LARGE DEVIATION PRINCIPLE FOR S-UNIMODAL MAPS WITH FLAT CRITICAL POINTS

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To the memory of Yoichiro Takahashi (1946 – 2019)

Abstract. We study a topologically exact, negative Schwarzian unimodal map without neutral periodic points whose critical point is non-recurrent and flat. Assuming that the critical order is polynomial or logarithmic, we establish the Large Deviation Principle and provide a partial description of the minimizers of the rate function. We apply our main results to a certain parametrized family of unimodal maps in the same topological conjugacy class, and determine the sets of minimizers.

1. Introduction

Consider a dynamical system $f\colon X \to X$ of a compact Riemannian manifold X. The theory of large deviations deals with the behavior of the empirical mean

$$\delta_x^n = \frac{1}{n}\left(\delta_x + \delta_{f(x)} + \cdots + \delta_{f^{n-1}(x)}\right) \quad \text{as } n \to \infty,$$

where δ_x denotes the Dirac measure at x. We put a Lebesgue measure $|\cdot|$ on X as a reference measure, and investigate the asymptotic behavior of the empirical mean for Lebesgue almost every initial condition. Let $\mathcal{M}$ denote the space of Borel probability measures on X endowed with the topology of weak* convergence. We say *the (level-2) Large Deviation Principle* (LDP) holds if there exists a lower semi-continuous function $I = I(f;\cdot)\colon \mathcal{M} \to [0,\infty]$ which satisfies the following:

- (lower bound) for any open subset $\mathcal{G}$ of $\mathcal{M}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log |\{x \in X \colon \delta_x^n \in \mathcal{G}\}| \geq -\inf_{\mathcal{G}} I; \tag{1.1}$$

- (upper bound) for any closed subset $\mathcal{C}$ of $\mathcal{M}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\{x \in X \colon \delta_x^n \in \mathcal{C}\}| \leq -\inf_{\mathcal{C}} I, \tag{1.2}$$

where $\log 0 = -\infty$ and $\inf \emptyset = \infty$. The function I is called *a rate function.* Since $\mathcal{M}$ is metrizable, if the LDP holds, then the rate function is unique. A measure $\mu \in \mathcal{M}$ satisfying $I(\mu) = 0$ is called *a minimizer* of I. In rough terms, the LDP implies that under iteration each empirical mean gets close to the set of minimizers. Hence, it is important to determine the set of minimizers.

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For a transitive uniformly hyperbolic system with Hölder continuous derivative, the LDP was established by Takahashi [22], Orey and Pelikan [17], Kifer [12]. For the level-1 LDP, see Young [26]. In these results, the rate function I is given by

$$I(\mu)=\begin{cases}-h(\mu)+\int\sum_{\chi_i(x)>0}\chi_i(x)d\mu(x) & \text{if } \mu \text{ is } f\text{-invariant;}\\ \infty & \text{otherwise,}\end{cases}$$

where $h(\mu)$ denotes the Kolmogorov-Sinaĭ entropy of μ relative to f, and $\sum_{\chi_i(x)>0}\chi_i(x)$ is the sum of positive Lyapunov exponents at x counted with multiplicity. The minimizer is unique and it is the Sinaĭ-Ruelle-Bowen measure [2, 20].

For non-hyperbolic systems, few results concerning the LDP were available until recently. For interval maps with neutral fixed points, Pollicott et al. [18, 19] proved several results that are closely related to the LDP. The method in [4] implies that the LDP holds for some non-hyperbolic systems which are very close to uniformly hyperbolic ones, such as almost Anosov systems, interval maps with neutral fixed points, and topologically exact unimodal maps with non-recurrent non-flat critical point. In [6], the LDP was established for certain non-uniformly expanding quadratic maps under strong assumptions on the hyperbolicity and recurrence of the orbit of the critical point. Substantial progress has been made in [5] which establishes the LDP for any topologically exact multimodal map with non-flat critical points and Hölder continuous derivatives. On the LDP for renormalizable unimodal maps with non-flat critical points, see [23].

In this paper we establish the LDP for unimodal maps with non-recurrent flat critical points, i.e., critical points at which all derivatives vanish. In [5], all critical points are assumed to be non-flat (see e.g., [8] for the definition), and this assumption is crucial as explained below.

1.1. **Statements of the main results.** Let $X=[0,1]$ and $f\colon X\to X$ be *a unimodal map*, i.e., a C^1 map whose critical set $\{x\in X\colon Df(x)=0\}$ consists of a single point $c\in(0,1)$ that is an extreme point. *An S-unimodal map f* is a unimodal map of class C^3 on $X\setminus\{c\}$ with negative Schwarzian derivative such that if $x\in\partial X$ is a fixed point then $|Df(x)|>1$. Any neutral periodic point of an S-unimodal map is attracting, and hence a topologically transitive S-unimodal map does not have a neutral periodic point. Denote by $\omega(c)$ the omega-limit set of c. We say the critical point c is *non-recurrent* if $c\notin\omega(c)$.

For an S-unimodal map with non-recurrent flat critical point having only hyperbolic repelling periodic points, Benedicks and Misiurewicz [1] constructed a sigma-finite invariant measure that is absolutely continuous with respect to the Lebesgue measure. Zweimüller [27] proved some statistical properties of the invariant measure, including a polynomial bound on the decay of correlations for maps with a flat critical behavior of the type $\exp(-|x-c|^{-\alpha})$ ($\alpha>0$). For a parametrized family of S-unimodal maps with this type of critical behavior with $\alpha<1/8$, Thunberg [24] proved the existence of a positive measure set of parameters for which the derivatives of the corresponding maps exhibit exponential growth along the orbits of the critical values. This positive measure set contains a dense subset corresponding to maps with non-recurrent critical points. The same type of flat critical behavior, which we refer to as of polynomial order, was also considered by Dobbs [9, Corollary 1.3].

In the following, for a flat critical point c we assume there exists a C^1 function ℓ on $X \setminus \{c\}$ such that the following holds:

(i) $\ell(x) \to \infty$ and $|D\ell(x)| \to \infty$ as $x \to c$. Here, $x \to c$ indicates both $x \to c^+$ and $x \to c^-$;

(ii) there exist C^1 diffeomorphisms ξ, η of $\mathbb{R}$ such that $\xi(c) = 0 = \eta(f(c))$ and $|\xi(x)|^{\ell(x)} = \eta(f(x))$ for all x near c.

In other words, up to C^1 changes of coordinates around c and $f(c)$ we have $f(x) = f(c) - |x-c|^{\ell(x)}$. The function ℓ determines how quickly $Df(x)$ decreases to 0 as $x \to c$. We work with two specific rates of growth for ℓ. We say the flat critical point c is *of polynomial order* if there exists a C^1 function v on X such that $v(c) > 0$ and for all x near c,

$$\ell(x) = |x-c|^{-v(x)}. \tag{1.3}$$

We say c is *of logarithmic order* if there exist a C^1 function u on X and $\alpha > 0$ such that $u(c) > 0$ and for all x near c,

$$\ell(x) = u(x)|\log|x-c||^{\alpha}.$$

We say f is *topologically exact* if for any non-empty open subset U of X there exists an integer $n \geq 1$ such that $f^n(U) = X$. If f is topologically exact, then it is topologically transitive. All our main results hold for topologically exact S-unimodal maps with non-recurrent flat critical points of polynomial or logarithmic order. To simplify our exposition we restrict ourselves to the case of polynomial order.

Let $\mathcal{M}(f)$ denote the set of elements of $\mathcal{M}$ which are f-invariant. For an S-unimodal map f and $\nu \in \mathcal{M}(f)$ define *the Lyapunov exponent* $\chi(\nu)$ by

$$\chi(\nu) = \int \log|Df| d\nu.$$

From the result of Bruin and Keller [3], $\chi(\nu) \geq 0$ holds[1] for any $\nu \in \mathcal{M}(f)$ if all periodic points of f are hyperbolic repelling. Let $F\colon \mathcal{M} \to [-\infty, 0]$ denote the free energy

$$F(\nu) = \begin{cases} h(\nu) - \chi(\nu) & \text{if } \nu \in \mathcal{M}(f); \\ -\infty & \text{otherwise.} \end{cases}$$

The entropy is upper semi-continuous, and for our map f the Lyapunov exponent is not lower semi-continuous [3, Proposition 2.8]. Hence, the lower semi-continuity of $-F$ is not a reasonable assumption in our investigation of the LDP. We introduce the lower semi-continuous regularization I of $-F$ by

$$I(\mu) = -\inf_{\mathcal{G} \ni \mu} \sup_{\mathcal{G}} F, \tag{1.4}$$

where the infimum is taken over all open subsets $\mathcal{G}$ of $\mathcal{M}$ containing μ.

[1] The proof there does not use the non-flatness of the critical point.

Theorem 1.1. *Let $f\colon X \to X$ be a topologically exact S-unimodal map with non-recurrent flat critical point of polynomial order. Then the Large Deviation Principle holds. The rate function is given by I in* (1.4).

Flat critical points behave like neutral fixed points by trapping nearby orbits for a very long period of time, and hence can influence statistical properties of the map. By the result of Benedicks and Misiurewicz [1], for a map in Theorem A there exists a sigma-finite invariant measure that is absolutely continuous with respect to the Lebesgue measure. This measure is unique up to a multiplicative constant, and is a finite measure if and only if $\int \log|Df(x)|dx > -\infty$. If finite, then its normalization is denoted by $\mu_{\rm ac}$, and is called *an acip.* Many of the statistical properties of f depend on whether or not the map has an acip. For further details, see Zweimüller [27].

We now describe the set of minimizers of the rate function. For an S-unimodal map with non-flat critical point satisfying the Collet-Eckmann condition [7], the minimizer is unique and it is the acip, see Appendix A for details. The theorem below shows that the same characterization does not hold for maps with flat critical points. A measure $\mu \in \mathcal{M}$ is called *a post-critical measure* if there exists an increasing sequence $\{m_i\}_{i\geq 0}$ of positive integers such that $\delta_c^{m_i}$ converges in the weak* topology to μ as $i \to \infty$. As $\mathcal{M}$ is compact, post-critical measures exist. Each post-critical measure is f-invariant with its support contained in $\omega(c)$.

Theorem 1.2. *Let $f\colon X \to X$ be as in Theorem 1.1. Then the following hold:*

(a) *any post-critical measure is a minimizer.*
(b) *if $\mu \in \mathcal{M}(f)$ is a minimizer and $\mu(\omega(c)) = 0$, then μ is the acip.*

Theorem 1.2 implies the following consequences. If the acip exists, then it is a minimizer. Since the set of minimizers is a convex set, it contains convex combinations of the acip and post-critical measures. If $f|_{\omega(c)}$ is uniquely ergodic, then any minimizer is a convex combination of the acip and the unique post-critical measure. If the acip does not exist, any minimizer is supported on $\omega(c)$.

1.2. **Application.** To illustrate our main results, consider a parametrized family $\{f_b\}_{b>0}$ of unimodal maps given by

$$f_b(x) = \begin{cases} -2^{2^b}\,|x-1/2|^{|x-1/2|^{-b}} + 1 & \text{for } x \in [0,1] \setminus \{1/2\}; \\ 1 & \text{for } x = 1/2. \end{cases}$$

Here, $1/2$ is a flat critical point of polynomial order. A tedious computation shows that f_b has negative Schwarzian derivative, for example, if $b \geq 1/\sqrt{6}$. Note that $f_b(0) = f_b(1) = 0$. A direct calculation shows that $Df_b(0) > 1$. Then, from Singer's Theorem [21] all periodic points are hyperbolic repelling. Hence, f_b is topologically conjugate to the full tent map, and so is topologically exact. By Theorem 1.1, the LDP holds. Since $\int \log|Df_b(x)|dx > -\infty$ holds if and only if $b < 1$, f_b has an acip (denoted by $\mu_{{\rm ac},b}$) if and only if $b < 1$. The typical behavior changes at $b = 1$:

- for $1/\sqrt{6} \leq b < 1$, the measure $\delta_{x,b}^n = (1/n)\sum_{i=0}^{n-1}\delta_{f_b^i(x)}$ converges in the weak* topology as $n \to \infty$ to $\mu_{{\rm ac},b}$ for Lebesgue a.e. $x \in X$;

- for $b \geq 1$, $\delta_{x,b}^n$ converges in the weak* topology as $n \to \infty$ to the Dirac measure δ_0 at 0 for Lebesgue a.e. $x \in X$.

Theorem 1.2 yields a complete characterization of the set of minimizers:

- for $1/\sqrt{6} \leq b < 1$, $I(f_b; \mu) = 0$ if and only if μ is a convex combination of δ_0 and $\mu_{\mathrm{ac},b}$;
- for $b \geq 1$, $I(f_b; \mu) = 0$ if and only if $\mu = \delta_0$.

For each $b \in [1/\sqrt{6}, 1)$, let $p_b^+ \in X$ denote the orientation-reversing fixed point of f_b, and p_b^- the preimage of p_b^+ by f_b that differs from p_b^+. The first return map to the interval (p_b^-, p_b^+) defines an inducing scheme to which the acip $\mu_{\mathrm{ac},b}$ lifts. From the result of Zweimüller [27], this inducing scheme has polynomial tail with respect to the Lebesgue measure, uniformly over all b contained in each compact subinterval of $[1/\sqrt{6}, 1)$. Then, the result of Freitas and Todd [10] on statistical stability implies that $b \in [1/\sqrt{6}, 1) \mapsto \mu_{\mathrm{ac},b} \in \mathcal{M}$ is continuous in the L^1 norm. Considering the first return map to (p_b^-, p_b^+), it is not hard to show that the acip of f_b for $b < 1$ converges in the weak* topology to δ_0 as $b \to 1$. As a consequence, the set of minimizers depends continuously on b. This type of change also occurs for the Manneville-Pomeau maps, see Appendix B for details.

1.3. **Methods of proofs of the main results.** A proof of Theorem 1.1 is briefly outlined as follows. The lower bound is already known to hold for a broad class of smooth interval maps including those in Theorem 1.1, see [5, Proposition 2.1]. A strategy for the upper bound is to construct "good" finite hyperbolic subsystems, as developed in [4]. This strategy was taken in [5], but there is one key difference from [5].

The class of maps treated in this paper is disjoint from those treated in [5]. In [5], all critical points are assumed to be non-flat, and the following estimate was used in the construction of good finite hyperbolic subsystems (see [5, Lemma 3.2]): for any interval $\widehat{U}$ contained in a small neighborhood of the critical set and any subinterval U of $\widehat{U}$,

$$\frac{|f(U)|}{|f(\widehat{U})|} \leq C_0 \frac{|U|}{|\widehat{U}|},$$

where $C_0 > 0$ is a uniform constant. This estimate obviously fails for maps with flat critical points. To dispense with this estimate altogether, we use an inducing scheme with distinctive property (see Proposition 2.3).

The rest of this paper consists of two sections. In Section 2 we construct inducing schemes with distinctive property for maps in Theorem 1.1. We then use them to construct good finite hyperbolic subsystems and complete the proof of Theorem 1.1. Theorem 1.2 is proved in Section 3. To prove Theorem 1.2(a), we use the inducing scheme with distinctive property and show that any post-critical measure is weak*-approximated by measures supported on periodic orbits with arbitrarily small Lyapunov exponents. To prove Theorem 1.2(b), we evaluate the free energy along sequences of measures which approximate minimizers, carefully analyzing the lack of lower semi-continuity of Lyapunov exponents.

2. On the proof of the LDP

In Section 2.1 we introduce inducing schemes and describe their basic properties. In Section 2.2 we construct an inducing scheme with distinctive property. In Section 2.3 we prove Theorem 1.1.

2.1. **Inducing schemes.** Let f be a unimodal map, U an interval of X and $n \geq 1$ an integer. Each connected component of $f^{-n}(U)$ is called a *pullback* of U by f^n. A pullback J of U by f^n is called *diffeomorphic* if $f^n : J \to U$ is a diffeomorphism. We say an open subinterval Y of X is *nice* if $Y \cap \bigcup_{n=1}^{\infty} f^n(\partial Y) = \emptyset$ holds.

Assume the critical point c of f is non-recurrent. Define

$$\Lambda = \overline{\{f^n(c) \colon n \geq 1\}}. \tag{2.1}$$

Then Λ is a hyperbolic set. Let Y be a nice interval which contains c and satisfies $\overline{Y} \cap \Lambda = \emptyset$. All pullbacks of Y are diffeomorphic and mutually disjoint. *The first entry time to* Y is a function $R\colon X \to \mathbb{Z}_{>0} \cup \{\infty\}$ defined by

$$R(x) = \inf\left(\{n \geq 1\colon f^n(x) \in Y\} \cup \{\infty\}\right).$$

The restriction of R to Y is denoted by $R|_Y$ and called *the first return time*. If R is constant on an interval $W \subset X$, then this common value is denoted by $R(W)$, and if moreover W is a pullback of Y by $f^{R(W)}$ then W is called *a first-entry pullback*. Let $\mathcal{W}$ denote the collection of all first-entry pullbacks which are contained in Y. The triplet $(Y, \mathcal{W}, R)$ is called *an inducing scheme*. If f is transitive, then $\overline{Y} = \overline{\bigcup_{J \in \mathcal{W}} J}$ holds.

Lemma 2.1. *Let f be a topologically transitive S-unimodal map with non-recurrent critical point. Then $\chi(\mu) > 0$ holds for any $\mu \in \mathcal{M}(f)$.*

Proof. From the ergodic decomposition theorem it suffices to consider the case where μ is ergodic. Let $(Y, \mathcal{W}, R)$ be an inducing scheme. If $\mu(Y) = 0$, then from Mañé's hyperbolicity theorem [15, Theorem A] $\chi(\mu) > 0$ holds. Assume $\mu(Y) > 0$. Define $\widehat{f} : \bigcup_{J \in \mathcal{W}} J \to Y$ by $\widehat{f}(x) = f^{R(J_x)}(x)$ where J_x is the element of $\mathcal{W}$ containing x. Denote by $\widehat{\mu}$ the normalized restriction of μ to Y. Then $\widehat{\mu}$ is $\widehat{f}$-invariant for which $\int R d\widehat{\mu}$ is finite and $\mu = (1/\int R d\widehat{\mu}) \sum_{J \in \mathcal{W}} \sum_{n=0}^{R(J)-1} (f^n)_*(\widehat{\mu}|_J)$. The Koebe Principle [8, Chapter IV. Theorem 1.2] implies that some iterate of $\widehat{f}$ is uniformly expanding, and thus $\chi(\mu) > 0$ holds. □

2.2. **Inducing scheme with distinctive property.** We shall construct an inducing scheme $(Y, \mathcal{W}, R)$ which allows us to glue orbits of part of the tail set $\{R|_Y > n\} = \{x \in Y \colon R(x) > n\}$ to the nice interval Y to form a pullback of Y whose first return time to Y is approximately n. In addition, we request that the size of this pullback is not too small. A limited form of this distinctive property is a consequence of [8, Chapter V. Lemma 3.3] stated as follows.

Proposition 2.2. *Let f be a topologically exact S-unimodal map with a non-recurrent critical point c and let $(Y, \mathcal{W}, R)$ be an inducing scheme. There exist $C > 0$ and $N_0 \geq 1$ such that for any integer $n \geq 0$ and any connected component*

A of $\{R|_Y > n\}$ not containing c, there exists $J \in \mathcal{W}$ which is contained in A and satisfies

$$n < R(J) \leq n + N_0 \quad \textit{and} \quad \frac{|J|}{|A|} \geq C.$$

The assumption $c \notin A$ is important in Proposition 2.2. If c is flat and $c \in A$, then the same conclusions do not hold. In order to treat the case $c \in A$ we need the following version of Proposition 2.2.

Proposition 2.3 (A distinctive property of an inducing scheme)**.** *Let f be as in Proposition 2.2, and moreover assume the critical point is flat of polynomial order. There exist an inducing scheme $(Y, \mathcal{W}, R)$ and a constant $C > 0$ with the following property: for any $\varepsilon > 0$ there exists $N_1 \geq 1$ such that for any integer $n \geq N_1$ and the connected component A of $\{R|_Y > n\}$ containing c, there exists $J \in \mathcal{W}$ which is contained in A and satisfies*

$$n < R(J) \leq (1+\varepsilon)n \quad \textit{and} \quad \frac{|J|}{|A|} \geq \frac{C}{n}.$$

Proof. Since f is topologically exact, the critical point c is accumulated by periodic points from both sides. There exists a nice interval

$$Y = (a^-, a^+)$$

which contains c and satisfies $\overline{Y} \cap \Lambda = \emptyset$, and $f^{R_1}(a^-) = a^-$ or $f^{R_1}(a^+) = a^+$ where

$$R_1 = \min\{n \geq 1 \colon Y \cap f^n(Y) \neq \emptyset\}. \tag{2.2}$$

With no loss of generality we may assume $f^{R_1}(a^-) = a^-$. In what follows we show that the inducing scheme $(Y, \mathcal{W}, R)$ satisfies the desired properties. The following notation is in use: for two distinct non-empty subsets A and B of X, $A < B$ indicates $\sup A \leq \inf B$. We finish the proof of Proposition 2.3 assuming the conclusions of the next two lemmas. The latter one is concerned with the order of flatness of c.

Lemma 2.4. *There exist a sequence $\{J_k\}_{k \geq 1}$ of pairwise disjoint open subintervals of Y, a non-decreasing sequence $\{R_k\}_{k \geq 1}$ of positive integers, constants $\theta_0, \theta_1 > 0$, such that the following hold for any $k \geq 1$:*

(a) *$\{c\} < J_{k+1} < J_k$ and $\overline{J_{k+1}} \cap \overline{J_k} \neq \emptyset$.*
(b) *$|J_k| \geq \theta_0(|J_k| + |J_{k+1}|)$.*
(c) *if $J_k \in \mathcal{W}$ then $R(J_k) = R_k$.*
(d) *if $J_{k+1} \notin \mathcal{W}$ then $J_k \in \mathcal{W}$ and $\min_{J_{k+1}} R \geq R_k$.*
(e) *$R_{k+2} - R_k \geq 1$.*
(f) *$R_{k+1} - R_k \leq \theta_1$.*

Lemma 2.5. *There exists $\lambda > 0$ such that for any $\varepsilon > 0$,*

$$\limsup_{k \to \infty} \frac{\sum_{i \geq k(\varepsilon)} |J_i|}{\sum_{i \geq k} |J_i|} \leq (1 - \lambda\varepsilon)^{\frac{1}{\max v}},$$

where $k(\varepsilon) = k + \lfloor \varepsilon k \rfloor$.

Let $\varepsilon > 0$, let $n \geq 4\theta_1/\varepsilon$ be an integer and let A denote the connected component of $\{R|_Y > n\}$ containing c. Put $A^+ = (c, a^+) \cap A$. We proceed assuming $|A| \leq 2|A^+|$, and lastly indicate necessary minor modifications to treat the other case. Put

$$L = \{k \geq 1\colon J_k \in \mathcal{W}, J_k \subset A^+\}.$$

By Lemma 2.4(a)(d), L is non-empty. Put

$$\underline{n} = \min L \quad \text{and} \quad E_n = [\underline{n}, \underline{n} + \varepsilon n/(2\theta_1)] \cap L.$$

We have $\min E_n = \underline{n}$, and Lemma 2.4(d) gives $\max E_n \geq \underline{n} + \varepsilon n/(2\theta_1) - 2$. If $J_{\underline{n}-1} \in \mathcal{W}$ then $R(J_{\underline{n}-1}) \leq n$ by Lemma 2.4(a), and $R(J_{\underline{n}}) \leq R(J_{\underline{n}-1}) + \theta_1$ by Lemma 2.4(c)(f). If $J_{\underline{n}-1} \notin \mathcal{W}$, then $J_{\underline{n}-2} \in \mathcal{W}$ by Lemma 2.4(d). We have $R(J_{\underline{n}-2}) \leq n$, for otherwise $R(J_{\underline{n}-2}) > n$ and so $J_{\underline{n}-1} \subset A^+$ by Lemma 2.4(a)(d), a contradiction to the definition of $\underline{n}$. Hence $R(J_{\underline{n}}) \leq R(J_{\underline{n}-2}) + 2\theta_1 \leq n + 2\theta_1$. In all cases we have

$$R(J_{\underline{n}}) \leq n + 2\theta_1.$$

Using this and Lemma 2.4(c)(f), for each $k \in E_n$ we have

$$R(J_k) \leq R(J_{\underline{n}}) + (k - \underline{n})\theta_1 \leq n + 2\theta_1 + \frac{\varepsilon n}{2} \leq (1 + \varepsilon)n.$$

Hence we obtain

$$\bigcup_{k \in E_n} J_k \subset A^+ \cap \{n < R \leq (1+\varepsilon)n\}. \tag{2.3}$$

We have

$$\begin{aligned}
\sum_{k \in E_n} |J_k| &\geq \theta_0 \sum_{k \in E_n} (|J_k| + |J_{k+1}|) \quad \text{by Lemma 2.4(b)} \\
&\geq \theta_0 \sum_{k=\min E_n}^{\max E_n - 1} |J_k| \quad \text{by Lemma 2.4(d)} \\
&= \theta_0 \sum_{k \geq \min E_n} |J_k| \left(1 - \frac{\sum_{k \geq \max E_n} |J_k|}{\sum_{k \geq \min E_n} |J_k|}\right) \qquad (2.4) \\
&\geq \theta_0 |A^+| \left(1 - \frac{\sum_{k \geq \max E_n} |J_k|}{\sum_{k \geq \min E_n} |J_k|}\right) \\
&\geq \frac{\theta_0 |A^+|}{2} \frac{\lambda\varepsilon}{\max v} \quad \text{by Lemma 2.5,}
\end{aligned}$$

where the last inequality holds for sufficiently large n.

To finish the proof of Proposition 2.3, pick an interval of maximal length in $\{J_k \colon k \in E_n\}$ and denote it by J. By (2.3), $n < R(J) \leq (1+\varepsilon)n$. From (2.4) there exist constants $C = C(\theta_0, \theta_1) > 0$ and $N_1 = N_1(\theta_0, \theta_1, \varepsilon) \geq 1$ such that

$$\frac{|J|}{|A|} \geq \frac{|J|}{2|A^+|} \geq \frac{\sum_{k \in E_n} |J_k|}{2|A^+| \# E_n} \geq \frac{\theta_0 \lambda \varepsilon}{4(\varepsilon n/(2\theta_1) + 1) \max v} \geq \frac{C}{n},$$

for any $n \geq N_1$. If $|A| > 2|A^+|$, then for $A^- = (a^-, c) \cap A$ we have $|A| \leq 2|A^-|$. We repeat the above argument replacing A^+ by A^-, and J_k by the pullback of $f(J_k)$ by f which is not J_k. □

It is left to prove Lemma 2.4 and Lemma 2.5. We need the next lemma for the proof of Lemma 2.4.

Lemma 2.6. *Let W_1, W_2 be distinct first-entry pullbacks with $\{c\} < W_1 < W_2$ or $W_1 < W_2 < \{c\}$ such that $R(W_1) = R(W_2)$. There exists a first-entry pullback W such that $W_1 < W < W_2$ and $R(W) < R(W_1)$.*

Proof. Set $m = R(W_1) = R(W_2)$, and let U be the minimal open interval containing W_1 and W_2. Let $n \geq 1$ be the smallest integer such that $c \in f^n(U)$. We must have that $n < m$. If $1 \leq k \leq n-1$ then $Y \cap f^k(W_1 \cup W_2) = \emptyset$, and thus $Y \cap f^k(U) = \emptyset$. Since W_1, W_2 are first-entry pullbacks, $Y \cap f^n(W_1 \cup W_2) = \emptyset$. Define W to be the pullback of Y by f^n which is contained in U. □

Proof of Lemma 2.4. For a subset U of X, a first-entry pullback $W \subset U$ is called *the minimal pullback in U* if $R(W') > R(W)$ holds for any other first-entry pullback $W' \subset U$. Set

$$c^+ = \inf\{f^n(c) \colon n \geq 1, f^n(c) > c\}.$$

The assumption $f^{R_1}(a^-) = a^-$ implies $a^+ < c^+$. Since $f^n(a^+) \notin Y$ and $f^n(c^+) \notin Y$ for any $n \geq 1$, Mañé's hyperbolicity theorem implies the existence of a first-entry pullback in (a^+, c^+). Define V_1 to be the minimal pullback in (a^+, c^+).

In what follows we construct a finite sequence $V_1, V_2, \ldots$ of first-entry pullbacks by induction. Let $i \geq 1$ and suppose $V_1, \ldots, V_i$ have been defined. From Lemma 2.6, for any first-entry pullback W such that $V_i < W$, $R(V_i) < R(W)$ holds, or else there exists a first-entry pullback W such that $V_i < W$ and $R(V_i) > R(W)$. In the first case we stop the construction. In the second case, we define V_{i+1} to be the first-entry pullback such that $V_i < V_{i+1}$, $R(V_i) > R(V_{i+1})$, and any first-entry pullback W such that $V_i < W < V_{i+1}$ satisfies $R(W) > R(V_i)$. Since V_1 is the minimal pullback in (a^+, c^+), we have $(a^+, c^+) < V_{i+1}$. By Lemma 2.6, we end up with a sequence $\{V_i\}_{i=1}^T$ of first-entry pullbacks such that $V_1 < V_2 < \cdots < V_T$ and the following hold:

- ∘ if we write $V_i = (u_i, v_i)$ and $V_{i+1} = (u_{i+1}, v_{i+1})$, then V_{i+1} is the minimal pullback in (v_i, v_{i+1}).
- ∘ if W is a first-entry pullback such that $V_T < W$ then $R(V_T) < R(W)$.

The construction implies that the sequence of first entry pullbacks with these properties is unique.

Set

$$\theta_1 = R(V_1).$$

For each $1 \leq i \leq T$, let V_{-i} denote the pullback of $f(V_i)$ by f such that $V_{-i} < \{c\}$. Since boundary points of a nice interval are not contained in any first-entry pullback, the following holds:

(∗) for any nice interval $U \subset X \setminus Y$ such that $a^+ \in \partial U$ and $V_1 \subset U$ (resp. $a^- \in \partial U$ and $V_{-1} \subset U$), the minimal pullback W in U belongs to $\{V_i\}_{i=1}^T$ (resp. $\{V_{-i}\}_{i=1}^T$) and satisfies $R(W) \leq \theta_1$.

Indeed, the minimal pullback W in U is V_{i_0} where $i_0 = \max\{1 \leq i \leq T : V_i \subset U\}$.

We now construct $\{J_k\}_{k\geq 1}$, $\{R_k\}_{k\geq 1}$ by induction on k as follows. Define J_1 to be the pullback of Y by f^{R_1} which is contained in Y and satisfies $\{c\} < J_1$. In (2.2), R_1 has already been defined. Let $k \geq 1$ and assume that J_i, R_i have been defined for $i = 1, \ldots, k$, such that $f^{R_k}(Y \setminus (\bigcup_{i=1}^k J_i)) \cap Y = \emptyset$. Since the interval $f^{R_k}(Y \setminus (\bigcup_{i=1}^k J_i))$ is nice and contains V_1 or V_{-1} depending on whether $f^{R_k}(c) > c$ or $f^{R_k}(c) < c$, from $(*)$ the minimal pullback W in $f^{R_k}(Y \setminus (\bigcup_{i=1}^k J_i))$ belongs to either $\{V_{-i}\}_{i=1}^T$ or $\{V_i\}_{i=1}^T$. Let J_{-i} denote the pullback of $f(J_i)$ by f such that $J_{-i} < \{c\}$. Let J denote the pullback of Y by $f^{R_k+R(W)}$ which is contained in $Y \setminus (\bigcup_{i=1}^k J_{-i} \cup J_i)$ and satisfies $\{c\} < J$. If $\partial J_k \cap \partial J \neq \emptyset$, then set $J_{k+1} = J$ and $R_{k+1} = R_k + R(W)$. Otherwise, set $J_{k+2} = J$ and define J_{k+1} to be the maximal open interval sandwiched by J_k and J_{k+2}. Furthermore, set $R_{k+1} = R_{k+2} = R_k + R(W)$. This completes the construction of $\{J_k\}_{k\geq 1}$ and $\{R_k\}_{k\geq 1}$. To check (a) and (c)-(f) is straightforward from the construction.

To prove (b), we choose $\tau \in (0,1)$ such that for each $k \geq 1$ there exists a diffeomorphic pullback of the concentric open interval with $f^{R_k}(J_k \cup J_{k+1})$ of length $(1+2\tau)|f^{R_k}(J_k \cup J_{k+1})|$ by f^{R_k} which contains $J_k \cup J_{k+1}$, and Λ does not intersect the concentric open interval with Y of length $(1+2\tau)|Y|$. The first condition is fulfilled for sufficiently small τ because Λ intersects neither of $\bigcup_{i=1}^T V_{-i}$, Y, $\bigcup_{i=1}^T V_i$.

In what follows we treat three cases separately. Put $K = (\tau/(1+\tau))^2$ and

$$\Delta = \begin{cases} \inf\{|x-y| \colon x \in Y, y \in V_{-1} \cup V_1\} & \text{if } a^+ \notin V_1; \\ \min\{|V_{-1}|, |V_1|\} & \text{otherwise.} \end{cases}$$

Case 1: $J_k \in \mathcal{W}$ and $J_{k+1} \in \mathcal{W}$. Then $f^{R_k}(J_k) = Y$ and $f^{R_k}(J_{k+1}) \in \{V_{-1}, V_1\}$. The Koebe Principle gives

$$\frac{|J_k|}{|J_k \cup J_{k+1}|} \geq K \min\{|Y|, \Delta\}. \tag{2.5}$$

Case 2: $J_k \in \mathcal{W}$ and $J_{k+1} \notin \mathcal{W}$. Then $f^{R_k}(J_k) = Y$, $f^{R_k}(J_{k+2}) \in \{V_{-i}, V_i\}$ for some $1 \leq i \leq T$, and $f^{R_k}(J_{k+1})$ is the maximal open subinterval sandwiched by $f^{R_k}(J_{k+2})$ and Y. Hence $|f^{R_k}(J_{k+1})| \geq \Delta$ holds. The Koebe Principle gives (2.5).

Case 3: $J_k \notin \mathcal{W}$ and $J_{k+1} \in \mathcal{W}$. Then $f^{R_k}(J_{k+1}) \in \{V_{-i}, V_i\}$ for some $1 \leq i \leq T$, and $f^{R_k}(J_k)$ is the maximal open subinterval sandwiched by $f^{R_k}(J_{k+1})$ and Y. Hence $|f^{R_k}(J_k)| \geq \Delta$ holds. The Koebe Principle gives

$$\frac{|J_k|}{|J_k \cup J_{k+1}|} \geq K \min\{\Delta, |V_{-i}|, |V_i|\}. \tag{2.6}$$

Lemma 2.4(b) now follows from (2.5) and (2.6). □

Proof of Lemma 2.5. Write $J_k = (a_{k+1}, a_k) = (c+b_{k+1}, c+b_k)$. A direct calculation using the local form of f near c of polynomial order (see (1.3) for the definition) gives

$$\frac{\sum_{i\geq k(\varepsilon)}|J_i|}{\sum_{i\geq k}|J_i|}=\frac{b_{k(\varepsilon)}}{b_k}=\frac{\ell(a_k)^{\frac{1}{v(a_k)}}}{\ell(a_{k(\varepsilon)})^{\frac{1}{v(a_{k(\varepsilon)})}}}$$
$$=\left(\frac{\ell(a_k)}{\ell(a_{k(\varepsilon)})}\right)^{\frac{1}{v(a_{k(\varepsilon)})}}\cdot\ell(a_k)^{\frac{1}{v(a_k)}-\frac{1}{v(a_{k(\varepsilon)})}}. \tag{2.7}$$

The first equality follows from Lemma 2.4(a). Put $D_k=|Df^{R_k}(f(c))|$. The local form of f near c implies that $b_k^{\ell(a_k)}D_k$ is bounded away from zero and infinity uniformly on $k\geq 1$. Hence

$$\log b_k\approx-\frac{\log D_k}{\ell(a_k)},$$

where $\approx$ indicates that the ratio of the two numbers converges to 1 as $k\to\infty$. Now we assume

$$b_{k(\varepsilon)}\geq b_k^{1+\varepsilon^2} \tag{2.8}$$

for infinitely many k. Otherwise the desired inequality obviously holds. Let k be such that (2.8) holds. Then

$$\frac{\log D_{k(\varepsilon)}}{\log D_k}\frac{\ell(a_k)}{\ell(a_{k(\varepsilon)})}\approx\frac{\log b_{k(\varepsilon)}}{\log b_k}\leq 1+\varepsilon^2.$$

Lemma 2.4(e) gives $R_{k(\varepsilon)}-R_k\geq\lfloor\lfloor\varepsilon k\rfloor/2\rfloor$. From Mañé's hyperbolicity theorem, there is a constant $\lambda_0>0$ independent of ε such that $D_{k(\varepsilon)}\geq e^{\lambda_0\varepsilon k}D_k$ holds for sufficiently large k. Then there is a constant $\lambda>0$ independent of ε such that for sufficiently large k we have

$$\frac{\log D_k}{\log D_{k(\varepsilon)}}\leq\frac{\log D_k}{\log D_k+\lambda_0\varepsilon k}\leq 1-2\lambda\varepsilon,$$

and therefore

$$\frac{\ell(a_k)}{\ell(a_{k(\varepsilon)})}\leq 1-\lambda\varepsilon. \tag{2.9}$$

By the mean value theorem applied to the C^1 function $1/v$, there exists a constant $K>0$ such that

$$\frac{1}{v(a_k)}-\frac{1}{v(a_{k(\varepsilon)})}\leq K(b_k-b_{k(\varepsilon)}). \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.7) gives

$$\frac{b_{k(\varepsilon)}}{b_k}\leq(1-\lambda\varepsilon)^{\frac{1}{\max v}}\,\ell(a_k)^{K(b_k-b_{k(\varepsilon)})}.$$

For the second term,

$$\ell(a_k)^{K(b_k-b_{k(\varepsilon)})}=b_k{}^{-(b_k-b_{k(\varepsilon)})Kv(a_k)}\leq\left(b_k^{b_k}\right)^{-K\max v},$$

which converges to 1 as $k\to\infty$. We obtain the desired inequality. □

2.3. **Proof of Theorem 1.1.** Let f be a topologically exact S-unimodal map with non-recurrent flat critical point of polynomial order. The lower bound (1.1) is already known, see [5, Proposition 2.1]. Since f is topologically exact, the upper bound (1.2) follows from the proposition below, see [5, Sections 4.2 and 4.3] for details. For each function $\phi\colon X \to \mathbb{R}$ and an integer $n \geq 1$, write $S_n\phi$ for the Birkhoff sum $\sum_{k=0}^{n-1} \phi \circ f^k$.

Proposition 2.7. *Let $(Y, \mathcal{W}, R)$ be an inducing scheme for which the conclusion of Proposition 2.3 holds. For any $\varepsilon > 0$, an integer $l \geq 1$, continuous functions $\phi_1, \ldots, \phi_l\colon X \to \mathbb{R}$ and $\alpha_1, \ldots, \alpha_l \in \mathbb{R}$ the following holds:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \left| \left\{ x \in Y \colon \frac{1}{n} S_n\phi_i(x) \geq \alpha_i \quad 1 \leq \forall i \leq l \right\} \right|$$
$$\leq \sup \left\{ F(\mu)\colon \int \phi_i d\mu > \alpha_i - \varepsilon \quad 1 \leq \forall i \leq l \right\}.$$

Proof. For each integer $n \geq 0$, let $\widehat{\mathscr{D}}_n$ denote the set of connected components of the set $Y \setminus \{x \in Y \colon f^j(x) \in \partial Y$ for some $0 \leq j \leq n\}$. Notice that $f^j(W) \neq X$ for any $W \in \widehat{\mathscr{D}}_n$ and any $0 \leq j \leq n$.

Let $\varepsilon > 0$, $l \geq 1$ be an integer, $\phi_1, \ldots, \phi_l\colon X \to \mathbb{R}$ be continuous functions and $\alpha_1, \ldots, \alpha_l \in \mathbb{R}$. Let $\mathscr{D}_n$ denote the collection of $W \in \widehat{\mathscr{D}}_n$ such that there exists $y_W \in W$ such that $(1/n)S_n\phi_i(y_W) \geq \alpha_i$ for any $1 \leq i \leq l$. Obviously,

$$\left| \left\{ x \in Y \colon \frac{1}{n} S_n\phi_i(x) \geq \alpha_i \quad 1 \leq \forall i \leq l \right\} \right| \leq \sum_{W \in \mathscr{D}_n} |W|. \tag{2.11}$$

Put $\varepsilon_0 = \varepsilon/(4(1 + \max_{1 \leq i \leq l}(\sup|\phi_i| + |\alpha_i|)))$. Fix $\delta > 0$ such that if $|x - y| \leq \delta$ then $|\phi_i(x) - \phi_i(y)| \leq \varepsilon_0$ holds for any $1 \leq i \leq l$. Since f is topologically exact, there exists an integer $n' \geq 1$ such that for any $n > n'$ and $W \in \mathscr{D}_n$, $|f^j(W)| \leq \delta$ holds for any $0 \leq j \leq n - n' - 1$.

Let $N_1 = N_1(\varepsilon_0) \geq 1$ be the integer for which the conclusion of Proposition 2.3 holds with ε there replaced by ε_0. Since there are only a finite number of connected components of $\{R|_Y > n\}$ with $n < N_1$, reducing the constant $C > 0$ in Proposition 2.3 if necessary we have the following: for any $n < N_1$, any connected component A of $\{R|_Y > n\}$ and any $J \in \mathcal{W}$ with the smallest first return time among those which are contained in A,

$$n < R(J) \leq n + N_0 \quad \text{and} \quad \frac{|J|}{|A|} \geq \frac{C}{n},$$

where $N_0 \geq 1$ is the one in Proposition 2.2.

If W is a pullback of Y, then the integer $r \geq 1$ such that $f^r(W) = Y$ is unique. This $r = r(W)$ is called *an inducing time of* W. Let $\tau \in (0, 1)$ be such that Λ in (2.1) does not intersect the concentric open interval with Y of length $(1 + 2\tau)|Y|$. Put $K = (\tau/(1 + \tau))^2$.

Lemma 2.8. *If $n \geq 1$ is a sufficiently large integer, for each $W \in \mathscr{D}_n$ there exists a pullback W_* of Y which is contained in W and satisfies the following:*

$$n < r(W_*) \leq n + \max\{N_0, \varepsilon_0 n\}; \tag{2.12}$$

$$\frac{|W_*|}{|W|} \geq \frac{KC}{n}; \tag{2.13}$$

$$\frac{1}{r(W_*)} S_{r(W_*)}(x) > \alpha_i - \varepsilon \quad \textit{for any } x \in \overline{W_*} \textit{ and any } 1 \leq i \leq l. \tag{2.14}$$

Proof. Let $n > n'$. Put $m(W) = \max\{k \leq n \colon f^k(W) \subset Y\}$. Then $f^{m(W)}(W)$ coincides with one of the connected components of $\{R|_Y > n - m(W)\}$, denoted by A. By Proposition 2.3 it is possible to choose $J \in \mathcal{W}$ which is contained in A and satisfies

$$n - m(W) < R(J) \leq n - m(W) + \max\{N_0, \varepsilon_0(n - m(W))\}$$

and

$$\frac{|J|}{|A|} \geq \frac{C}{n}.$$

Now, define W_* to be the pullback of J by $f^{m(W)}$ which is contained in W. Since $r(W_*) = m(W) + R(J)$, we have

$$n \leq r(W_*) \leq \max\{n + N_0, (1 + \varepsilon_0)n\}$$

and

$$\frac{|W_*|}{|W|} \geq K \frac{|f^{m(W)}(W_*)|}{|f^{m(W)}(W)|} = K \frac{|J|}{|A|} \geq \frac{KC}{n}.$$

Hence (2.12) and (2.13) hold.

For each $W \in \mathscr{D}_n$ choose $y_W \in W$ such that $(1/n)S_n\phi_i(y_W) \geq \alpha_i$ holds for any $1 \leq i \leq l$. Let $x \in \overline{W_*}$ and $1 \leq i \leq l$. Since $|f^j(x) - f^j(y_W)| \leq |f^j(\overline{W})| \leq \delta$ hold for any $0 \leq j \leq n - n' - 1$ we have

$$\begin{aligned} |S_n\phi_i(x) - S_n\phi_i(y_W)| \leq & |S_{n-n'}\phi_i(x) - S_{n-n'}\phi_i(y_W)| \\ & + |S_{n'}\phi_i(f^{n-n'}(x)) - S_{n'}\phi_i(f^{n-n'}(y_W))| \\ \leq & (n - n')\varepsilon_0 + 2n' \sup|\phi_i| \\ \leq & 2\varepsilon_0 n. \end{aligned}$$

The last inequality holds for sufficiently large n. Therefore

$$\begin{aligned} S_{r(W_*)}\phi_i(x) &= S_n\phi_i(y_W) + (S_n\phi_i(x) - S_n\phi_i(y_W)) + (S_{r(W_*)}\phi_i(x) - S_n\phi_i(x)) \\ &\geq n\alpha_i - 2\varepsilon_0 n - (r(W_*) - n)\sup|\phi_i| \\ &> r(W_*)(\alpha_i - \varepsilon). \end{aligned}$$

This implies (2.14). □

The next lemma follows from the proof of the variational principle [25, Section 9.3]. See also [5, Lemma 4.5].

Lemma 2.9. *Let $t, q \geq 1$ be integers, and let $L_1, \ldots, L_t$ be distinct pullbacks of Y by f^q contained in Y. There exists an f^q-invariant Borel probability measure $\widehat{\mu}$ supported on $\bigcap_{j=0}^{\infty}(f^q)^{-j}\left(\overline{L_1 \cup \cdots \cup L_t}\right)$ such that the measure*

$$\mu = \frac{1}{q}(\widehat{\mu} + f_*(\widehat{\mu}) + \cdots + f_*^{q-1}(\widehat{\mu}))$$

is in $\mathcal{M}(f)$ and satisfies

$$\log\left(|L_1|+\cdots+|L_t|\right)\le qF(\mu)+\log\frac{|Y|}{K}.$$

Let $n>n'$ be a sufficiently large integer for which the conclusion of Lemma 2.8 holds. From (2.13) we have

$$\begin{aligned}\sum_{W\in\mathscr{D}_n}|W| &\le \frac{n}{KC}\sum_{W\in\mathscr{D}_n}|W_*|\\ &\le \frac{n}{KC}\sum_{s=n}^{\lfloor(1+\varepsilon_0)n\rfloor}\sum_{\substack{W\in\mathscr{D}_n\\ r(W_*)=s}}|W_*|\\ &\le \frac{n(\varepsilon_0 n+1)}{KC}\sum_{\substack{W\in\mathscr{D}_n\\ r(W_*)=s_0}}|W_*|\end{aligned}\tag{2.15}$$

for some $n\le s_0\le\lfloor(1+\varepsilon_0)n\rfloor$. Notice that W_* ($W\in\mathscr{D}_n$) are pairwise disjoint since each W_* is contained in W. From Lemma 2.9 there exists $\mu\in\mathcal{M}(f)$ such that $\int\phi_i d\mu>\alpha_i-\varepsilon$ holds for any $1\le i\le l$ and

$$\frac{1}{s_0}\log\sum_{\substack{W\in\mathscr{D}_n\\ r(W_*)=s_0}}|W_*|\le F(\mu)+\frac{1}{s_0}\log\frac{|Y|}{K}.$$

Since $s_0\ge n$ and $F(\mu)\le 0$ we have

$$\frac{1}{n}\log\sum_{\substack{W\in\mathscr{D}_n\\ r(W_*)=s_0}}|W_*|\le\frac{s_0}{n}F(\mu)+\frac{1}{n}\log\frac{|Y|}{K}\le F(\mu)+\frac{1}{n}\log\frac{|Y|}{K}.\tag{2.16}$$

The desired inequality in Proposition 2.7 follows from (2.11), (2.15) and (2.16). □

3. Descriptions of minimizers

In this section we give partial descriptions of minimizers for the rate functions for maps in Theorem 1.1. Sections 3.1 and 3.2 provide analytic estimates. In Section 3.3 we prove Theorem 1.2.

3.1. **Recovering expansion.** For two positive functions $a(x)$ and $b(x)$ defined on (subsets of) neighborhoods of the critical point c, the expression $a(x)\sim b(x)$ indicates that $a(x)/b(x)$ is bounded and bounded away from 0. Put

$$\Phi(x)=\left|D\ell(x)\log|x-c|+\frac{\ell(x)}{x-c}\right|\qquad\text{for } x\in X\setminus\{c\}.$$

The two terms on the right-hand side have the same sign: positive for $x>c$ and negative for $x<c$. Note that $\Phi(x)\to\infty$ as $x\to c$, and that $|Df(x)|\sim|f(x)-f(c)|\Phi(x)$.

Lemma 3.1. *Let $(Y, \mathcal{W}, R)$ be an inducing scheme with $Y = (a^-, a^+)$. For any $x \in Y$ such that $R(x) < \infty$, and either $f^i(x), f^i(c) \le a^-$ or $f^i(x), f^i(c) \ge a^+$ for each $1 \le i \le R(x) - 1$, we have*

$$R(x) \sim \ell(x) \log |x - c|^{-1} \quad \textit{and} \quad |Df^{R(x)}(x)| \sim \Phi(x).$$

Proof. Let $x \in Y$. From Mañé's hyperbolicity theorem, the distortion of iterates of f outside of Y is uniformly bounded: there exists a constant $C \ge 1$ such that for any $z \in X$ in between $f(x)$ and $f(c)$,

$$C^{-1} \le \frac{|Df^{R(x)-1}(z)|}{|Df^{R(x)-1}(f(c))|} \le C. \tag{3.1}$$

Up to C^1 changes of coordinates around c and $f(c)$ we have $f(x) = f(c) - |x-c|^{\ell(x)}$. Using (3.1) we obtain

$$|f^{R(x)}(x) - f^{R(x)}(c)| \sim |x - c|^{\ell(x)} |Df^{R(x)-1}(f(c))|.$$

There exist $N \ge 1$ and $\lambda_0 > 0$ such that for any $x \in Y$ with $R(x) \ge N$ we have

$$e^{\lambda_0 (R(x)-1)} \le |Df^{R(x)-1}(f(c))| \le (\sup |Df|)^{R(x)-1},$$

and thus

$$|x - c|^{\ell(x)} e^{\lambda_0(R(x)-1)} \le |f^{R(x)}(x) - f^{R(x)}(c)| \le |x - c|^{\ell(x)} (\sup |Df|)^{R(x)+N}. \tag{3.2}$$

Let $\tau \in (0, 1)$ be such that Λ in (2.1) does not intersect the concentric open interval with Y of length $(1 + 2\tau)|Y|$. Since $f^{R(x)}(x) \in Y$ and $f^{R(x)}(c)$ does not belong to the concentric closed interval with Y of length $(1 + 2\tau)|Y|$, we have

$$\tau |Y| \le |f^{R(x)}(x) - f^{R(x)}(c)| \le 1. \tag{3.3}$$

The first estimate in Lemma 3.1 follows from (3.2) and (3.3).

For the second one, note that

$$|Df(x)| \sim |x - c|^{\ell(x)} \Phi(x) \sim |f(x) - f(c)| \Phi(x). \tag{3.4}$$

Combining (3.1), (3.3) and (3.4) we obtain

$$\begin{aligned} |Df^{R(x)}(x)| &= |Df^{R(x)-1}(f(x))||Df(x)| \\ &\sim \frac{|f^{R(x)}(x) - f^{R(x)}(c)|}{|f(x) - f(c)|} |f(x) - f(c)| \Phi(x) \sim \Phi(x). \end{aligned}$$ □

3.2. **Partial lower semi-continuity of Lyapunov exponents.** The next lemma asserts that the Lyapunov exponent is lower semi-continuous along sequences of measures whose limit measures have no weight on the omega-limit set $\omega(c)$ of c.

Lemma 3.2. *Let f be a topologically exact S-unimodal map with a non-recurrent critical point c. Let $\{\mu_k\}_{k \ge 1}$ be a sequence of ergodic measures in $\mathcal{M}(f)$ which converges to a measure $\mu \in \mathcal{M}(f)$ in the weak* topology as $k \to \infty$ such that $\mu(\omega(c)) = 0$. Then*

$$\liminf_{k \to \infty} \chi(\mu_k) \ge \chi(\mu).$$

Proof. For $\delta > 0$ let $B_\delta = (c-\delta, c+\delta) \cap X$. In what follows we assume $\mu(B_\delta) > 0$ for any $\delta > 0$. The other case can be treated with minor modifications. Put

$$\Lambda_\delta = \{x \in X\colon \inf\{|x-y|\colon y \in \Lambda\} < \delta\},$$

where $\Lambda = \overline{\{f^n(c)\colon n \geq 1\}}$ as in (2.1). We assume δ is small enough so that the following hold:

$$f(B_\delta) \subset \Lambda_\delta; \tag{3.5}$$

$$\inf_{B_\delta \setminus \{c\}} \Phi \geq 1/\delta^2. \tag{3.6}$$

Note that (3.5) holds due to the contraction of f near c, and (3.6) holds because the flat critical point c is of polynomial order. Set $U_\delta = B_\delta \cup \Lambda_\delta$. We evaluate derivatives along orbits carefully analyzing their returns to U_δ. For each $x \in U_\delta \setminus \Lambda$ define

$$q(x) = \min\{i \geq 1\colon f^i(x) \notin U_\delta\}.$$

Since Λ is a hyperbolic set, for δ small enough, $q(x)$ is well-defined for all $x \in U_\delta \setminus \Lambda$. If $x \in B_\delta$ then from $|f^{q(x)}(x) - f^{q(x)}(c)| > \delta$ and (3.6) we have

$$\begin{aligned} |Df^{q(x)}(x)| &\sim |Df^{q(x)-1}(f(x))||f(x)-f(c)|\Phi(x) \\ &\sim |f^{q(x)}(x) - f^{q(x)}(c)|\Phi(x) > 1/\delta. \end{aligned}$$

Hence the following holds:

(i) if $x \in B_\delta$ then $|Df^{q(x)}(x)| > 1$.

We fix $\delta', \delta'' \in (0, \delta)$ with $\delta'' < \delta'$ such that

(ii) if $x \in \Lambda_{\delta'}$ then $|Df^{q(x)}(x)| > 1$, and

(iii) if $x \in \Lambda_{\delta,\delta'}$ and $|Df^{q(x)}(x)| \leq 1$ then $f^n(x) \in \Lambda_{\delta,\delta''}$ for $0 \leq n \leq q(x)-1$, where $\Lambda_{\delta,\delta'} = \overline{\Lambda_\delta \setminus \Lambda_{\delta'}}$.

These choices are feasible since Λ is a hyperbolic set. Define a function $\varphi_\delta\colon X \to \mathbb{R}$ by

$$\varphi_\delta(x) = \begin{cases} \max\{\log|Df(x)|, -1/\delta\} & \text{if } x \in B_\delta; \\ \log|Df(x)| & \text{if } x \notin B_\delta. \end{cases}$$

Since B_δ is open, $\mu(B_\delta) > 0$ and $\mu_k \to \mu$ in the weak* topology, there exists $k_\delta \geq 1$ such that $\mu_k(B_\delta) > 0$ holds for any $k \geq k_\delta$. For each $k \geq k_\delta$ we fix $x_k \in X$ such that for $\phi = \log|Df|$ and the indicator functions $\phi = \mathbb{1}_{B_\delta}$, $\mathbb{1}_{\Lambda_{\delta,\delta''}}$, and $\phi = \mathbb{1}_{U_\delta} \cdot \varphi_\delta$ we have

$$\lim_{n\to\infty} \frac{1}{n} S_n\phi(x_k) = \int \phi \, d\mu_k. \tag{3.7}$$

By (3.7) for $\phi = \mathbb{1}_{B_\delta}$ and $\mu_k(B_\delta) > 0$, the orbit of x_k visits B_δ infinitely often. By (3.5) we can define an infinite sequence of integers $0 = n_0 < q_0 < n_1 < n_1 + q_1 < n_2 < \cdots$ inductively by $q_0 = q(x_k)$, $n_{l+1} = \inf\{n > n_l + q_l\colon f^n(x_k) \in U_\delta\}$ and $q_{l+1} = q(f^{n_{l+1}}(x_k))$ $(l = 0, 1, \ldots)$. For each l, let $\Sigma_0, \Sigma_1, \Sigma_2$ (resp. $\Sigma_0', \Sigma_1', \Sigma_2'$) denote the sums of $\log|Df^{q_i}(f^{n_i}(x_k))|$ (resp. $\sum_{n=n_i}^{n_i+q_i-1}\varphi_\delta(f^n(x_k))$ over the following sets of the subscripts i, respectively:

$$\{0\leq i\leq l\colon f^{n_i}(x_k)\in\Lambda_{\delta,\delta'}, |Df^{q_i}(f^{n_i}(x_k))|\leq 1\};$$
$$\{0\leq i\leq l\colon f^{n_i}(x_k)\in B_\delta\cup\Lambda_{\delta'}\};$$
$$\{0\leq i\leq l\colon f^{n_i}(x_k)\in\Lambda_{\delta,\delta'}, |Df^{q_i}(f^{n_i}(x_k))|> 1\}.$$

If the corresponding set of i is empty, define the sum to be 0. Notice that $\Sigma_0=\Sigma_0'$ and $\Sigma_2=\Sigma_2'$. Let $\varepsilon>0$. For l large enough we have

$$\frac{\Sigma_0'+\Sigma_1'+\Sigma_2'}{n_l+q_l}<\int \mathbb{1}_{U_\delta}\cdot\varphi_\delta d\mu_k+\varepsilon\leq\int_{\Lambda_\delta}\varphi_\delta d\mu_k+\varepsilon,$$

where the first inequality holds for sufficiently large l from (3.7) for $\phi=\mathbb{1}_{U_\delta}\cdot\varphi_\delta$, and the second inequality is because φ_δ is negative on B_δ. Then

$$\begin{aligned}\frac{\Sigma_1'+\Sigma_2'}{n_l+q_l}&<\int_{\Lambda_\delta}\varphi_\delta d\mu_k-\frac{\Sigma_0'}{n_l+q_l}+\varepsilon\\ &\leq\int_{\Lambda_\delta}\varphi_\delta d\mu_k+\frac{M(\delta)}{n_l+q_l}\#\{n_1\leq n<n_l+q_l\colon f^n(x_k)\in\Lambda_{\delta,\delta''}\}+\varepsilon\\ &<\int_{\Lambda_\delta}\varphi_\delta d\mu_k+M(\delta)(\mu_k(\Lambda_{\delta,\delta''})+\varepsilon)+\varepsilon,\end{aligned}\tag{3.8}$$

where $M(\delta)=\max\{0,-\inf_{\Lambda_\delta}\log|Df|\}$. We have used (iii) to estimate Σ_0' from below. By (3.7) for $\phi=\mathbb{1}_{\Lambda_{\delta,\delta''}}$, the last inequality in (3.8) holds for sufficiently large l. Since $\Sigma_1\geq 0$ from (i) (ii) and $\Sigma_2'\geq 0$, we have

$$\begin{aligned}\log|Df^{n_l+q_l}(x_k)|&\geq\log|Df^{n_l+q_l}(x_k)|-\Sigma_1\\ &=\sum_{n=0}^{n_l+q_l-1}\varphi_\delta(f^n(x_k))-\Sigma_1'\\ &\geq\sum_{n=0}^{n_l+q_l-1}\varphi_\delta(f^n(x_k))-\Sigma_1'-\Sigma_2'.\end{aligned}\tag{3.9}$$

Dividing (3.9) by n_l+q_l, plugging (3.8) into the result and letting $l\to\infty$ yields

$$\begin{aligned}\chi(\mu_k)&=\lim_{l\to\infty}\frac{1}{n_l+q_l}\log|Df^{n_l+q_l}(x_k)|\\ &\geq\int\varphi_\delta d\mu_k-\int_{\Lambda_\delta}\varphi_\delta d\mu_k-M(\delta)(\mu_k(\Lambda_{\delta,\delta''})+\varepsilon)-\varepsilon.\end{aligned}\tag{3.10}$$

Since $\Lambda_{\delta,\delta''}$ is a closed set with $\mu(\Lambda_{\delta,\delta''})=0$, we have $\limsup_{k\to\infty}\mu_k(\Lambda_{\delta,\delta''})\leq\mu(\Lambda_{\delta,\delta''})=0$. Letting $k\to\infty$ in (3.10) gives

$$\liminf_{k\to\infty}\chi(\mu_k)\geq\int\varphi_\delta d\mu-\int_{\Lambda_\delta}\log|Df|d\mu-M(\delta)\varepsilon-\varepsilon.$$

As $\delta\to 0$, the first integral converges to $\chi(\mu)$. Since $\mu(\Lambda)=0$ which follows from $\mu(\omega(c))=0$ and $\mu(\Lambda\setminus\omega(c))=0$, the second integral converges to 0. Moreover $M(\delta)$ stays bounded. Since $\varepsilon>0$ is arbitrary the desired inequality holds. □

3.3. **Proof of Theorem 1.2.** Let f be a topologically exact S-unimodal map with a non-recurrent flat critical point c of polynomial order. To prove Theorem 1.2(a), let $(Y, \mathcal{W}, R)$ be the inducing scheme in Proposition 2.3. Let $\varepsilon > 0$. For each $n \geq N_1$, take $J_n \in \mathcal{W}$ which is contained in the connected component of $\{R|_Y > n\}$ containing c and satisfies $n < R(J_n) \leq (1+\varepsilon)n$.

Let $\mu \in \mathcal{M}(f)$ be a post-critical measure. We show that μ is approximated in the weak* topology by measures supported on periodic orbits with arbitrarily small Lyapunov exponents. Since μ is a post-critical measure, there exists a sequence $\{m_i\}_{i\geq 0}$ of positive integers such that $m_i \nearrow \infty$ and $\delta_c^{m_i} \to \mu$ in the weak* topology as $i \to \infty$. For each $i \geq 0$ let $n_i \geq N_1$ denote the largest integer such that $R(J_{n_i}) \leq m_i$. Put $k_i = R(J_{n_i})$ and define x_i to be the fixed point of f^{k_i} in J_{n_i}. Since c is of polynomial order, we have

$$\lim_{x \to c} \frac{\log |D\ell(x)|}{\ell(x) \log |x-c|^{-1}} = 0. \tag{3.11}$$

The estimates in Lemma 3.1 and (3.11) together imply

$$\lim_{i\to\infty} \chi(\delta_{x_i}^{k_i}) = \lim_{i\to\infty} \frac{1}{k_i} \log |Df^{k_i}(x_i)| = 0.$$

Let $\phi\colon X \to \mathbb{R}$ be continuous. Fix $\delta > 0$ such that if $x, y \in X$ and $|x-y| \leq \delta$ then $|\phi(x)-\phi(y)| \leq \varepsilon$. Since f is topologically exact, there exists $N(\delta) \geq 1$ such that if $J \in \mathcal{W}$ and $R(J) > N(\delta)$ then $|f^n(J)| \leq \delta$ holds for any $n \in \{1, \ldots, R(J) - N(\delta)\}$. If $k_i > N(\delta)$, then $|S_{k_i}\phi(x_i) - S_{k_i}\phi(c)| \leq k_i\varepsilon + N(\delta)\sup|\phi|$. We have

$$n_i < k_i = R(J_{n_i}) \leq m_i < R(J_{n_i+1}) \leq (1+\varepsilon)(n_i+1).$$

Therefore $m_i - k_i \leq \varepsilon(n_i+1)$, and $|(1/k_i)S_{k_i}\phi(x_i) - (1/m_i)S_{m_i}\phi(c)| \leq 2\varepsilon$ holds for sufficiently large i, namely $|(1/k_i)S_{k_i}\phi(x_i) - (1/m_i)S_{m_i}\phi(c)| \to 0$ as $i \to \infty$. Since $\left|(1/m_i)S_{m_i}\phi(c) - \int \phi d\mu\right| \to 0$ it follows that $\left|(1/k_i)S_{k_i}\phi(x_i) - \int \phi d\mu\right| \to 0$ as required. This completes the proof of Theorem 1.2(a).

It is left to prove Theorem 1.2(b). Let $\mu \in \mathcal{M}(f)$ be a minimizer such that $\mu(\omega(c)) = 0$. From the argument in [5, Section 2] there is a sequence $\{\mu_k\}_{k\geq 1}$ of ergodic measures in $\mathcal{M}(f)$ which converges in the weak* topology to μ with $\lim_{k\to\infty} F(\mu_k) = 0$. By Lemma 3.2 and the upper semi-continuity of Lyapunov exponent, $\lim_{k\to\infty} \chi(\mu_k) = \chi(\mu)$ holds. We have

$$0 = \lim_{k\to\infty} F(\mu_k) \leq \limsup_{k\to\infty} h(\mu_k) - \lim_{k\to\infty} \chi(\mu_k) \leq F(\mu).$$

Hence $F(\mu) = 0$. By Dobbs' extension [9, Theorem 1.5] of Ledrappier's characterization of acips [13], μ is the acip. □

Appendix A. Minimizers for Collet-Eckmann maps

An S-unimodal map f with a non-flat critical point c satisfies *the Collet-Eckmann condition* [7] if

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n(f(c))| > 0.$$

The LDP holds for a topologically exact S-unimodal map satisfying the Collet-Eckmann condition, and the rate function is given as in (1.4), see [5]. The Collet-Eckmann condition implies the existence of a unique acip, see e.g., [8].

Theorem A. *Let f be a topologically exact S-unimodal map with non-flat critical point satisfying the Collet-Eckmann condition. The acip of f is the unique minimizer.*

Proof. Let $\mu_{\rm ac}$ denote the acip of f. By the result of Keller and Nowicki [11, Theorem 1.2], for a function $\phi\colon X \to \mathbb{R}$ of bounded variation with the positive limiting variance $\sigma_\phi^2 > 0$, for sufficiently small $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \left\{ x \in X\colon \left| \frac{1}{n} S_n\phi(x) - \int \phi d\mu_{\rm ac} \right| > \varepsilon \right\} \right| < 0.$$

Now, let $\mu \in \mathcal{M}(f) \setminus \{\mu_{\rm ac}\}$. Let $\phi\colon X \to \mathbb{R}$ be Lipschitz continuous with $\int \phi d\mu \neq \int \phi d\mu_{\rm ac}$. Then $\phi \neq \psi \circ f - \psi$ holds for any $\psi \in L^2(\mu_{\rm ac})$, and thus $\sigma_\phi^2 > 0$, see Liverani [14]. Put $\varepsilon = (1/2)|\int \phi d\mu - \int \phi d\mu_{\rm ac}|$. The set $\{\nu \in \mathcal{M}\colon |\int \phi d\nu - \int \phi d\mu_{\rm ac}| > \varepsilon\}$ is an open neighborhood of μ. The lower bound (1.1) gives

$$-I(\mu) \leq \liminf_{n\to\infty} \frac{1}{n} \log \left| \left\{ x \in X\colon \left| \frac{1}{n} S_n\phi(x) - \int \phi d\mu_{\rm ac} \right| > \varepsilon \right\} \right|.$$

The right-hand side is negative, and so $I(\mu) > 0$. □

Appendix B: LDP for intermittent interval maps

Consider the map $f_\alpha\colon [0,1] \to [0,1]$ given by $f_\alpha(x) = x + x^{1+\alpha}$ (mod 1) where $f_\alpha(0) = 0$, the value of f_α at its discontinuity is 0, $f_\alpha(1) = 1$ and $\alpha > 0$.

Theorem B. *The Large Deviation Principle holds for f_α with the rate function*

$$I(f_\alpha; \mu) = \begin{cases} \chi(\mu) - h(\mu) & \text{if } \mu \text{ is } f_\alpha\text{-invariant;} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let Y denote the domain of the branch of f_α not containing 0 and by $R|_Y$ the first return time to Y. From [16, Lemma 2.1] and the mean value theorem, there exists a constant $C > 0$ independent of $n \geq 1$ such that

$$\frac{|\{R|_Y = n+1\}|}{|\{R|_Y > n\}|} \geq |\{R|_Y = n+1\}| \geq C n^{-\frac{2(1+\alpha)}{\alpha}}.$$

Then the argument in Section 2.3 shows the LDP. Since the free energy is upper semi-continuous, there is no need for regularization. □

There exists an acip if and only if $\alpha < 1$, and it is unique. For $\alpha < 1$, $I(f_\alpha; \mu) = 0$ if and only if μ is a convex combination of δ_0 and the acip. For $\alpha \geq 1$, δ_0 is the unique minimizer.

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