j)) \in \Omega_1^j\}, \\ \Omega_2^j &:= \{\Phi(\sigma(\ell)) : \ell = N_1, \dots, j-1\} \quad \text{and} \quad N_2 := \min\{j > N_1 : \Phi(\sigma(j)) \in \Omega_2^j\}. \end{aligned}$$

Similarly, we define

$$\begin{aligned} \Omega_{2m+1}^j &:= \{\Phi(\sigma(\ell)) : \ell = N_{2m}, \dots, j-1\}, \\ N_{2m+1} &:= \min \left\{ j > N_{2m} : \Phi(\sigma(j)) \in \bigcup_{i=0}^{m-1} \Omega_{2i+1}^{N_{2i+1}} \cup \Omega_{2m+1}^j \right\}, \\ \Omega_{2m}^j &:= \{\Phi(\sigma(\ell)) : \ell = N_{2m-1}, \dots, j-1\}, \text{ and} \\ N_{2m} &:= \min \left\{ j > N_{2m-1} : \Phi(\sigma(j)) \in \bigcup_{i=1}^{m-1} \Omega_{2m}^{N_{2m}} \cup \Omega_{2m}^j \right\}. \end{aligned}$$

Then there is an $M < q_\alpha$ such that the sets $\{A_{\sigma(N_i)}, \dots, A_{\sigma(N_{i+1}-1)}\}_{i=0}^{M-1}$ and $\{A_{\sigma(N_M)}, \dots, A_{\sigma(n)}\}$ separate the h_j ’s in the sense that no h_j is repeated inside any one of these sets, and moreover the union of “even” sets contains exactly one copy of each h_j and similarly with the “odd” sets. We can write

$$\begin{aligned} & F_n^\sigma(k_1 \otimes \dots \otimes k_{p_\alpha} \otimes h_1 \otimes h_1 \otimes \dots \otimes h_{q_\alpha} \otimes h_{q_\alpha}) \\ &= \sum_{e_1, \dots, e_M} \left\{ \langle e_1, \left[\left[A_{\sigma(1)}, A_{\sigma(2)}, \dots \right], A_{\sigma(N_1-1)} \right] \right\rangle \right. \\ & \quad \times \left(\prod_{i=1}^{M-1} \langle e_{i+1}, \left[\left[e_i, A_{\sigma(N_i)}, \dots \right], A_{\sigma(N_{i+1}-1)} \right] \rangle \right) \left. \left[\left[e_M, A_{\sigma(N_M)}, \dots \right], A_{\sigma(n)} \right] \right\}. \end{aligned}$$

If M is even, then

$$\begin{aligned} \mathcal{A} &= \langle e_1, \left[\left[A_{\sigma(1)}, A_{\sigma(2)}, \dots \right], A_{\sigma(N_1-1)} \right] \rangle \\ & \times \left(\prod_{i=1}^{M/2-1} \langle e_{2i+1}, \left[\left[e_{2i}, A_{\sigma(N_{2i})}, \dots \right], A_{\sigma(N_{2i+1}-1)} \right] \rangle \right) \left[\left[e_M, A_{\sigma(N_M)}, \dots \right], A_{\sigma(n)} \right] \end{aligned}$$

is a function of $h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M$ and k_i for $i \in I$ some subset of $\{1, \dots, n\}$, and

$$\mathcal{B} = \prod_{i=1}^{M/2} \langle e_{2i}, \left[\left[e_{2i-1}, A_{\sigma(N_{2i-1})}, \dots \right], A_{\sigma(N_{2i}-1)} \right] \rangle$$

is a function of $h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M$ and k_i for $i \in I^c$. Thus,

$$\|\hat{F}_n^{\sigma, \alpha}\|_2^2 \leq \left(\sum_{k_i: i \in I, h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M} \|A\|_{\mathfrak{g}_{CM}}^2 \right) \left(\sum_{k_i: i \in I^c, h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M} |\mathcal{B}|^2 \right)$$

which is finite again by Proposition 3.6. Similarly, if M is odd,

$$\begin{aligned} \mathcal{A} = & \langle e_1, [[[A_{\sigma(1)}, A_{\sigma(2)}], \dots], A_{\sigma(N_1-1)}] \rangle \\ & \times \prod_{i=1}^{(M-1)/2} \langle e_{2i+1}, [[[e_{2i}, A_{\sigma(N_{2i})}], \dots], A_{\sigma(N_{2i+1}-1)}] \rangle \end{aligned}$$

and

$$\mathcal{B} = \left(\prod_{i=1}^{(M-1)/2} \langle e_{2i}, [[[e_{2i-1}, A_{\sigma(N_{2i-1})}], \dots], A_{\sigma(N_{2i}-1)}] \rangle \right) \langle [e_M, A_{\sigma(N_M)}], \dots, A_{\sigma(n)} \rangle$$

are both functions of $h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M$ and some k_i , and

$$\|\hat{F}_n^{\sigma, \alpha}\|_2^2 \leq \left(\sum_{k_i: i \in I, h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M} |A|_{\mathfrak{g}_{CM}}^2 \right) \left(\sum_{k_i: i \in I^c, h_1, \dots, h_{q_\alpha}, e_1, \dots, e_M} \|\mathcal{B}\|_{\mathfrak{g}_{CM}}^2 \right)$$

is again finite by Proposition 3.6 and this completes the proof. \square

Propositions 2.5 and 4.1 now allow us to make the following definition.

Definition 4.2. A *Brownian motion* on G is the continuous G -valued process defined by

$$g_t = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} \sum_{m=\lceil n/2 \rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} \int_{\Delta_{p_\alpha}(t)} f_\alpha(s, t) \hat{F}_n^{\sigma, \alpha}(dB_{s_1} \otimes \dots \otimes dB_{s_{p_\alpha}}),$$

where

$$c_n^\sigma = (-1)^{e(\sigma)} / n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix},$$

$\hat{F}_n^{\sigma, \alpha}$ is as defined in equation (4.5), and f_α is as defined below equation (4.6). For $t > 0$, let $\nu_t = \text{Law}(g_t)$ be the *heat kernel measure at time t* , a probability measure on G .

Proposition 4.3 (Finite-dimensional approximations). *For G_π a finite-dimensional Lie subgroup of G_{CM} , let π denote orthogonal projection of G_{CM} onto G_π and let g_t^π be the continuous process on G_π defined by*

$$g_t^\pi = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} \sum_{m=\lceil n/2 \rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} \int_{\Delta_{p_\alpha}(t)} f_\alpha(s, t) \hat{F}_n^{\sigma, \alpha}(d\pi B_{s_1} \otimes \dots \otimes d\pi B_{s_{p_\alpha}}),$$

where the stochastic integrals are defined as in Proposition 2.6. Then g_t^π is Brownian motion on G_π . In particular, for $G_\ell = G_{\pi_\ell}$ an increasing sequence of finite-dimensional Lie subgroups such that the associated orthogonal projections π_ℓ are increasing to $I_{\mathfrak{g}_{CM}}$, let $g_t^\ell = g_t^{\pi_\ell}$. Then, for all $t < \infty$,

$$(4.8) \quad \lim_{\ell \rightarrow \infty} \mathbb{E} \|g_t^\ell - g_t\|_{\mathfrak{g}}^2 = 0.$$

Proof. First note that g_t^π solves the Stratonovich equation $\delta g_t^\pi = L_{g_t^\pi} \delta \pi B_t$ with $g_0^\pi = \mathbf{e}$, see [9, 11, 6] where $\langle \pi B \rangle_t$ is a standard \mathfrak{g}_π -valued Brownian motion. Thus, g_t^π is a G_π -valued Brownian motion.

By equation (4.7) and its preceding discussion,

$$g_t^\ell = \sum_{n=1}^r \sum_{\sigma \in \mathcal{S}_n} \sum_{m=\lceil n/2 \rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} \sum_{a=0}^{q_\alpha} b_\alpha^a t^a J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t,$$

and thus, to verify (4.8), it suffices to show that

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \|\pi_\ell B_t - B_t\|_{\mathfrak{g}}^2 = 0$$

and

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left\| J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t - J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t \right\|^2 = 0,$$

for all $n \in \{2, \dots, r\}$, $\sigma \in \mathcal{S}_n$ and $\alpha \in \mathcal{J}_n$.

So let $\mu_t = \text{Law}(B_t)$. Then it is known that, if V is a finite-dimensional subspace of \mathfrak{g}_{CM} and π_V is the orthogonal projection from \mathfrak{g}_{CM} to V , then π_V admits a μ_t -a.s. unique extension to \mathfrak{g} . Moreover, if V_n is an increasing sequence of finite-dimensional subspaces, then

$$\lim_{n \rightarrow \infty} \mathbb{E} \|\pi_{V_n} B_t - B_t\|_{\mathfrak{g}}^2 = 0;$$

see for example Section 8.3.3 of [37].

By Proposition 4.1, $\hat{F}_n^{\sigma, \alpha}$ is Hilbert-Schmidt, and recall that \tilde{f}_α is a deterministic polynomial function in s . Thus $J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})$ and $J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})$ are \mathfrak{g}_{CM} -valued martingales as defined in Proposition 2.6, and Proposition 2.6 gives the desired convergence as well (in \mathfrak{g}_{CM} and thus in \mathfrak{g}). \square

Remark 4.4. In fact, for each of the stochastic integrals $J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})$, it is possible to prove the stronger convergence that, for all $p \in [1, \infty)$,

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left[\sup_{\tau \leq t} \left\| J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_\tau - J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_\tau \right\|^p \right] = 0,$$

for all $n \in \{2, \dots, r\}$, $\sigma \in \mathcal{S}_n$ and $\alpha \in \mathcal{J}_n$. Again, Proposition 2.6 gives the limit for $p = 2$ and thus for $p \in [1, 2]$. For $p > 2$, Doob's maximal inequality implies it suffices to show that

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left\| J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t - J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t \right\|^p = 0.$$

Since each $J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})$ and $J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})$ has chaos expansion terminating at degree n , a theorem of Nelson (see Lemma 2 of [34] and pp. 216-217 of [33]) implies that, for each $j \in \mathbb{N}$, there exists $c_j < \infty$ such that

$$\mathbb{E} \left\| J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t - J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t \right\|^{2j} \leq c_j \left(\mathbb{E} \left\| J_n^\ell(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t - J_n(\tilde{f}_\alpha \hat{F}_n^{\sigma, \alpha})_t \right\|^2 \right)^j.$$

In a similar way, one may prove the following convergence for the Brownian motions under right translations by elements of G_{CM} .

Proposition 4.5. *For any $y \in G_{CM}$,*

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \|g_t^\ell y - g_t y\|_{\mathfrak{g}}^2 = 0.$$

where $g_t y$ is the measurable right group action of $y \in G_{CM}$ on $g_t \in G$, as in Proposition 3.4.

Remark 4.6. Note that, while the present paper focuses on the case where μ is non-degenerate and B is Brownian motion on G , the above construction and finite-dimensional approximations would all follow with essentially no modification if one considered instead a Gaussian measure μ whose support was, for example, a subspace \mathfrak{h} of \mathfrak{g} such that \mathfrak{h} generates the span of \mathfrak{g} via the Lie bracket.

4.2. Quasi-invariance and log Sobolev. We are now able to prove Theorem 1.1, which states that the heat kernel measure $\nu_t = \text{Law}(g_t)$ is quasi-invariant under left and right translation by elements of G_{CM} and gives estimates for the Radon-Nikodym derivatives of the “translated” measures. Given the results so far, the proof could be given as an application of Theorem 7.3 and Corollary 7.4 of [16]. However, we provide here a full proof for the reader’s convenience.

Proof of Theorem 1.1. Fix $t > 0$ and π_0 an orthogonal projection onto a finite-dimensional subspace G_0 of \mathfrak{g}_{CM} . Let $h \in G_0$, and $\{\pi_n\}_{n=1}^\infty$ be an increasing sequence of projections such that $G_0 \subset \pi_n G_{CM}$ for all n and $\pi_n|_{G_{CM}} \uparrow I_{G_{CM}}$. Let $J_t^{n,r}(h, \cdot)$ denote the Radon-Nikodym derivative of $\nu_t^n \circ R_h^{-1}$ with respect to ν_t^n . Then for each n and for any $q \in [1, \infty)$, we have the following integrated Harnack inequality

$$\left(\int_{G_n} (J_t^{n,r}(h, g))^q d\nu_t^n(g) \right)^{1/q} \leq \exp \left(\frac{(q-1)k}{2(e^{kt} - 1)} d_n(e, h)^2 \right)$$

where k is the uniform lower bound on the Ricci curvature as in Proposition 3.10 and d_n is Riemannian distance on G_n ; see for example Theorem 1.6 of [16].

By Proposition 4.5, we have that for any $f \in C_b(G)$, the class of bounded continuous functions on G

$$(4.9) \quad \begin{aligned} \int_G f(gh) d\nu_t(g) &= \mathbb{E}[f(g_t h)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[f(g_t^n h)] = \lim_{n \rightarrow \infty} \int_{G_n} (f \circ i_n)(gh) d\nu_t^n(g), \end{aligned}$$

where $i_n : G_n \rightarrow G$ denotes the inclusion map. Note that for any n

$$\begin{aligned} \int_{G_n} |(f \circ i_n)(gh)| d\nu_t^n(g) &= \int_{G_n} J_t^{n,r}(h, g) |(f \circ i_n)(g)| d\nu_t^n(g) \\ &\leq \|f \circ i_n\|_{L^{q'}(G_n, \nu_t^n)} \exp \left(\frac{k(q-1)}{2(e^{kt} - 1)} d_n(e, h)^2 \right), \end{aligned}$$

where q' is the conjugate exponent to q . Allowing $n \rightarrow \infty$ in this last inequality yields

$$(4.10) \quad \int_G |f(gh)| d\nu_t(g) \leq \|f\|_{L^{q'}(G, \nu_t)} \exp \left(\frac{k(q-1)}{2(e^{kt} - 1)} d(e, h)^2 \right),$$

by equation (4.9) and the fact that the length of a path in G_{CM} can be approximated by the lengths of paths in the finite-dimensional projections. That is, for any π_0 and $\varphi \in C^1([0, 1], G_{CM})$ with $\varphi(0) = \mathbf{e}$, there exists an increasing sequence $\{\pi_n\}_{n=1}^\infty$ of orthogonal projections such that $\pi_0 \subset \pi_n$, $\pi_n|_{\mathfrak{g}_{CM}} \uparrow I_{\mathfrak{g}_{CM}}$, and

$$\ell_{CM}(\varphi) = \lim_{n \rightarrow \infty} \ell_{G_{\pi_n}}(\pi_n \circ \varphi).$$

To see this, let φ be a path in G_{CM} . Then one may show that

$$\ell_{G_{\pi_n}}(\pi_n \circ \varphi) = \int_0^1 \left\| \pi_n \varphi'(s) + \sum_{\ell=1}^{r-1} c_\ell \text{ad}_{\pi_n \varphi(s)}^\ell \pi_n \varphi'(s) \right\|_{\mathfrak{g}_{CM}} ds$$

for appropriate coefficients c_ℓ ; see for example Section 3 of [32]. Thus, we have proved that (4.10) holds for $f \in C_b(G)$ and $h \in \cup_\pi G_\pi$. As this union is dense in G by Proposition 3.9, dominated convergence along with the continuity of $d(e, h)$ in h implies that (4.10) holds for all $h \in G_{CM}$.

Since the bounded continuous functions are dense in $L^q(G, \nu_t)$ (see for example Theorem A.1 of [25]), the inequality in (4.10) implies that the linear functional $\varphi_h : C_b(G) \rightarrow \mathbb{R}$ defined by

$$\varphi_h(f) = \int_G f(gh) d\nu_t(g)$$

has a unique extension to an element, still denoted by φ_h , of $L^q(G, \nu_t)^*$ which satisfies the bound

$$|\varphi_h(f)| \leq \|f\|_{L^q(G, \nu_t)} \exp\left(\frac{k(q-1)}{2(e^{kt}-1)} d(e, h)^2\right)$$

for all $f \in L^q(G, \nu_t)$. Since $L^q(G, \nu_t)^* \cong L^q(G, \nu_t)$, there then exists a function $J_t^r(h, \cdot) \in L^q(G, \nu_t)$ such that

$$(4.11) \quad \varphi_h(f) = \int_G f(g) J_t^r(h, g) d\nu_t(g),$$

for all $f \in L^q(G, \nu_t)$, and

$$\|J_t^r(h, \cdot)\|_{L^q(G, \nu_t)} \leq \exp\left(\frac{k(q-1)}{2(e^{kt}-1)} d(e, h)^2\right).$$

Now restricting (4.11) to $f \in C_b(G)$, we may rewrite this equation as

$$(4.12) \quad \int_G f(g) d\nu_t(gh^{-1}) = \int_G f(g) J_t^r(h, g) d\nu_t(g).$$

Then a monotone class argument (again use Theorem A.1 of [25]) shows that (4.12) is valid for all bounded measurable functions f on G . Thus, $d(\nu_t \circ R_h^{-1})/d\nu_t$ exists and is given by $J_t^r(h, \cdot)$, which is in L^q for all $q \in (1, \infty)$ and satisfies the desired bound.

A parallel argument gives the analogous result for $d(\nu_t \circ L_h^{-1})/d\nu_t$. Alternatively, one could use the right translation invariance just proved along with the facts that ν_t inherits invariance under the inversion map $g \mapsto g^{-1}$ from its finite-dimensional projections and that $d(e, h^{-1}) = d(e, h)$. \square

The following also records the straightforward fact that the heat kernel measure does not charge G_{CM} .

Proposition 4.7. *For all $t > 0$, $\nu_t(G_{CM}) = 0$.*

Proof. This follows trivially from the fact that g_t is the sum of a Brownian motion B_t on \mathfrak{g} with a finite sequence of stochastic integrals taking values in \mathfrak{g}_{CM} . \square

Thus, G_{CM} maintains its role as a dense subspace of G of measure 0 with respect to the distribution of the ‘‘group Brownian motion’’.

Definition 4.8. A function $f : G \rightarrow \mathbb{R}$ is said to be a (smooth) cylinder function if $f = F \circ \pi$ for some finite-dimensional projection π and some (smooth) function $F : G_\pi \rightarrow \mathbb{R}$. Also, f is a cylinder polynomial if $f = F \circ \pi$ for F a polynomial function on G_π .

Theorem 4.9. Given a cylinder polynomial f on G , let $\nabla f : G \rightarrow \mathfrak{g}_{CM}$ be the gradient of f , the unique element of \mathfrak{g}_{CM} such that

$$\langle \nabla f(g), h \rangle_{\mathfrak{g}_{CM}} = \tilde{h}f(g) := f'(g)(L_{g*}h_e),$$

for all $h \in \mathfrak{g}_{CM}$. Then for k as in Proposition 3.10,

$$\int_G (f^2 \ln f^2) d\nu_t - \left(\int_G f^2 d\nu_t \right) \cdot \ln \left(\int_G f^2 d\nu_t \right) \leq 2 \frac{1 - e^{-kt}}{k} \int_G \|\nabla f\|_{\mathfrak{g}_{CM}}^2 d\nu_t.$$

Proof. Following the method of Bakry and Ledoux applied to G_P (see Theorem 2.9 of [17] for the case needed here) shows that

$$\mathbb{E} [(f^2 \ln f^2)(g_t^\pi)] - \mathbb{E} [f^2(g_t^\pi)] \ln \mathbb{E} [f^2(g_t^\pi)] \leq 2 \frac{1 - e^{-k_\pi t}}{k_\pi} \mathbb{E} \|\nabla^\pi f\|_{\mathfrak{g}_\pi}^2,$$

for k_π as in equation (3.2). Since the function $x \mapsto (1 - e^{-x})/x$ is decreasing and $k \leq k_\pi$ for all finite-dimensional projections π , this estimate also holds with k_π replaced with k . Now applying Proposition 4.3 to pass to the limit as $\pi \uparrow I$ gives the desired result. \square

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