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An upper bound for higher order eigenvalues of symmetric graphs

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Abstract. In this paper, we derive an upper bound for higher eigenvalues of the normalized Laplace operator associated with a symmetric finite graph in terms of lower eigenvalues.

1. Introduction

Let G be a connected, finite, simple and undirected graph on N vertices. Let Δ be the normalized Laplace operator assiciated with G. The operator $-\Delta$ is identified with a positive semidefinite real symmetric matrix of size N. Denote by $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$ all eigenvalues of Δ counted with multiplicities. For any connected graph, we have $\lambda_0 = 0$ and its multiplicity is 1. All the eigenvalues lie in the interval $[0, 2]$. We consider the following question: Are there other contraints on the spectrum $\{\lambda_i\}_{i=0}^{N-1}$? In particular, is λ_{k+1} controlled by preceding eigenvalues, $\lambda_1, \ldots, \lambda_k$? This question is a discrete analogue of the so-called Payne-Pólya-Weinberger's inequality. For the Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \uparrow \infty$ of the Laplacian on a bounded domain in the Euclidean plane, Payne-Pólya-Weinberger [5, 6] proved that

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{i=1}^{k} \lambda_i.$$

This result is extended to arbitrary dimension by Thompson [7]. Later, Hile and Protter [3] and Yang [8] proved sharper inequalities. In particular, Yang [8] proved that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i. \tag{1.1}$$

Chung and Oden proposed to study of the discrete analogue of these results. For the Dirichlet eigenvalues $\{\lambda_i\}_{i\geq 1}$ of the normalized Laplacian on a connected finite subgraph in the integer lattice of rank n, Hua, Lin and Su [4] proved that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i.$$

What can be said about the Laplacian without boundary conditions? Unlike the

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case of the Dirichlet boundary condition, 0 is always an eigenvalue. For the eigenvalues $\{\lambda_i\}_{i\geq 0}$ with $\lambda_0 := 0$ of the Laplacian on a compact Riemannian homogeneous manifold, Cheng and Yang [1] proved that

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(4\lambda_i+\lambda_1). \tag{1.2}$$

In this paper, we consider a discrete analogue of (1.2). More precisely, for a finite *symmetric* graph, i.e., a graph whose isomorphism group acts transitively on the set of pairs of adjacency vertices, we prove a discrete analogue of (1.2).

THEOREM 1.1. *Let G be an symmetric finite graph with N vertices. Denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$ the all eigenvalues of the normalized Laplace operator. Then, for any non-zero eigenvalue λ of Δ, we have*

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2(1-\lambda_i) \leq \sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)(2(2-\lambda)\lambda_i+\lambda).$$

By using Chebyshev's sum inequality, we obtain an upper bound of λ_{k+1} in terms of $\lambda_1, \ldots, \lambda_k$.

THEOREM 1.2. *In the same setting as Theorem 1.1, we have*

$$\lambda_{k+1} \leq \frac{(k+1)\lambda_1 + \sum_{i=1}^{k}((5-2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k}(1-\lambda_i)}.$$

Let $\mu_1 := \lambda_1$ and m be the multiplicity of μ_1. If G is not a complete graph, then we can consider $\mu_2 := \lambda_{m+1}$, i.e., the second smallest positive eigenvalue. We have a upper bound for the ratio μ_2/μ_1 in terms of the multiplicity of μ_1.

COROLLARY 1.3. *In the same setting as Theorem 1.1, let m be the multiplicity of μ_1 and put $\mu_2 := \lambda_{m+1}$. Then, we have*

$$\frac{\mu_2}{\mu_1} \leq 3m+1.$$

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2. Preliminaries

In this section, unless otherwise stated, we assume that all graphs are connected, finite, simple and undirected. We recall some basic facts on the theory of eigenvalues of a regular graph. For details, see e.g. [2]. Let $G = (V, E)$ be a d-regular graph, $d \geq 1$, and put $N := \#V$. If two vertices $x, y \in V$ are adjacent, then we denote this situation by $x \sim y$. Note that since G is undirected, $x \sim y$ if and only if $y \sim x$. The *normalized*

Laplace operator Δ acting on the space $C(V)$ of functions on V is defined by

$$\Delta u(x) := \frac{1}{d} \sum_{y \sim x} (u(y) - u(x)), \, u \in C(V), x \in V.$$

The normalized Laplace operator is identified with the real-symmetric matrix $D^{-1}A - I$, where D is the scalar matrix with diagonal entries d, A is the adjacency matrix of G and I is the identity matrix. A complex number λ is called an *eigenvalue* of Δ if there exists $u \in C(V) \setminus \{0\}$ such that $\Delta u + \lambda u = 0$ holds. In this case, the function u is called an *eigenfunction* with eigenvalue λ. For an eigenvalue λ of Δ, we denote by W_λ the space of all functions $u \in C(V)$ satisfying $\Delta u + \lambda u = 0$ and we call the dimension of W_λ *multiplicity* of λ. Let us denote the eigenvalues of Δ by $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$, counted with multiplicity. We define an inner product $\langle \cdot, \cdot \rangle$ on $C(V)$ by

$$\langle u, v \rangle := \sum_{x \in V} u(x)v(x)d.$$

We denote by $\|\cdot\|$ the norm induced by the inner product $\langle \cdot, \cdot \rangle$. We list up some elementary facts on eigenvalues and eigenfunctions without proofs.

- 0 is an eigenvalue of multiplicity 1 and constant functions are eigenfunctions with eigenvalue 0.
- All eigenvalues lie in the interval $[0, 2] \subset \mathbb{R}$.
- There exists an orthonormal basis $\{u_i\}_{i=0}^{N-1}$ of $C(V)$ such that each function u_i is an eigenfunction with eigenvalue λ_i.

By the min-max formula, each eigenvalue λ_k has a variational characterization:

$$\lambda_k = \inf \left\{ \frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d \sum_V u^2} \;\middle|\; u \neq 0, \langle u, u_i \rangle = 0, \, i = 0, \ldots, k-1 \right\},$$

where the symbol $\sum_{x \sim y}$ means the summation over all unordered pairs (x, y) such that $x \sim y$. In particular, we have

$$\lambda_1 = \inf \left\{ \frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d \sum_V u^2} \;\middle|\; u \neq 0, \sum_V u = 0 \right\}. \tag{2.1}$$

The following lemma is well-known. For the proof, see e.g. [2].

LEMMA 2.1. *For any regular graph G but a complete graph, we have*

$$\lambda_1 \leq 1.$$

REMARK 2.2. *If G is the complete graph of degree d, then $\lambda_1 = 1 + 1/d$.*

Let $\Gamma \colon C(V) \times C(V) \to C(V)$ be the *carré du champ operator* associated to Δ, i.e., for $u, v \in C(V)$,

$$\Gamma(u, v) := \frac{1}{2} (\Delta(uv) - (\Delta u)v - u\Delta v).$$

For two vertices $x, y \in V$ with $x \sim y$, we define the *difference operator* $\nabla_{xy} \colon C(V) \to C(V)$ by

$$\nabla_{xy} u := u(y) - u(x),\ u \in C(V).$$

By a simple calculation, we have

$$\Gamma(u, v)(x) = \frac{1}{2d} \sum_{y \sim x} (\nabla_{xy} u)(\nabla_{xy} v),\ x \in V.$$

The carré du champ $\Gamma(u, v)$ is an analogy of $\langle \nabla u, \nabla v \rangle$ in the context of Riemannian geometry, where ∇ is the gradient operator. We list up some identities for Γ.

Lemma 2.3. *Let $u, v, v_1, v_2 \in C(V)$.*

1. $\langle u, \Delta v \rangle = -\sum_V \Gamma(u, v) d$.
2. *For any $x \in V$, we have*

$$\begin{aligned}\Gamma(u, v_1 v_2)(x) = \Gamma(u, v_1) v_2(x) + \Gamma(u, v_2) v_1(x) \\ + \frac{1}{2d} \sum_{y \sim x} (\nabla_{xy} u)(\nabla_{xy} v_1)(\nabla_{xy} v_2).\end{aligned}$$

In particular,

$$\sum_V \Gamma(u, v_1 v_2) = \sum_V (\Gamma(u, v_1) v_2 + \Gamma(u, v_2) v_1).$$

Making use of the min-max formula and appropriate trial functions, we have the following lemma.

Lemma 2.4. *Let $k \geq 1$ be an integer. For any function $h \in C(V)$, we have*

$$\frac{1}{2} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \Phi_i(h) \leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \| 2\Gamma(h, u_i) + u_i \Delta h \|^2,$$

where $\Phi_i(h) = \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2$.

Proof. Let $h \in C(V)$. For $i = 0, \ldots, k$, define $\varphi_i \in C(V)$ as the orthogonal projection of hu_i to the subspace spanned by $\{u_{k+1}, \ldots, u_{N-1}\}$, i.e.,

$$\varphi_i := hu_i - \sum_{j=0}^{k} a_{ij} u_j,\ a_{ij} := \langle hu_i, u_j \rangle.$$

Clearly the function φ_i is perpendicular to $u_0, \ldots, u_k$. The min-max formula yields

$$\lambda_{k+1} \|\varphi_i\|^2 \leq \frac{1}{2} \sum_{x \sim y} (\nabla_{xy} \varphi_i)^2 = \sum_V \Gamma(\varphi_i, \varphi_i) d. \tag{2.2}$$

From (1) in Lemma 2.3 and the fact that $\langle \varphi_i, u_j \rangle = 0$ for $j = 0, \ldots, k$, we have

$$\begin{aligned}
\sum_V \Gamma(\varphi_i, \varphi_i) d &= -\langle \varphi_i, \Delta \varphi_i \rangle \\
&= -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h - \lambda_i u_i h + \sum_{j=0}^{k} a_{ij} \lambda_j u_j \rangle \\
&= -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h - \lambda_i u_i h \rangle \\
&= -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle + \lambda_i \|\varphi_i\|^2.
\end{aligned}$$

From (2.2), we obtain

$$(\lambda_{k+1} - \lambda_i) \|\varphi_i\|^2 \leq -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle. \tag{2.3}$$

Let A_i be the right hand side of (2.3). We estimate A_i in two ways. First, we claim that

$$A_i = \frac{1}{2} \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 + \sum_{j=0}^{k} (\lambda_i - \lambda_j) a_{ij}^2. \tag{2.4}$$

To see (2.4), we use Lemma 2.3. By the definition of φ_i,

$$A_i = \sum_{j=0}^{k} a_{ij} \langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle - d \sum_V (h u_i^2 \Delta h + 2 h u_i \Gamma(h, u_i)).$$

The first term is equal to $\sum_{j=0}^{k} (\lambda_i - \lambda_j) a_{ij}^2$. Indeed, by the definition of $\Gamma(h, u_i)$ and Lemma 2.3, we have

$$\begin{aligned}
\langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle &= \langle u_j, \Delta(h u_i) + \lambda_i h u_i \rangle \\
&= \lambda_i a_{ij} - \langle \lambda_j u_j, h u_i \rangle \\
&= (\lambda_i - \lambda_j) a_{ij}.
\end{aligned} \tag{2.5}$$

The second term is equal to $\sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 / 2$. Indeed,

$$\begin{aligned}
-\langle h u_i^2, \Delta h \rangle &= \sum_V \Gamma(h u_i^2, h) d \\
&= \sum_V (h \Gamma(u_i^2, h) + u_i^2 \Gamma(h, h)) d \\
&= \frac{1}{2} \sum_{x \sim y} \left((\nabla_{xy} u_i)^2 h(x) (\nabla_{xy} h) + u_i(x)^2 (\nabla_{xy} h)^2 \right) \\
&\quad + \sum_V 2 h u_i \Gamma(h, u_i) d \\
&= \frac{1}{2} \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 + \sum_V 2 h u_i \Gamma(h, u_i) d.
\end{aligned}$$

Second, we claim that

$$(\lambda_{k+1}-\lambda_i)A_i \le \|u_i\Delta h + 2\Gamma(u_i,h)\|^2 - \sum_{j=0}^{k}(\lambda_i-\lambda_j)^2 a_{ij}^2. \tag{2.6}$$

From the definition of A_i, we have

$$A_i = -\langle \varphi_i, 2\Gamma(h,u_i) + u_i\Delta h - \sum_{j=0}^{k}(\lambda_i-\lambda_j)a_{ij}u_j\rangle.$$

Applying the Cauchy-Schwartz inequality to the definition of A_i and taking (2.3) and (2.5) into account, we have

$$(\lambda_{k+1}-\lambda_i)A_i^2 \le A_i(\|2\Gamma(h,u_i)+u_i\Delta h\|^2 - \sum_{j=0}^{k}(\lambda_i-\lambda_j)^2 a_{ij}^2).$$

From (2.4) and (2.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{x\sim y}u_i(x)u_i(y)(\nabla_{xy}h)^2 + \sum_{i,j=0}^{k}(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)a_{ij}^2\\
&\le \sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)\|2\Gamma(h,u_i)+u_i\Delta h\|^2 - \sum_{i,j=0}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2 a_{ij}^2.
\end{aligned}$$

Since $\sum_{i,j=0}^{k}(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)a_{ij}^2 = -\sum_{i,j=0}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2 a_{ij}^2$, we complete the proof. □

3. Proof of main results

In this section, we prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. In the subsection 3.1, we show some symmetries of eigenfunctions on a symmetric graph, which are used to prove Theorem 1.1.

3.1. Symmetries of eigenfunctions on a symmetric graph

We derive some properties of eigenfunctions on a symmetric graph. In particular, it seems that symmetry is essential in Lemma 3.2. A graph $G=(V,E)$ is said to be *symmetric* if for any two edges $(x,y),(x',y')\in E$, there exists an automorphism γ of G such that $x'=\gamma x$ and $y'=\gamma y$ hold. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. Note that symmetric graphs are vertex-transitive, i.e., $\mathrm{Aut}(G)$ acts transitively on V, and thus regular. We say that a vector subspace W of $C(V)$ is *invariant* if for any $u\in W$ and $\gamma\in\mathrm{Aut}(G)$, $\gamma u\in W$, where γu is defined by $\gamma u(x) := u(\gamma x), x\in V$.

Lemma 3.1. *Let $G=(V,E)$ be a vertex-transitive graph. Let W be an invariant vector subspace of $C(V)$ of dimension m and let $\{u_\alpha\}_{\alpha=1}^m$ be an orthonormal basis of W. Then, the function $|u_1|^2+\cdots+|u_m|^2$ is constant and its value is $m/d\#V$.*

Proof. Put $f(x) := |u_1(x)|^2+\cdots+|u_m(x)|^2$. By the invariance of W, the family $\{\gamma u_\alpha\}_{\alpha=1}^m$ is also an orthonormal basis of W for any $\gamma\in\mathrm{Aut}(G)$. For fixed $x\in V$, it is easy to see that the sum $|u_1(x)|^2 + \cdots + |u_m(x)|^2$ is independent of the choice of an orthonormal basis $\{u_\alpha\}$. Thus,

$$f(\gamma x) = \sum_{\alpha=1}^{m} |\gamma u_\alpha(x)|^2 = \sum_{\alpha=1}^{m} |u_\alpha(x)|^2 = f(x).$$

The transitivity of the action of $\mathrm{Aut}(G)$ yields that f is constant. Let C be the value of $|u_1(x)|^2 + \cdots + |u_m(x)|^2$. By multiplying d and summing over $x \in V$, we have

$$Cd\#V = \sum_{\alpha=1}^{m} \sum_{x \in V} |u_\alpha(x)|^2 d = m.$$

$\square$

Lemma 3.2. *Let G be a symmetric graph. Let λ be an eigenvalue of Δ and let $\{u_\alpha\}_{\alpha=1}^m$ be an orthonormal basis of W_λ. Then, the function $g(x, y) := \sum_{\alpha=1}^m |\nabla_{xy} u_\alpha|^2, x \sim y$, is constant and its value is $m\lambda/\#E$.*

Proof. Since W_λ is an invariant vector subspace of $C(V)$, the family $\{\gamma u_\alpha\}_{\alpha=1}^m$ is also an orthonormal basis of W_λ for any $\gamma \in \mathrm{Aut}(G)$. Since the sum $\sum_{\alpha=1}^m |\nabla_{xy} u_\alpha|^2$ is independent of the choice of an orthonormal basis $\{u_\alpha\}$, we have

$$g(\gamma x, \gamma y) = \sum_{\alpha=1}^{m} |\nabla_{xy}(\gamma u_\alpha)|^2 = \sum_{\alpha=1}^{m} |\nabla_{xy} u_\alpha|^2 = g(x, y).$$

The symmetry of G yields that g is constant. Let C' be the value of g. By summing over $x \sim y$, we have

$$2C'\#E = \sum_{\alpha=1}^{m} \sum_{x \sim y} |\nabla_{xy} u_\alpha|^2 = 2\lambda m.$$

$\square$

Corollary 3.3. *Let G be a symmetric graph. Let λ and $\{u_\alpha\}$ be as in Lemma 3.2. Then, the function $f_3(x, y) = \sum_{\alpha=1}^m u_\alpha(x) \nabla_{xy} u_\alpha$ is constant and its value is $-\lambda m/2\#E$.*

Proof. The constancy of f_3 immediately follows from Lemma 3.1 and Lemma 3.2. Let C_3 be the value of f_3. By summing over $x \sim y$, we have

$$2C_3\#E = \sum_{\alpha=1}^{m} \sum_{x \sim y} u_\alpha(x) \nabla_{xy} u_\alpha.$$

By interchanging x and y,

$$\sum_{x \sim y} u_\alpha(x) \nabla_{xy} u_\alpha = -\sum_{x \sim y} u_\alpha(y) \nabla_{xy} u_\alpha.$$

Thus, we obtain

$$2C_3\#E = \frac{1}{2}\sum_{\alpha=1}^{m}\sum_{x\sim y}(u_\alpha(x) - u_\alpha(y))\nabla_{xy}u_\alpha = -\lambda m.$$

$\square$

3.2. Proof of main results

We prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. First, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{u_\alpha\}_\alpha$ be an orthonormal basis of E_μ. Then, we have

$$\begin{aligned}\sum_{\alpha=1}^{m}\sum_{x\sim y}u_i(x)u_i(y)|\nabla_{xy}u_\alpha|^2 &= \frac{\lambda m}{\#E}\sum_{x\sim y}u_i(x)u_i(y)\\ &= \frac{\lambda m}{\#E}\sum_{x\in V}u_i(x)d\cdot\frac{1}{d}\sum_{y\sim x}u_i(y)\\ &= \frac{\lambda m}{\#E}(1-\lambda_i)\sum_{x\in V}u_i(x)^2 d\\ &= \frac{\lambda m}{\#E}(1-\lambda_i). \qquad (3.1)\end{aligned}$$

Next, we evaluate $\sum_\alpha \|2\Gamma(u_\alpha, u_i) + u_i\Delta u_\alpha\|^2$. By Jensen's inequality, we have

$$4\Gamma(u_i,u_\alpha)(x)^2 = \left(\frac{1}{d}\sum_{y\sim x}(\nabla_{xy}u_i)(\nabla_{xy}u_\alpha)\right)^2 \le \frac{1}{d}\sum_{y\sim x}(\nabla_{xy}u_i)^2(\nabla_{xy}u_\alpha)^2,$$

which yields

$$4\sum_{\alpha=1}^{m}\sum_{x\in V}\Gamma(u_i,u_\alpha)(x)^2 d \le \frac{2\lambda\lambda_i m}{\#E}. \qquad (3.2)$$

By Lemma 3.1, we have

$$\sum_{\alpha=1}^{m}\sum_{x\in V}(u_i(x)\Delta u_\alpha(x))^2 d = \frac{\lambda^2 m}{2\#E}. \qquad (3.3)$$

By Lemma 3.3,

$$\begin{aligned}-4\lambda\sum_{\alpha=1}^{m}\sum_{x\in V}u_i(x)u_\alpha(x)\Gamma(u_\alpha,u_i)(x)d &= \frac{\lambda^2 m}{\#E}\sum_{x\in V}u_i(x)\sum_{y\sim x}\nabla_{xy}u_i\\ &= -\frac{\lambda^2\lambda_i m}{\#E}. \qquad (3.4)\end{aligned}$$

By letting $h = u_\alpha$ in Lemma 2.4, summing over $\alpha = 1,\dots,m$ and taking (3.1), (3.2), (3.3) and (3.4) into account, we obtain

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2(1-\lambda_i) \le \sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)(2(2-\lambda)\lambda_i+\lambda).$$

□

In order to prove Theorem 1.2, we need some lemmas.

LEMMA 3.4 (CHEBYSHEV'S SUM INEQUALITY). *Let $N \ge 1$ be an integer and $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ two sequences of real numbers. If both of $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ are non-increasing, then*

$$\frac{1}{N}\sum_{i=1}^{N} a_i b_i \ge \left(\frac{1}{N}\sum_{i=1}^{N} a_i\right)\left(\frac{1}{N}\sum_{i=1}^{N} b_i\right).$$

LEMMA 3.5. *For any $0 \le k \le N-1$,*

$$\sum_{i=0}^{k}(1-\lambda_i) \ge 0$$

and the equality holds if and only if $k = N-1$.

PROOF. Let A be the adjacency matrix of G and $\nu_0 \ge \nu_1 \ge \cdots \ge \nu_{N-1}$ be all eigenvalues of A. Since any diagonal entry of A is 0, $\sum_{i=0}^{N-1}\nu_i$ is also 0 and $\sum_{i=0}^{k}\nu_i \ge 0$ for any k, with the equality holds if and only if $k = N-1$. By the relation between Δ and A, we have

$$\sum_{i=0}^{k}(1-\lambda_i) = \frac{1}{d}\sum_{i=0}^{k}\nu_i \ge 0$$

and the equality holds if and only if $k = N-1$. □

Next, we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. By letting $\lambda = \lambda_1$ in Theorem 1.1, we have

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i^2-(\lambda_{k+1}-2\lambda_1+5)\lambda_i+\lambda_{k+1}-\lambda) \le 0.$$

Clearly, $\lambda_{k+1}-\lambda_i$ is non-increasing in i. Put $f(x) := x^2-(\lambda_{k+1}-2\lambda_1+5)x$. Then, the function f is non-increasing in the interval $(-\infty, (\lambda_{k+1}-2\lambda_1+5)/2]$. From Lemma 2.1, $(\lambda_{k+1}-2\lambda_1+5)/2 \ge 2$. Since $0 \le \lambda_i \le 2$, $\lambda_i^2-(\lambda_{k+1}-2\lambda_1+5)\lambda_i+\lambda_{k+1}-\lambda$ is non-increasing in i. We may use Lemma 3.4 and thus

$$\left(\lambda_{k+1}-\sum_{i=0}^{k}\frac{\lambda_i}{k+1}\right)\left(\sum_{i=0}^{k}\frac{(1-\lambda_i)\lambda_{k+1}+\lambda_i^2-(5-2\lambda_1)\lambda_i}{k+1}-\lambda_1\right) \le 0.$$

If $k \geq m(\lambda_1)$, then $\lambda_{k+1} - \sum_{i=0}^{k} \lambda_i/(k+1)$ is strictly positive. In this case, we have

$$\frac{1}{k+1}\sum_{i=0}^{k}((1-\lambda_i)\lambda_{k+1} + \lambda_i^2 - (5-2\lambda_1)\lambda_i - \lambda_1) \leq 0.$$

By Lemma 3.5, we obtain

$$\lambda_{k+1} \leq \frac{(k+1)\lambda_1 + \sum_{i=1}^{k}((5-2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k}(1-\lambda_i)}.$$

This inequality also holds for $k < m(\lambda_1)$. □

Proof of Corollary 1.3. If $k = m(\lambda_1)$, then $\lambda_{k+1} = \mu_2$ and $\lambda_1 = \cdots = \lambda_{k-1} = \mu_1$. By Theorem 1.2, we have

$$\frac{\mu_2}{\mu_1} \leq \frac{6m+1-3m\mu_1}{m+1-m\mu_1}.$$

Let $g(x) := (6m+1-3mx)/(m+1-mx)$. The function g is increasing. By Lemma 2.1,

$$\frac{\mu_2}{\mu_1} \leq g(1) = 3m+1.$$

□

4. On the non-triviality of Corollary 1.3

In this section, we consider symmetric graphs, other than complete graphs. Let μ_1 and μ_2 be the first and the second smallest positive eigenvalue, respectively. If $(3m+1)\mu_1$ is not less than 2, then the inequality in Corollary 1.3 is trivial since $\mu_2 \leq 2$ always holds. In this section, we see that there exist infinitely many graphs such that $(3m+1)\mu_1$ is strictly less than 2.

Let C_N, $N \geq 3$, be the cycle graph with N vertices. Cycle graphs are symmetric. The spectra of cycle graphs are well-known.

Lemma 4.1. *The smallest positive eigenvalue of the normalized Laplace operator associated with C_N is $1 - \cos(2\pi/N)$ and its multiplicity is 2.*

Since $1 - \cos(2\pi/N)$ is decreasing in N and tends to 0 as $N \to \infty$, there exists a number N_0 such that $(3m+1)\mu_1 = 7(1-\cos(2\pi/N))$ is strictly less than 2 for any $N \geq N_0$. In fact, we can take $N_0 = 9$.

References

[1] Q.-M. Cheng and H. Yang, *Estimates on eigenvalues of Laplacian*, Math. Ann. **331** (2005), no. 2, 445–460.

[2] F. R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.

[3] G. N. Hile and M. H. Protter, *Inequalities for eigenvalues of the Laplacian*, Indiana Univ. Math. J. **29** (1980), no. 4, 523–538.

[4] B. Hua, Y. Lin, and Y. Su, *Payne-Pólya-Weinberger, Hile-Protter and Yang's inequalities for Dirichlet Laplace eigenvalues on integer lattices*, arXiv:1710.05799 (2017).

[5] L. E. Payne, G. Pólya, and H. F. Weinberger, *On the ratio of consecutive eigenvalues*, J. Math. and Phys. **35** (1956), 289–298.

[6] L. E. Payne, G. Pólya, and H. F. Weinberger, *Sur le quotient de deux fréquences propres consécutives*, C. R. Acad. Sci. Paris **241** (1955), 917–919.

[7] C. J. Thompson, *On the ratio of consecutive eigenvalues in N-dimensions*, Studies in Appl. Math. **48** (1969), 281–283.

[8] H. Yang, *An estimate of the difference between consecutive eigenvalues*, Preprint IC/91/60 of ICTP ,Trieste (1995).

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