ON THE MULTICANONICAL SYSTEMS OF QUASI-ELLIPTIC SURFACES

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ABSTRACT. We consider the multicanonical systems $|mK_S|$ of quasi-elliptic surfaces with Kodaira dimension 1 in characteristic 2. We show that for any $m \geq 6$ $|mK_S|$ gives the structure of quasi-elliptic fiber space, and 6 is the best possible number to give the structure for any such surfaces.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p \geq 0$, and let S be a nonsingular complete algebraic surface with Kodaira dimension 1 defined over k. Then, S has a structure of genus 1 fibration $\varphi : S \longrightarrow B$. We denote by K_S a canonical divisor of S and we consider the multicanonical system $|mK_S|$. As is well known, the multicanonical system $|mK_S|$ gives the genus 1 fibration if m is large enough. In Katsura and Ueno [5] and Katsura [3] (see also Iitaka [2]), we considered the following question:

Question 1.1. (1) Does there exist a positive integer M such that if $m \geq M$, the multicanonical system $|mK_S|$ gives a structure of genus 1 fibration for any elliptic surface S over k with Kodaira dimension 1?

(2) What is the smallest M which satisfies this property?

For this question, we have the following theorem.

Theorem 1.2. (1) *For the complex analytic elliptic surfaces,* $M = 86$ *and* 86 *is best possible (cf. Iitaka* [2]*).*

(2) *For the algebraic elliptic surfaces, if the characteristic* $p = 0$ *or* $p \geq 3$*, then* $M = 14$ *and* 14 *is best possible (Katsura and Ueno* [5] *and Katsura* [3]*).*

(3) *For the algebraic elliptic surfaces, if the characteristic* $p = 2$*, then* $M = 12$ *and* 12 *is best possible (Katsura* [3]*).*

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If $p = 2$ or 3, there are two kinds of genus 1 fibrations, namely, the elliptic fibration and the quasi-elliptic fibration (cf. Bombieri and Mumford [1]). In these cases, we can also consider the same question for quasi-elliptic surfaces with Kodaira dimension 1. In characteristic 3, we already showed the following results (Katsura [4]).

Theorem 1.3. *For the quasi-elliptic surfaces in characteristic 3, we have* $M = 5$*, and* 5 *is best possible.*

Therefore, the remaining case of the question for the surfaces with Kodaira dimension 1 is the one in characteristic 2, and in this paper we show the following theorem. It finishes the answer to the question above for surfaces with Kodaira dimension 1 which S. Iitaka considered in the case of complex analytic elliptic surfaces in 1970 (cf. [2]).

Theorem 1.4. *For the quasi-elliptic surfaces in characteristic 2, we have* $M = 6$ *and* 6 *is best possible.*

In Section 2, we summarize basic facts on the theory of vector fields in positive characteristic and some results on quasi-elliptic surfaces. In Section 3, we give a criterion for a vector field that makes a singularity on the quotient of curve. In Section 4, we construct a quasi-elliptic surface over an elliptic curve with only one tame multiple fiber and examine the structure of its multicanonical system. In Section 5, we examine the multicanonical system of quasi-elliptic surfaces in characteristic 2 and show our main theorem.

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2. Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, and let S be a nonsingular complete algebraic surface defined over k. A non-zero rational vector field D on S is called p-closed if there exists a rational function f on S such that $D^p = fD$.

We use a vector field to construct a quotient surface of S. Let $\{U_i = \mathrm{Spec} A_i\}$ be an affine open covering of S and we set $A_i^D = \{D(\alpha) = 0 | \alpha \in A_i\}$. Then, affine surfaces $\{U_i^D = \mathrm{Spec} A_i^D\}$ glue together to define a normal quotient surface S^D.

We now recall some results on vector fields by Rudakov and Shafarevich [8, Section 1]. Now, we assume that D is p-closed. Then, we know that the natural morphism $\pi : S \longrightarrow S^D$ is a purely inseparable morphism of degree p. If the affine open covering $\{U_i\}$ of S is fine enough, then taking local coordinates x_i, y_i on U_i, we see that there exist $f_i, g_i \in A_i$ and a rational function h_i such that the divisors defined by $f_i = 0$ and by $g_i = 0$ have no common divisor and that the vector field D is expressed as

$$D = h_i \left(f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i} \right) \quad \text{on } U_i.$$

Divisors (h_i) on U_i give a global divisor (D) on S, and zero-cycles defined by the ideal (f_i, g_i) on U_i give a global zero cycle $\langle D \rangle$ on S. A point contained in the support of $\langle D \rangle$ is called an isolated singular point of D. ([8, Theorem 1, Corollary]). Rudakov and Shafarevich showed that S^D is nonsingular if and only if $\langle D \rangle = 0$. When S^D is nonsingular, they also showed a canonical divisor formula

$$K_S \sim \pi^* K_{S^D} + (p-1)(D), \tag{2.1}$$

where $\sim$ means linear equivalence.

Now, we consider an irreducible curve C on S and we set $C' = \pi(C)$. Take an affine open set U_i above such that $C \cap U_i$ is non-empty. The curve C is said to be integral with respect to the vector field D if D is tangent to C at a general point of $C \cap U_i$. Rudakov-Shafarevich showed the following proposition (cf. [8, Proposition 1]):

Proposition 2.1. (i) *If C is integral, then $C = \pi^{-1}(C')$ and $C^2 = pC'^2$.*
(ii) *If C is not integral, then $pC = \pi^{-1}(C')$ and $pC^2 = C'^2$.*

Now, let $\varphi : S \longrightarrow B$ be a quasi-elliptic surface. We denote by g the genus of the curve B. As was shown in Katsura [4], we have $Alb(S) \cong J(B)$, and $\chi(\mathcal{O}_S) \geq (1-g)/3$ (See also Lang [6] and Raynaud [7]). Here, $Alb(S)$ is the Albanese variety of S and $J(B)$ is the Jacobian variety of B. As a corollary, we know that if $g = 1$, then $\chi(\mathcal{O}_S) \geq 0$, and that if $g = 0$, then $\chi(\mathcal{O}_S) \geq 1$. We will freely use these inequalities in Section 5.

3. Cuspidal points

From here on, let k be an algebraically closed field of characteristic 2, if otherwise mentioned. Let S be a nonsingular complete algebraic surface over k, and let D be a non-zero 2-closed rational vector field on S. Let U be an affine open set of S, and x, y be local coordinates of U. Then, as in Section 2, D is given by

$$D = h(f\partial/\partial x + g\partial/\partial y),$$

where f, g are regular functions on U such that $f = 0$ has no common curves with $g = 0$, and where h is a rational function on S.

Lemma 3.1. *Under the assumption above, $D(fg) = 0$ holds.*

Proof. We set $\alpha = hf$ and $\beta = hg$. Since there exists a rational function γ such that $D^2 = \gamma D$, we have

$$\begin{aligned}\alpha\alpha_x + \beta\alpha_y &= \gamma\alpha,\\ \alpha\beta_x + \beta\beta_y &= \gamma\beta.\end{aligned}$$

Therefore, by direct calculation, we have $D(\alpha\beta) = 0$. Since $\alpha\beta = h^2 fg$, we conclude $D(fg) = 0$. □

Corollary 3.2. $D(f/g) = 0$.

Proof. We have $D(f/g) = D(fg/g^2) = (1/g^2)D(fg) = 0$. □

Definition 3.3. Let D be a non-zero rational vector field on a nonsingular surface S, and C be a nonsingular irreducible curve on S. Let P be a point on C which is not an isolated singular point of D. If D is non-integral on C and integral at a point P on C, we call P a cuspidal point of the vector field D.

Proposition 3.4. *Under the notation in Definition 3.3, we consider the projection $\pi : S \longrightarrow S^D$. Then, the image $\pi(P)$ of the cuspidal point P is a singular point of the curve $\pi(C)$.*

Proof. Let O_P be the local ring of the cuspidal point P and let x, y be a system of parameters of O_P. Let $x = 0$ be a local equation of C at the point P. By the definition of cuspidal points, there exist elements α, β, γ and δ of O_P and a constant $c \in k$ such that $\beta \not\equiv 0$ and $c \neq 0$, and such that $f = \alpha x + \beta y$ and $g = \gamma x + \delta y + c$. Since the situation is local, we may omit h from D. By Corollary 3.2, we see that $D(x + (f/g)y) = 0$. Since $g(P) \neq 0$, $x + (f/g)y$ is contained in O_P. Considering the completion $\hat{O}_P$ of O_P, we have $\hat{O}_P \cong k[[x, y]]$. Since $k[[x, y]]^D \supset k[[y^2, x + (f/g)y]]$ and $\dim_k k[[x, y]]^D/k[[x^2, y^2]] = \dim_k k[[y^2, x + (f/g)y]]/k[[x^2, y^2]] = 2$, we have $k[[x, y]]^D = k[[y^2, x + (f/g)y]]$. Although by the general theory of the vector field the point $\pi(P)$ is a nonsingular point of S^D, this result also shows that S^D is nonsingular at $\pi(P)$. We set $X = x^2$, $Y = y^2$ and $Z = x + (f/g)y$, and let $\tilde{f}$, $\tilde{g}$ be elements of O_P whose coefficients are the squares of the ones of f, g, respectively. Let S' be a surface defined by the equation

$$Z^2 = X + (\tilde{f}/\tilde{g})Y.$$

Since the degrees of the algebraic extensions $k(S)/k(S^D)$ and $k(S)/k(S')$ of fields are 2 and $k(S^D) \supset k(S')$ holds, we have $k(S^D) = k(S')$, that is, S' is birationally equivalent to S^D. Since $\tilde{g}(P) = c^2 \neq 0$, S' is nonsingular at the point $(X, Y, Z) = (0, 0, 0)$. Therefore, by the Zariski main theorem the surface S^D is isomorphic to S' around $\pi(P)$. The curve $\pi(C)$ is defined by $X = 0$ at the point $\pi(P)$. Therefore, the equation of the curve $\pi(C)$ at $\pi(P)$ on the plane $X = 0$ is given by

$$Z^2(\tilde{\delta}|_{X=0}Y + c^2) = \tilde{\beta}|_{X=0}Y^2.$$

Here, the notation of $\tilde{\beta}$ and $\tilde{\delta}$ are similar to $\tilde{f}$ and $\tilde{g}$. This equation for the curve $\pi(C)$ shows that $\pi(P)$ is a singular point of $\pi(C)$. □

4. A construction of a quasi-elliptic surface

Let E be an elliptic curve and $\{U_0, U_\infty\}$ be an affine open covering and let U_0 (resp. U_∞) be given by the equation

$$y^2 + y = x^3 \text{ (resp. } z^2 + z = w^3).$$

The change of coordinates is given by

$$y = 1/z, \; x = w/z.$$

Let $\{V_0, V_\infty\}$ ($V_0 \cong V_\infty \cong \mathbf{A}^1$: an affine line) be affine open covering of the projective line $\mathbf{P}^1$ and t (resp.s) be a coordinate of V_0 (resp. V_∞). The change of coordinates is given by

$$t = 1/s.$$

We consider the algebraic surface $S = E \times \mathbf{P}^1$. Then, $\{U_i \times V_j \mid i = 0, \infty; j = 0, \infty\}$ gives an affine open covering of S. We have a projection

$$\psi : S \longrightarrow E.$$

Let C_∞ be the curve on S defined by $s = 0$. We consider the following rational vector field D on $U_0 \times V_0$.

$$(I) \quad D = y\frac{\partial}{\partial x} + (x^2 + x^2 t + t^4)\frac{\partial}{\partial t}.$$

Then, D gives a rational vector field on S and on each affine chart it is concretely given as follows:

$$\begin{aligned}
(II) \quad D &= \tfrac{1}{z^2}\{z\tfrac{\partial}{\partial w} + (w^2 + w^2 t + z^2 t^4)\tfrac{\partial}{\partial t}\} \\
&= \tfrac{1}{w^4}\{(z+1)w\tfrac{\partial}{\partial w} + ((z+1)^2 + (z+1)^2 t + w^4 t^4)\tfrac{\partial}{\partial t}\} \\
&\quad \text{on } U_\infty \times V_0 \\
(III) \quad D &= \tfrac{1}{s^2}\{ys^2\tfrac{\partial}{\partial x} + (x^2 s^4 + x^2 s^3 + 1)\tfrac{\partial}{\partial s}\} \\
&\quad \text{on } U_0 \times V_\infty \\
(IV) \quad D &= \tfrac{1}{z^2 s^2}\{zs^2\tfrac{\partial}{\partial w} + (w^2 s^4 + w^2 s^3 + z^2)\tfrac{\partial}{\partial s}\} \\
&= \tfrac{1}{w^4 s^2}\{(z+1)ws^2\tfrac{\partial}{\partial w} + ((z+1)^2 s^4 + (z+1)^2 s^3 + w^4)\tfrac{\partial}{\partial s}\} \\
&\quad \text{on } U_\infty \times V_\infty
\end{aligned}$$

Since $\frac{\partial y}{\partial x} = x^2$, we have $D^2 = x^2 D$. Therefore, the rational vector field D is 2-closed. The isolated singularities of D on each affine chart are as follows.

On $U_0 \times V_0$	$P : (x, y, t) = (0, 0, 0)$
On $U_\infty \times V_0$	$Q_1 : (w, z, t) = (0, 0, 1)$
On $U_0 \times V_\infty$	No isolated singular point
On $U_\infty \times V_\infty$	$R : (w, z, s) = (0, 0, 0), Q_2 : (w, z, s) = (0, 0, 1).$

On the surface S, Q_1 and Q_2 give the same point, and we denote it by Q. We set

$$\psi(P) = P', \psi(Q) = \psi(R) = Q', \psi^{-1}(P') = F_0, \psi^{-1}(Q') = F_\infty.$$

From here on, we use the same notation for the curve and the proper transform of the curve, if no confusion can occur. We blow-up at P, and denote the exceptional curve by G_1. Then, on the exceptional curve G_1 there exists one isolated singular point of the rational vector field D. We blow-up at the singular point, and denote the exceptional curve by G_2. Then, the vector field has no isolated singular point on G_2. Now, we blow-up at Q, and denote the exceptional curve by E_1. Then, the vector field has no isolated singular point on E_1. We again blow-up at R, and denote the exceptional curve by E_2. On the surface $\tilde{S}$ which we got by these blowing-ups the rational vector field D has no isolated singularities. We have the morphism

$$\tilde{\psi} : \tilde{S} \longrightarrow E$$

which is induced by ψ. Then, on $\tilde{S}$, by our construction we have the following lemma.

Lemma 4.1. *On $\tilde{S}$, we have the following results.*

(1) $\tilde{\psi}^{-1}(P') = F_0 + G_1 + 2G_2$, $\tilde{\psi}^{-1}(Q') = F_\infty + E_1 + E_2$.

(2) The curves F_0, G_1 and F_∞ are integral with respect to the vector field D. The curves G_2, E_1, E_2 and C_∞ are non-integral with respect to the vector field D.

(3) $F_0^2 = -2$, $G_1^2 = -2$, $G_2^2 = -1$, $F_\infty^2 = -2$, $E_1^2 = -1$, $E_2^2 = -1$.

(4) $(F_0, G_2) = (G_2, G_1) = 1$, $(F_0, G_1) = 0$.

(5) $(F_\infty, E_1) = (F_1, E_2) = (C_\infty, E_2) = 1$, $(F_\infty, C_\infty) = (E_1, E_2) = (C_\infty, E_1) = 0$.

(6) There is a cuspidal point of the vector field D on G_2. There is also a cuspidal point of the vector field D on E_2 where it intersects with C_∞.

We consider the quotient surface $\tilde{S}^D$ of $\tilde{S}$ by D. We have the projection

$$\pi : \tilde{S} \longrightarrow \tilde{S}^D$$

and a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \stackrel{\pi}{\longrightarrow} & \tilde{S}^D \\ \tilde{\psi}\downarrow & & \downarrow \psi' \\ E & \stackrel{F}{\longrightarrow} & E^{(2)}. \end{array}$$

Here, F is the Frobenius morphism and $E^{(2)}$ is the Frobenius image. We set $B = E^{(2)}$, $P'' = F(P')$ and $Q'' = F(Q')$. For a curve C on $\tilde{S}$, we denote the curve $\pi(C)$ on $\tilde{S}^D$ again by C, if no confusion can occur. By Lemma 4.1 and Proposition 3.4, we have the following lemma.

Lemma 4.2. *On $\tilde{S}^D$, we have the following results.*

(1) $\psi'^{-1}(P'') = 2F_0 + 2G_1 + 2G_2$, $\psi'^{-1}(Q'') = 2F_\infty + E_1 + E_2$.

(2) $F_0^2 = -1$, $G_1^2 = -1$, $G_2^2 = -2$, $F_\infty^2 = -1$, $E_1^2 = -2$, $E_2^2 = -2$, $C_\infty^2 = -2$.

(3) $(F_0, G_2) = (G_2, G_1) = 1$, $(F_0, G_1) = 0$.

(4) $(F_\infty, E_1) = (F_1, E_2) = 1$, $(C_\infty, E_2) = 2$, $(F_\infty, C_\infty) = (E_1, E_2) = (C_\infty, E_1) = 0$.

(5) G_2 and E_2 are rational cuspidal curves.

First, we blow-down F_0, G_1 and F_∞, and then E_1 becomes an exceptional curve of the first kind and so we blow-down it:

$$\eta : \tilde{S}^D \longrightarrow X.$$

Then, we have a quasi-elliptic surface

$$\varphi : X \longrightarrow B.$$

The fiber $\varphi^{-1}(P'')$ is the only one multiple fiber, and we have no other singular fiber.

Now, let's caluculate the canonical divisor K_X. First, we have

$$K_{\tilde{S}^D} \sim \eta^* K_X + F_0 + G_1 + E_1 + 2F_\infty.$$

Therefore, we have

$$\pi^* K_{\tilde{S}^D} \sim \pi^*\eta^* K_X + F_0 + G_1 + 2E_1 + 2F_\infty.$$

On $\tilde{S}$, by a direct calculation of D and $K_{\tilde{S}}$, we have

$$\begin{array}{l} (D) = -2C_\infty - 4F_\infty + G_1 + 4G_2 - 3E_1 - 3E_2, \\ K_{\tilde{S}} \sim -2C_\infty + G_1 + 2G_2 + E_1 - E_2. \end{array}$$

Putting these data in the canonical bundle formula by Rudakov-Shafarevich:

$$K_{\tilde{S}} \sim (D) + \pi^* K_{\tilde{S}^D},$$

we have

$$\pi^*\eta^* K_X \sim 2(F_\infty + E_1 + E_2) - (F_0 + G_1 + 2G_2).$$

Therefore, we have

$$\eta^* K_X \sim (2F_\infty + E_1 + E_2) - (F_0 + G_1 + G_2).$$

Hence, we have

$$K_X \sim E_2 - G_2 \approx G_2,$$

where $\approx$ means numerical equivalence. This means that there exists a divisor $\mathcal{L}$ on B such that

$$K_X \sim \varphi^*(\mathcal{L}) + G_2. \tag{4.1}$$

Therefore, the fiber $\varphi^{-1}(P'')$ is a tame multiple fiber.

Proposition 4.3. *The surface $\varphi : X \longrightarrow B$ which we constructed above is a quasi-elliptic surface with only one tame multiple fiber. It has no more singular fibers and $\chi(\mathcal{O}_X) = 0$ holds. The linear system $|6K_X|$ gives the structure of the quasi-elliptic surface, and the linear system $|5K_X|$ does not give the structure of the quasi-elliptic surface.*

Proof. Take a general fiber G. Then, we have $G^2 = 0$ and $(K_X, G) = 0$. Therefore, by the genus formula the virtual genus of G is 1. On the other hand, $\tilde{\psi} : \tilde{S} \longrightarrow E$ is a ruled surface. Therefore, G is not an elliptic curve. This means that $\varphi : X \longrightarrow B$ is a quasi-elliptic surface. By our construction, we have Betti numbers $b_1(X) = 2$ and $b_2(X) = 2$. Therefore, the Euler number $c_2(X) = 1 - 2 + 2 - 2 + 1 = 0$. Since $K_X^2 = 0$, we have $\chi(\mathcal{O}_X) = 0$ by Noether's formula. Since we have $\mathrm{H}^0(X, \mathcal{O}_X(6K_X)) \cong \mathrm{H}^0(B, \mathcal{O}_B(3P''))$ and the divisor $3P''$ is very ample on B, the linear system $|6K_X|$ gives the structure of the quasi-elliptic surface. Since $\mathrm{H}^0(X, \mathcal{O}_X(5K_X)) \cong \mathrm{H}^0(B, \mathcal{O}_B(2P''))$ and the divisor $2P''$ is not very ample on B, the linear system $|5K_X|$ does not give the structure of the quasi-elliptic surface. □

Remark 4.4. In the above, we calculate the canonical divisor K_X by the construction of our quasi-elliptic surface. We give here one more proof for (4.1). On the quasi-elliptic surface $\varphi : X \longrightarrow B$, the cusp locus C_∞ is a elliptic curve and we have $C_\infty^2 = -1$ by considering the structure of blow-down. Therefore, by the genus formula, we have $(K_X, C_\infty) = 1$. On the other hand, by the canonical bundle formula for the quasi-elliptic surface X, we have

$$K_X \sim \varphi^*(\mathcal{L}) + aG_2$$

with a line bundle $\mathcal{L}$ on B and $a = 0$ or 1. Since $1 = (K_X, C_\infty) = 2\deg \mathcal{L} + a$, we conclude $a = 1$ and $\deg \mathcal{L} = 0$, which shows (4.1).

5. Multicanonical systems

Let $\varphi : S \longrightarrow B$ be a quasi-elliptic surface over an algebraically closed field k of characteristic $p > 0$. Such a surface exists only in characteristic $p = 2$ or 3. In this case, the multiplicity of a multiple fiber is equal to p (cf. Bombieri-Mumford [1]). We denote by pF_i $(i = 1, \ldots, \lambda)$ the multiple fibers. Then, the canonical divisor formula is given by

$$K_S \sim \varphi^*(K_B - \mathbf{f}) + \sum_{i=1}^{\lambda} a_i F_i,$$

where $\mathbf{f}$ is a divisor on B and $-\deg \mathbf{f} = \chi(\mathcal{O}_S) + t$ with $t =$ length of the torsion part of $R^1\varphi_*\mathcal{O}_S$, and $0 \leq a_i \leq p - 1$. For details, see Bombieri-Mumford [1].

We denote by g the genus of the base curve B. Then, we have the following theorem.

Theorem 5.1. *Assume $p = 2$. Then, for any quasi-elliptic surface $\varphi : S \to B$ with Kodaira dimension $\kappa(S) = 1$ over k and for any $m \geq 6$ $|mK_S|$ gives the unique structure of quasi-elliptic surface, and 6 is the best possible number.*

Proof. The method of the proof is similar to the one in Iitaka [2] and Katsura-Ueno [5] (see also Katsura [3] [4]). The Kodaira dimension of S is equal to 1 if and only if

$$(*) \quad 2g - 2 + \chi(\mathcal{O}_S) + t + \sum_{i=1}^{\lambda}(a_i/m_i) > 0.$$

Therefore, we need to find the least integer m such that

$$(**) \quad m(2g - 2 + \chi(\mathcal{O}_S) + t) + \sum_{i=1}^{\lambda}[ma_i/m_i] \geq 2g + 1$$

holds under the condition $(*)$. Here, $[r]$ means the integral part of a real number r. We have the following 6 cases:

Case (I) $g \geq 2$
Case (II-1) $g = 1, \chi(\mathcal{O}_S) + t \geq 1$
Case (II-2) $g = 1, \chi(\mathcal{O}_S) = 0, t = 0$
Case (III-1) $g = 0, \chi(\mathcal{O}_S) + t \geq 3$
Case (III-2) $g = 0, \chi(\mathcal{O}_S) + t = 2$
Case (III-3) $g = 0, \chi(\mathcal{O}_S) = 1, t = 0$

Case (I) We have $2g - 2 + \chi(\mathcal{O}_S) \geq 5(g - 1)/3$. Hence, if $m \geq 3$, $(**)$ holds.

Case (II-1) If $m \geq 3$, $(**)$ holds.

Case (II-2) All multiple fibers are tame in this case. If $m \geq 6$, $(**)$ holds by $p = 2$. As we constructed in Section 4, there exists a quasi-elliptic surface with only one tame multiple fiber of type II and $\chi(\mathcal{O}_S) = 0$ over an elliptic curve. Therefore, we need $m \geq 6$.
Case (III-1) $(**)$ holds for $m \geq 1$.
Case (III-2) Since $\chi(\mathcal{O}_S) \geq 1$, we have $t \leq 1$. Therefore, the number of wild fibers is less than or equal to 1. If there exists at least one tame multiple fiber then $(**)$ holds for $m \geq 2$. If there exist no tame fibers and only one wild fiber, then by Katsura-Ueno [5] Lemma 2.4, this case is excluded in the case of $p = 2$.
Case (III-3) By $p = 2$, $(**)$ holds for $m \geq 4$.

The result on the best possible number follows from the example in Section 4. □

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