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Homogenization of symmetric Dirichlet forms

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Abstract. We consider a homogenization problem for symmetric jump-diffusion processes by using the Mosco convergence and the two-scale convergence of the corresponding Dirichlet forms. Moreover, we show the weak convergence of the processes.

1. Introduction

Homogenization of elliptic operators with periodically oscillating coefficients (or diffusion processes) is formulated as follows:

$$(1.1)\qquad \begin{cases} -\mathrm{div}\Big(A\Big(\frac{\cdot}{\delta}\Big)\nabla u_\delta\Big)(x) = f(x), & x \in D, \\ u_\delta(x) = \quad 0, & x \in \partial D \end{cases}$$

where $\delta > 0$ is a small parameter used to represent the heterogeneities of the body D, a bounded open set of $\mathbb{R}^d$ and f is a given function or functional as source. $A(x) = (a_{ij}(x))$ is a $d \times d$-matrix-valued function defined on $\mathbb{R}^d$ so that $x \mapsto a_{ij}(x)$ is Y-periodic for each i, j in the sense that, for $k \in \mathbb{Z}^d,\ \ell = 1, 2, \ldots, d$,

$$a_{ij}(x + k\boldsymbol{e}_\ell) = a_{ij}(x) \quad \text{a.e. } x \in Y = (0,1)^d.$$

Here $\boldsymbol{e}_\ell$ is the ℓ-th directional unit vector of $\mathbb{R}^d$. The main homogenization procedure consists in finding possible limit(s) u_0 to the sequence $\big(u_\delta\big)_{\delta>0}$ and in identifying the problem(s) which u_0 solves.

On the other hand, suppose we are given a function $b(x)$ and a Lévy measure $\nu(dh)$ on $\mathbb{R}^d$, and consider a linear operator

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d} \Big(u(x+h) - u(x) - \nabla u(x) \cdot h\mathbf{1}_{B(1)}(h)\Big) b(h)\nu(dh)$$

for some functions u. Then the homogenization of jump-type processes is as follows: Under suitable conditions on $b(x)$ in addition to the periodicity, does the process associated with $\mathcal{L}$, after appropriate scalings $t \mapsto \varphi(\delta)t$ and $x \mapsto \phi(\delta)x$, converge (in the law sense) to some process as $\delta \to 0$?

In order to tackle these problems, various methods are considered such as G-convergence, H-convergence, two-scale convergence, energy method and so on (see *e.g.* [1, 3, 5, 21]). These are used to investigate the properties of $\big(u_\delta\big)$ and the limit u_0.

In this paper, we consider the problem for symmetric jump-diffusion processes via Dirichlet form theory and the two-scale convergence is used to show the convergence of

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Dirichlet forms corresponding to the jump-diffusion processes. As for the existence of solution(s), it is quite easily seen under the transience assumption. To be more specific, let (E, d) be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L^2(E; m)$. Denote by $\mathcal{C}$ a special standard core of $(\mathcal{E}, \mathcal{F})$, which is a subset of $C_0(E)$, the set of all continuous functions with compact support. Assume here that the diffusion (or energy) measures are absolute continuous with respect to m and the killing measure vanishes in the Beurling-Deny decomposition of $(\mathcal{E}, \mathcal{F})$ (see [14]): for $u, v \in \mathcal{F}$,

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^{(c)}(u,v) + \mathcal{E}^{(j)}(u,v) \\ &= \frac{1}{2}\int_E \Gamma(u,v)(x)m(dx) + \iint_{x\neq y} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)J(x,dy)m(dx)\end{aligned}$$

and $(\mathcal{E}, \mathcal{F})$ is assumed to be transient, that is, $(\mathcal{F}_e, \mathcal{E})$ is itself a Hilbert space, equivalently, there exists a function $g \in L^1(E; m)$ so that $g(x) > 0$ m-a.e. $x \in E$ and

$$\int_E |u(x)|g(x)m(dx) \le \sqrt{\mathcal{E}(u,u)}, \qquad u \in \mathcal{F}_e. \tag{1.2}$$

Here $\mathcal{F}_e$ is the extended Dirichlet space of $\mathcal{F}$. Moreover we assume that Γ sends the set $\mathcal{C} \times \mathcal{C}$ into $L^1(E; m)$ and the map $x \mapsto \int_{y \neq x} \big(1 \wedge d(x,y)^2\big) J(x, dy)$ belongs to $L^1_{\mathsf{loc}}(E; m)$.

In order to consider the problem, for any $\delta > 0$, suppose we are given a bilinear functional Γ_δ defined on $\mathcal{C} \times \mathcal{C}$ into $L^1(E; m)$ such that the normal contraction operates on Γ_δ (i.e., $\varphi(u) \in \mathcal{C}$ and $\Gamma_\delta(\varphi(u), \varphi(u)) \le \Gamma_\delta(u, u)$ holds m-a.e. for $u \in \mathcal{C}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $|\varphi(x) - \varphi(y)| \le |x - y|$ for $x, y \in \mathbb{R}$) and a positive measurable symmetric function $b_\delta(x, y) = b_\delta(y, x)$ defined on $E \times E$ so that, for some positive numbers α, β,

$$\begin{cases} \alpha\Gamma(u,u) \le \Gamma_\delta(u,u) \le \beta\Gamma(u,u) \quad m\text{-a.e.} \quad \text{for } \forall\, u \in \mathcal{C}, \\ \alpha \le b_\delta(x,y) \le \beta, \qquad m\otimes m\text{-a.e.} \end{cases} \tag{1.3}$$

Then the following quadratic forms on $L^2(E; m)$ produce regular symmetric Dirichlet forms on $L^2(X; m)$ having $\mathcal{C}$ as a common (standard) core for any $\delta > 0$:

$$\begin{aligned}\mathcal{E}^\delta(u,v) &:= \mathcal{E}^{(c),\delta}(u,v) + \mathcal{E}^{(j),\delta}(u,v) \\ &:= \frac{1}{2}\int_E \Gamma_\delta(u,v)dm + \iint_{x\neq y} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)b_\delta(x,y)J(x,dy)m(dx)\end{aligned} \tag{1.4}$$

for $u, v \in \mathcal{C}$. It follows from (1.3) that,

$$\alpha\mathcal{E}^{(k)}(u,u) \le \mathcal{E}^{(k),\delta}(u,u) \le \beta\mathcal{E}^{(k)}(u,u), \quad \text{for } \forall\, u \in \mathcal{C},\ k = c \text{ or } j,$$

accordingly, the domains of $\mathcal{E}^\delta$ are all equivalent and equal to $\mathcal{F}$, and their extended Dirichlet spaces are also equivalent and equal to $\mathcal{F}_e$.

Let us now denote by $(\mathcal{A}_\delta, \mathcal{D}(\mathcal{A}_\delta))$ the (L^2-)infinitesimal generator of $(\mathcal{E}^\delta, \mathcal{F})$ for each $\delta > 0$:

$$\mathcal{E}^\delta(u,v) = -(\mathcal{A}_\delta u, v), \qquad \text{for } u \in \mathcal{D}(\mathcal{A}_\delta),\ v \in \mathcal{F}.$$

A main problem in this paper is formulated as follows: *Is there a solution u_δ of the following equation for some function or functional f for each $\delta > 0$:*

$$-\mathcal{A}_\delta u_\delta = f, \tag{1.5}$$

and, then does the sequence of solutions $\{u_\delta\}_{\delta>0}$ have limit(s) u_0?

Indeed, we will give an affirmative answer to the question for a class of symmetric Dirichlet forms (see Section 4).

2. Existence of (weak) solutions

Since we have assumed that the reference Dirichlet form $(\mathcal{E}, \mathcal{F})$ is transient, that is, $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space, there exists a unique $u_\delta \in \mathcal{F}_e$ for each $f \in \mathcal{F}_e^*$, the dual space of $\mathcal{F}_e$ with respect to the Hilbert space $(\mathcal{F}_\delta, \mathcal{E})$, so that

$$\mathcal{E}^\delta(u_\delta, \varphi) = \langle f, \varphi \rangle, \quad \forall\, \varphi \in \mathcal{F}_e \tag{2.1}$$

by making use of the Lax-Milgram theorem to the bilinear continuous and coercive functional $\mathcal{E}^\delta$, where $\langle f, \varphi\rangle$ denotes the pairing of $(f, \varphi) \in \mathcal{F}_e^* \times \mathcal{F}_e$.

On the other hand, if f can be written as $f = h{\cdot}g$ for some bounded Borel function h, the 0-resolvent $G^\delta f\big(:= G_0^\delta f \in \mathcal{F}_e\big)$ of f with respect to the transient Dirichlet form $(\mathcal{E}^\delta, \mathcal{F})$ exists and satisfies that

$$\mathcal{E}^\delta(G^\delta f, v) = (f, v), \quad \forall\, v \in \mathcal{F}_e.$$

In this (weak form) sense, we also solve the equation (1.5) for such f: $u_\delta = G^\delta f$.

From (2.1) and the continuity of f, we see

$$\big|\mathcal{E}^\delta(u_\delta, \varphi)\big| = \big|\langle f, \varphi\rangle\big| \le \|f\|_{\mathcal{F}_e^*} \cdot \|\varphi\|_{\mathcal{E}}, \qquad \varphi \in \mathcal{F}_e.$$

Here $\|f\|_{\mathcal{F}_e^*}$ is the continuity constant of f as a functional in $(\mathcal{F}_e, \mathcal{E})$ and $\|\varphi\|_{\mathcal{E}} = \sqrt{\mathcal{E}(\varphi, \varphi)}$ for $\varphi \in \mathcal{F}_e$. Plugging $\varphi = u_\delta$ in the above inequality, we have

$$\alpha \mathcal{E}(u_\delta, u_\delta) \le \mathcal{E}^\delta(u_\delta, u_\delta) \le \|f\|_{\mathcal{F}_e^*} \sqrt{\mathcal{E}(u_\delta, u_\delta)}; \qquad \mathcal{E}(u_\delta, u_\delta) \le \frac{1}{\alpha^2}\|f\|_{\mathcal{F}_e^*}^2.$$

The inequality (1.2) also tells us that

$$\int_E |u_\delta(x)| g(x) m(dx) \le \sqrt{\mathcal{E}(u_\delta, u_\delta)} \le \frac{1}{\alpha}\|f\|_{\mathcal{F}_e^*}$$

and, therefore the sequence $\big(u_\delta\big)_{\delta>0}$ is uniformly bounded in $(\mathcal{F}_e, \mathcal{E})$ and in $L^1(E; gm)$. Since $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space and $L^1(E; gm)$ is a separable normed space, there exists a subsequence $\{u_{\delta_n}\} \subset \{u_\delta\}$ so that u_{δ_n} converges weakly in $(\mathcal{F}_e, \mathcal{E})$ and also weakly in $L^1(E; gm)$.

3. Dirichlet forms associated with symmetric jump-diffusions

In this section, we suppose there exist a symmetric nonnegative definite matrix-valued measurable function $A(x) = (a_{ij}(x))_{i,j}$ on $\mathbb{R}^d$ and a locally bounded function $\nu(h)$ on

$\mathbb{R}^d \setminus \{0\}$ so that

(A1) $\exists\, \alpha, \beta > 0$ s.t. $\alpha|\xi|^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \le \beta|\xi|^2, \quad x,\ \xi \in \mathbb{R}^d.$

(A2) $\nu(h) = \nu(-h) \ge 0$ for $h \neq 0$ and $\int_{h\neq 0} \big(1 \wedge |h|^2\big)\nu(h)dh < \infty.$

Suppose further that there exists a nonnegative measurable function $b(x,h)$ on $\mathbb{R}^d \times \mathbb{R}^d$ so that

(A3) $0 \le b(x,h) \le \beta, \quad (x,h) \in \mathbb{R}^d \times \mathbb{R}^d.$

Consider the following quadratic form on $L^2(\mathbb{R}^d)$:

$$
\begin{aligned}
\mathcal{E}(u,v) &:= \mathcal{E}^c(u,v) + \mathcal{E}^j(u,v) \\
&:= \frac{1}{2}\int_{\mathbb{R}^d} A(x)\nabla u(x)\cdot\nabla v(x)dx \\
&\qquad + \frac{1}{2}\iint_{x\neq y} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)b(x,y-x)\nu(y-x)dydx
\end{aligned}
$$

for $u, v \in C_0^\infty(\mathbb{R}^d)$, the set of all smooth functions on $\mathbb{R}^d$ with compact support. So we consider $\Gamma(u,v)(x) := A(x)\nabla u(x)\cdot\nabla v(x)$ for $u, v \in \mathcal{C} := C_0^\infty(\mathbb{R}^d)$ and $J(x,dy) := \nu(x-y)dy$ as reference data. It is known that $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is a closable, symmetric Markovian form on $L^2(\mathbb{R}^d)$ and the closure $(\mathcal{E},\mathcal{F})$ becomes a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$ (see [14]). Though the proof of the following lemma is shown similarly to that of Lemma 5.4 in [4] (see also [19, 11, 12]), we give it for the readers convenience.

Lemma 3.1. *Assume that* **(A1)-(A3)** *hold. Then*

$$\big\{u \in L^2(\mathbb{R}^d) :\ \mathcal{E}(u,u) < \infty\big\} = H^1(\mathbb{R}^d) = \mathcal{F},$$

where

$$H^1(\mathbb{R}^d) = \Big\{u \in L^2(\mathbb{R}^d) :\ \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^d),\ i = 1, 2, \ldots, d\Big\}.$$

Proof: Take $u \in L^2(\mathbb{R}^d)$ so that $\mathcal{E}(u,u) < \infty$. Since $\mathcal{E}^c(u,u) < \infty$ and $\mathcal{E}^c(u,u)$ is comparable to $\|\,|\nabla u|\,\|^2$ by **(A1)**, we see that $u \in H^1(\mathbb{R}^d)$. Suppose now that $u \in H^1(\mathbb{R}^d)$. According to **(A3)**,

$$
\begin{aligned}
\mathcal{E}^j(u,u) &\le \frac{\beta}{2}\iint_{x\neq y} \big(u(x)-u(y)\big)^2\nu(y-x)dydx \\
&= \frac{\beta}{2}\iint_{h\neq 0} \big(u(x+h)-u(x)\big)^2\nu(h)dxdh \ = \frac{\beta}{2}\int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2\varphi(\xi)d\xi,
\end{aligned}
$$

where $\hat{u}$ is the Fourier transform of u and $\varphi(\xi)$ is the Lévy exponent associated with the Lévy measure $\nu(h)dh$ (see [14]):

$$\varphi(\xi) = \int_{h\neq 0} \big(1-\cos\langle \xi, h\rangle\big)\nu(h)dh, \quad \xi \in \mathbb{R}^d.$$

By **(A2)**, we have

$$\varphi(\xi) = \int_{h\neq 0} \big(1-\cos\langle \xi, h\rangle\big)\nu(h)dh \le c_1 \int_{h\neq 0} \big(|\xi|^2|h|^2 \wedge 1\big)\nu(h)dh \le c_2\big(|\xi|^2+1\big).$$

Using Plancherel's theorem, we find that for $u \in C_0^\infty(\mathbb{R}^d)$,

$$\iint_{h\neq 0} \big(u(x+h)-u(x)\big)^2 \nu(h)dxdh$$

$$= \frac{\beta}{2}\int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \varphi(\xi)d\xi \le c_2 \int_{\mathbb{R}^d} \big|\hat{u}(\xi)\big|^2 \big(1+|\xi|^2\big)ds = c_3\big(\|u\|^2 + \|\,|\nabla u|\,\|^2\big). \tag{3.1}$$

Then considering a limiting argument, it follows that

$$\mathcal{E}^j(u,u) \le \frac{\beta}{2}\iint_{h\neq 0} \big(u(x+h)-u(x)\big)^2 \nu(h)dxdh \le c_4\big(\|u\|^2 + \|\,|\nabla u|\,\|^2\big)$$

for $u \in H^1(\mathbb{R}^d)$. Since $\mathcal{E}^c(u,u)$ is comparable to $\|\,|\nabla u|\,\|^2$, we have $\mathcal{E}(u,u) < \infty$ for $u \in H^1(\mathbb{R}^d)$.

On the other hand, for $u \in C_0^\infty(\mathbb{R}^d)$,

$$\|u\|^2 + \|\,|\nabla u|\,\|^2 \le c_5\big(\mathcal{E}^c(u,u)+\|u\|^2\big) \le c_5\big(\mathcal{E}(u,u)+\|u\|^2\big) \le c_6\big(\|u\|^2+\|\,|\nabla u|\,\|^2\big). \tag{3.2}$$

This means that the $H^1(\mathbb{R}^d)$-norm is comparable to $\big(\mathcal{E}(\cdot,\cdot)+\|\cdot\|^2\big)^{1/2}$ on $C_0^\infty(\mathbb{R}^d)$. So taking the closure of $C_0^\infty(\mathbb{R}^d)$ implies $H^1(\mathbb{R}^d) = \mathcal{F}$ and then the proof is complete. □

Let $\{G_\lambda : \lambda > 0\}$ be the L^2-resolvent associated with $(\mathcal{E}, \mathcal{F})$. Under an additional condition, we show that G_λ has a strong Feller property:

Proposition 3.1. *Assume that* **(A1)**-**(A3)** *hold. Let $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > d$. Then there exists a positive constant $C > 0$ depending on d, p, λ and α such that*

$$\|G_\lambda f\|_{L^\infty(\mathbb{R}^d)} \le C\big\{\|f\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}\big\}. \tag{3.3}$$

Further if

$$1 < \exists\, \gamma < d/(d-1) \quad \textsf{s.t.} \quad \int_{0<|h|<1} |h|^\gamma \nu(h)dh < \infty, \tag{3.4}$$

then $G_\lambda f$ is Hölder continuous on $\mathbb{R}^d$ for any $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $d < p \le \gamma/(\gamma-1)$, and

$$G_\lambda(C_0(\mathbb{R}^d)) \subset C_\infty(\mathbb{R}^d). \tag{3.5}$$

Here $C_\infty(\mathbb{R}^d)$ denotes the space of all continuous functions on $\mathbb{R}^d$ vanishing at infinity.

Note that (3.3) implies that there exists a kernel $G_\lambda(x,y)$ so that

$$G_\lambda f(x) = \int_{\mathbb{R}^d} G_\lambda(x,y)f(y)\,dy, \quad \text{a.e. } x \in \mathbb{R}^d \tag{3.6}$$

holds. Let $\{T_t : t > 0\}$ be the L^2-semigroup associated with $(\mathcal{E}, \mathcal{F})$, i.e., $G_\lambda f = \int_0^\infty e^{-\lambda t} T_t f\, dt$. By means of Theorem 4.2.4 of [14], there is a density function $p_t(x, y)$ so that

$$T_t f(x) = \int_{\mathbb{R}^d} p_t(x, y) f(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}^d,\ t > 0 \tag{3.7}$$

holds. If the assumption (3.4) is satisfied, (3.6) and (3.7) hold true for any $x \in \mathbb{R}^d$.

In [8], Chen and Kumagai considered a non-local operator associated with $(\mathcal{E}, \mathcal{F})$ and showed a sharp estimate on $p_t(x, y)$ under the assumption **(A1)** on $A(x)$ and some assumptions on $J(x, y)$ $(= b(x, y - x)\nu(y - x))$. Our assumption on the jump part does not necessarily imply that of [8] and so Proposition 3.1 does not follow from the results of [8] directly. Employing methods from elliptic differential equations (see *e.g.* [16, 25, 28] etc.), we show Proposition 3.1. The proof will be given in Section 5.

Let $\mathsf{M} = (X_t, \mathbb{P}_x)$ be the symmetric Hunt process on $\mathbb{R}^d$ associated with $(\mathcal{E}, \mathcal{F})$. Note that under the assumptions **(A1)-(A3)**, the conditions (M-1), (M-2), (M-3) and (1.1) of [22] are satisfied in our case. Therefore, by means of Theorem 1.1 of [22], we immediately obtain the following.

Proposition 3.2. *Assume that* **(A1)-(A3)** *hold. Then the process* M *is conservative, that is,*

$$\mathbb{P}_x(X_t \in \mathbb{R}^d) = 1, \quad \text{q.e. } x \in \mathbb{R}^d,\ t > 0. \tag{3.8}$$

If (3.4) *is satisfied, then* (3.8) *holds true for any* $x \in \mathbb{R}^d$.

Note that, similar results to Propositions 3.1 and 3.2 have been obtained in [26] for a class of symmetric jump processes.

Let $Y = (0, 1)^d$ be the d-dimensional cube. Suppose henceforth that the function $A(x) = (a_{ij}(x))$ is Y*-periodic* in the sense that

$$A(x + k\boldsymbol{e}_i) = A(x), \quad \text{a.e. } x \in \mathbb{R}^d,\ k \in \mathbb{Z}, i = 1, 2, \ldots, d,$$

where $\boldsymbol{e}_i = {}^{\mathsf{T}}(0, \cdots, 0, \overset{i}{\check{1}}, 0, \ldots, 0) \in \mathbb{R}^d$ is the i-directional unit vector and the function $b(x, h)$ is also Y*-periodic* in h for each x.

For $\delta > 0$, set $A^\delta(x) := A(x/\delta)$ and $b^\delta(x, h) := b(x, h/\delta)$ and consider the quadratic forms $\mathcal{E}^{c,\delta}$ and $\mathcal{E}^{j,\delta}$ corresponding to $A^\delta(x)$ and $b^\delta(x, h)$ in the definitions of $\mathcal{E}^c$ and $\mathcal{E}^j$ in place of $A(x)$ and $b(x, h)$ respectively. Then $(\mathcal{E}^\delta, H^1(\mathbb{R}^d))$ is also a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$ for each $\delta > 0$, where

$$\mathcal{E}^\delta(u, v) = \mathcal{E}^{c,\delta}(u, v) + \mathcal{E}^{j,\delta}(u, v), \quad u, v \in H^1(\mathbb{R}^d).$$

Remark 3.1. (i) *The pair* $(\mathcal{E}^{c,\delta}, H^1(\mathbb{R}^d))$ *is itself a symmetric strongly local regular Dirichlet form on* $L^2(\mathbb{R}^d)$ *for each* $\delta > 0$. *Let us denote a symmetric diffusion process* $\tilde{\mathsf{M}} = (\tilde{X}(t), \mathbb{P}_x)$ *associated with* $(\mathcal{E}^c, H^1(\mathbb{R}^d))$. *For any* $\delta > 0$, *set*

$$X_t^{(\delta)} := X^{(\delta)}(t) := \delta \tilde{X}(t/\delta^2), \quad t > 0.$$

Then $(X_t^{(\delta)}, \mathbb{P}_x)$ *is also a symmetric diffusion process on* $\mathbb{R}^d$ *and denote by* $(\mathcal{E}^{c,(\delta)}, \mathcal{F}^{c,(\delta)})$ *the associated Dirichlet form on* $L^2(\mathbb{R}^d)$. *The* C_0*-semigroup* $\{T_t^{(\delta)}\}$ *of* $(\mathcal{E}^{c,(\delta)}, \mathcal{F}^{c,(\delta)})$ *is given by*

$$\begin{aligned} T_t^{(\delta)}u(x) &= \mathbb{E}\big[u\big(X^{(\delta)}(t)\big)\big|X^{(\delta)}(0) = x\big] = \mathbb{E}\Big[u\big(\delta\tilde{X}(t/\delta^2)\big)\big|\tilde{X}(0) = x/\delta\Big] \\ &= \tilde{T}_{t/\delta^2}u_\delta(x/\delta) \end{aligned}$$

for $x \in \mathbb{R}^d$. *Here* $u_\delta(x) := u(\delta x),\ x \in \mathbb{R}^d,\ \delta > 0$ *for a function* $u : \mathbb{R}^d \to \mathbb{R}$. *Note here that the Dirichlet form* $(\mathcal{E}^{c,(\delta)}, \mathcal{F}^{c,(\delta)})$ *can be characterized as*

$$\mathcal{E}^{c,(\delta)}(u, v) = \lim_{t\to 0} \frac{1}{t}\big(u - T_t^{(\delta)}u, v\big).$$

So, for $t > 0$,

$$\begin{aligned} \frac{1}{t}\big(u - T_t^{(\delta)}u, v\big) &= \frac{1}{t}\int_{\mathbb{R}^d} \Big(u(x) - T_t^{(\delta)}u(x)\Big)v(x)dx \\ &= \frac{1}{t}\int_{\mathbb{R}^d} \Big(u_\delta(x/\delta) - \tilde{T}_{t/\delta^2}u_\delta(x/\delta)\Big)v_\delta(x/\delta)dx \\ &= \frac{1}{\delta^2}\cdot\frac{1}{s}\int_{\mathbb{R}^d} \Big(u_\delta(y) - \tilde{T}_s u_\delta(y)\Big)v_\delta(y)\delta^d\, dy, \end{aligned}$$

where we changed of variables $x \mapsto \delta y$ *and* $t \mapsto \delta^2 s$ *in the last equality. Thus letting* $t \to 0$, *hence,* $s \to 0$, *we find*

$$\begin{aligned} \mathcal{E}^{c,(\delta)}(u, v) = \lim_{t\downarrow 0}\frac{1}{t}\big(u - T_t^{(\delta)}u, v\big) \;&=\; \delta^{d-2}\lim_{s\downarrow 0}\frac{1}{s}\int_{\mathbb{R}^d}\Big(u_\delta(y) - \tilde{T}_s u_\delta(y)\Big)v_\delta(y)\,dy \\ =\; \delta^{d-2}\mathcal{E}^c(u_\delta, v_\delta) \;&=\; \frac{\delta^{d-2}}{2}\int_{\mathbb{R}^d} A(x)\nabla u_\delta(x)\cdot\nabla v_\delta(x)dx \\ =\; \frac{1}{2}\int_{\mathbb{R}^d} A(x/\delta)\nabla u(x)\cdot\nabla v(x)dx \;&=\; \mathcal{E}^{c,\delta}(u, v). \end{aligned}$$

This means that the form $\mathcal{E}^{c,(\delta)}$ *is obtained by scaling* $t \mapsto t/\delta^2$ *and* $x \mapsto \delta x$ *from the diffusion process associated with* $\mathcal{E}^c$.

(ii) *Suppose now that a function* $b(x, h)$ *is given by* $b(x, h) = b(h)$ *for a continuous* Y*-periodic function* $b : \mathbb{R}^d \to \mathbb{R}$ *with* $0 < \alpha \le b(h) = b(-h) \le \beta$ *and* $\nu(x) = |x|^{-d-\kappa}, x \in \mathbb{R}^d \setminus \{0\}$ *for some* $0 < \kappa < 2$. *As in the Remark of* [27], *we see that the form* $\mathcal{E}^{j,\delta}$ *is obtained by scaling* $t \mapsto t/\delta^\kappa$ *and* $x \mapsto \delta x$ *from the process associated with* $\mathcal{E}^j$.

In both cases, $a_{ij}^\delta(x) := a_{ij}(x/\delta)$ *and* $b^\delta(h) := b(h/\delta)$ *converge to some functions* $\tilde{a}_{ij}$ *and* $\tilde{b}$ *weak-star in* $L^\infty(\mathbb{R}^d)$ *when* $\delta \to 0$ *respectively, which is much weaker than the* $L^1_{\mathsf{loc}}(\mathbb{R}^d)$*-convergence. Note that the corresponding forms* $\mathcal{E}^{c,\delta}$ *and* $\mathcal{E}^{j,\delta}$ *are obtained by different scalings. So we might think that we could not expect the uniqueness of limits of solutions* $\{u_\delta\}$ *in* (1.5) *nor identify the forms (or functionals) for which the limits of solutions satisfy, from the forms* $\{\mathcal{E}^{c,\delta} + \mathcal{E}^{j,\delta}\}$ *just assuming the weak-star convergence of* a_{ij}^δ *and* b^δ, *for example, at first sight.*

Let $\mathsf{M}^\delta = (X^\delta(t),\ \mathbb{P}_x^\delta)$ be the symmetric Hunt processes on $\mathbb{R}^d$ associated with $(\mathcal{E}^\delta, H^1(\mathbb{R}^d))$ for $\delta > 0$. Under the assumptions **(A1)-(A3)**, Propositions 3.1 and 3.2 hold for each $(\mathcal{E}^\delta, H^1(\mathbb{R}^d))$ or M^δ, and $X^\delta(\cdot) \in \mathbb{D}_{\mathbb{R}^d_\partial}$, where $\mathbb{R}^d_\partial$ is the one-point compactification of $\mathbb{R}^d$ and

$$\mathbb{D}_{\mathbb{R}^d_\partial} := \Big\{ f : [0,\infty) \to \mathbb{R}^d_\partial \,\Big|\, f \text{ is right continuous having left limits} \Big\}.$$

Proposition 3.3. *Assume the assumption* **(A1)-(A3)** *hold and further assume* (3.4). *Then the law* $\{\mathbb{P}_x^\delta\}_{\delta>0}$ *of* $\{X^\delta\}_{\delta>0}$ *are tight in the space* $\mathbb{D}_{\mathbb{R}^d_\partial}[0,T]$ *for any* $T > 0$, *where*

$$\mathbb{D}_{\mathbb{R}^d_\partial}[0,T] := \Big\{ f : [0,T] \to \mathbb{R}^d_\partial \,\Big|\, f \textit{ is right continuous having left limits} \Big\}.$$

Proof. We follow an argument similar to that of Proposition 3.4 of [7]. Given $t > 0$, the time-reversal operator r_t on the path space Ω is defined by

$$r_t(\omega)(s) := \begin{cases} \omega((t-s)-), & \text{if } \ 0 \le s \le t, \\ \omega(0), & \text{if } \ s \ge t. \end{cases}$$

Here, for $r > 0$, $\omega(r-) = \lim_{s\uparrow r} \omega(s)$ is the left limit at r and we use the convention that $\omega(0-) = \omega(0)$. Since M^δ is symmetric and conservative, $\mathbb{P}_m^\delta$ is invariant under the time-reversal operator r_t (see p.943 of [6]).

Let $u \in C_0^\infty(\mathbb{R}^d)$. Then, by means of Theorem 5.2.2 of [14] and Lemma 3.5 of [6], there is a martingale $M^{\delta,u}$ such that

$$u(X_t^\delta) - u(X_0^\delta) = M_t^{\delta,u} - \frac{1}{2}\left\{ M_{T-}^{\delta,u}(r_T) - M_{(T-t)-}^{\delta,u}(r_T) \right\},$$

for $0 \le t \le T$. $M^{\delta,u}$ is decomposed as $M^{\delta,u} = M^{\delta,u,c} + M^{\delta,u,j}$, where $M^{\delta,u,c}$ is the continuous part and $M^{\delta,u,j}$ is the purely discontinuous part of $M^{\delta,u}$, respectively (see (5.3.2) of [14]). Further for the quadratic variations $\langle M^{\delta,u,c}\rangle$ and $\langle M^{\delta,u,j}\rangle$, we have

$$\begin{aligned} \langle M^{\delta,u,c}\rangle_t &= \int_0^t A^\delta(X_s^\delta)\nabla u(X_s^\delta)\cdot\nabla u(X_s^\delta)\,ds, \\ \langle M^{\delta,u,j}\rangle_t &= \int_0^t \int_{\mathbb{R}^d} \left\{u(X_s^\delta) - u(y)\right\}^2 b^\delta(X_s^\delta, y - X_s^\delta)\nu(y - X_s^\delta)\,dy\,ds, \end{aligned}$$

(cf. (5.2.46), (5.3.9) of [14]). By the assumptions **(A1)-(A3)**,

$$\begin{aligned} &\langle M^{\delta,u,c}\rangle_t - \langle M^{\delta,u,c}\rangle_s \le \beta \||\nabla u|\|_\infty^2 (t-s), \\ &\langle M^{\delta,u,j}\rangle_t - \langle M^{\delta,u,j}\rangle_s \le \beta\left(4\|u\|_\infty^2 + \||\nabla u|\|_\infty^2\right)\int_{h\neq 0}(1\wedge|h|^2)\nu(h)\,dh\cdot(t-s), \end{aligned}$$

for $0 \le s < t \le T$. Since $\langle M^{\delta,u,c}, M^{\delta,u,j}\rangle = 0$, we get

$$\langle M^{\delta,u}\rangle_t - \langle M^{\delta,u}\rangle_s \le c(t-s),$$

for any $\delta > 0$, where c is a positive constant independent of $\delta > 0$. Therefore, by Proposition VI 3.26 of [20], $\{\langle M^{\delta,u}\rangle_t\}$ is C-tight, that is, $\{\langle M^{\delta,u}\rangle_t\}$ is tight in $\mathbb{D}_{\mathbb{R}}[0,T]$ equipped with the Skorohod topology and all limit points of $\{\langle M^{\delta,u}\rangle_t\}$ are laws of continuous processes. Then, by Theorem VI 4.13 of [20], the laws of $\{M^{\delta,u}\}$ is tight in $\mathbb{D}_{\mathbb{R}}[0,T]$. Since $\mathbb{P}^\delta_m$ is invariant under the time-reversal operator r_T when restricted to the time interval $[0,T]$, $\{u(X^\delta)\}$ is tight in $\mathbb{D}_{\mathbb{R}}[0,T]$. Thus, by virtue of Corollary 3.9.3 of [10], the laws $\{\mathbb{P}^\delta_m\}$ of $\{\mathsf{M}^\delta\}$ are tight in $\mathbb{D}_{\mathbb{R}^d_\partial}[0,T]$. Under the assumption (3.4), the law $\mathbb{P}^\delta_x$ is defined for every $x\in\mathbb{R}^d$ and hence the laws $\{\mathbb{P}^\delta_x\}$ of $\{\mathsf{M}^\delta\}$ are tight in $\mathbb{D}_{\mathbb{R}^d_\partial}[0,T]$. □

REMARK 3.2. *Following the same argument as in Section 3 of* [27] *we can directly check the tightness conditions of Theorem VI 3.21 of* [20] *for a class of symmetric jump processes treated in* [27], *and we obtain Proposition 3.3 corresponding to such jump processes with suitable scaling.*

4. Convergence of Dirichlet forms

In this section, we give an example of a sequence of Dirichlet forms for which it converges to a Dirichlet forms in the sense of Mosco as homogenization. To this end, let $A(x)=(a_{ij}(x))$ and $\nu(h)$ be the functions defined in the previous section, that is, $A(x)$ is a symmetric matrix-valued function on $\mathbb{R}^d$ that is Y-periodic and satisfies **(A1)** and $\nu(h)$ is a locally bounded function on $\mathbb{R}^d\setminus\{0\}$ satisfying **(A2)**. Further let $b(x,h)$ be a function given by $b(x,h)=b_1(x,h)+b_2(x,h)$ where $b_i(x,h)$ $(i=1,2)$ are nonnegative functions on $\mathbb{R}^d\times\mathbb{R}^d$, $b_1(x,h)$ is Y-periodic in h for each x and $b_2(x,h)$ is $Y\times Y$-periodic in (x,h), and $b(x,h)$ satisfies **(A3)**. Here a function $b(x,h)$ defined on $\mathbb{R}^d\times\mathbb{R}^d$ is said to be *$Y\times Y$-periodic* if the following condition is satisfied:

$$b(x+k\boldsymbol{e}_i,h+\ell\boldsymbol{e}_j)=b(x,h),\quad \text{a.e. } (x,h)\in\mathbb{R}^d\times\mathbb{R}^d,\ k,\ell\in\mathbb{Z},\ i,j=1,2,\dots,d.$$

Take a bounded domain D of $\mathbb{R}^d$ and put $A^\delta(x):=A(x/\delta)$ and $b^\delta(x,h)=b_1(x,h/\delta)+b_2(x/\delta,h/\delta)$ for $\delta>0$. Then consider the following quadratic form on $L^2(D):=L^2(D;dx)$: for $u,v\in C_0^\infty(D)$,

$$\begin{aligned}\mathcal{E}^\delta(u,v)&:=\mathcal{E}^{c,\delta}(u,v)+\mathcal{E}^{j,\delta}(u,v)\\ &:=\frac12\int_D A^\delta(x)\nabla u(x)\cdot\nabla v(x)dx\\ &\quad+\frac12\iint_{\substack{D\times D\\ x\neq y}}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)b^\delta(x,y-x)\nu(y-x)dxdy.\end{aligned}\tag{4.1}$$

Then by the conditions **(A1)**-**(A3)** and the inequalities (3.1) and (3.2) for the functions in $C_0^\infty(D)$ which are also considered as functions in $C_0^\infty(\mathbb{R}^d)$, we see

$$\begin{aligned}\frac{\alpha}{2}\int_D|\nabla u(x)|^2dx\le\mathcal{E}^\delta(u,u)&\le\frac{\beta}{2}\int_D|\nabla u(x)|^2dx+\frac{\beta}{2}\iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ x\neq y}}\big(u(x)-u(y)\big)^2\nu(y-x)dxdy\\ &\le\frac{\beta}{2}\int_D|\nabla u(x)|^2dx+c_3\Big(\int_{\mathbb{R}^d}u(x)^2dx+\int_{\mathbb{R}^d}|\nabla u(x)|^2dx\Big)\\ &=\Big(\frac{\beta}{2}+c_3\Big)\int_D|\nabla u(x)|^2dx+c_3\int_D u(x)^2dx,\end{aligned}$$

that is,

$$\mathcal{E}^\delta(u,u) + \|u\|^2_{L^2(D)} \approx \||\nabla u|\|^2_{L^2(D)} + \|u\|^2_{L^2(D)}, \quad u \in C_0^\infty(D) \tag{4.2}$$

So, taking the closure of $C_0^\infty(D)$, we find that $(\mathcal{E}^\delta, H_0^1(D))$ are regular transient symmetric Dirichlet forms on $L^2(D)$ for $\delta > 0$. Moreover $C_0^\infty(D)$ is a (common) core of $(\mathcal{E}^\delta, H_0^1(D))$ for $\delta > 0$. Though the following lemma is known and is also shown in [27, Lemma 2], we give it for the readers' convenience.

Lemma 4.1. *The sequence of functions* $\{b^\delta(x,h)\}_{\delta>0}$ *is weak-star converging to* $\bar{b}(x) := \int_Y b_1(x,h)dh + \iint_{Y\times Y} b_2(z,h)dzdh$ *in* $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$*, that is, for any* $\varphi \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$*, it follows that*

$$\lim_{\delta\to 0} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(b^\delta(x,h) - \bar{b}(x)\big)\varphi(x,h)dxdh = 0. \tag{4.3}$$

Proof: For simplicity, we show this lemma for the case $d = 1$ and the sequence $\delta_n = 1/n,\ n \in \mathbb{N}$. Since $C_0^1(\mathbb{R}\times\mathbb{R})$ is dense in $L^1(\mathbb{R}\times\mathbb{R})$, we are enough to show (4.3) for $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R})$. Let $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R})$. Then we see

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}} b_1(x,h/\delta_n)\varphi(x,h)dxdh \ = \int_{\mathbb{R}}\int_{\mathbb{R}} b_1(x,nh)\varphi(x,h)dxdh \\
&= \int_{\mathbb{R}} \frac{1}{n}\int_{\mathbb{R}} b_1(x,h)\varphi\Big(x,\frac{h}{n}\Big)dxdh = \int_{\mathbb{R}}\Big(\frac{1}{n}\sum_{\ell=-\infty}^{\infty}\int_\ell^{\ell+1} b_1(x,h)\varphi\Big(x,\frac{h}{n}\Big)dh\Big)dx \\
&= \int_{\mathbb{R}}\Big(\frac{1}{n}\sum_{\ell=-\infty}^{\infty}\int_0^1 b_1(x,h+\ell)\varphi\Big(x,\frac{h+\ell}{n}\Big)dh\Big)dx \\
&= \int_{\mathbb{R}}\Big(\frac{1}{n}\sum_{\ell=-\infty}^{\infty}\int_0^1 b_1(x,h)\varphi\Big(x,\frac{h+\ell}{n}\Big)dy\Big)dx \\
&= \int_{\mathbb{R}}\Big\{\int_0^1 b_1(x,h)\Big(\frac{1}{n}\sum_{\ell=-\infty}^{\infty}\varphi\Big(x,\frac{h+\ell}{n}\Big)dh\Big\}dx \\
&\to \int_{\mathbb{R}}\Big\{\int_0^1 b_1(x,h)\Big(\int_{\mathbb{R}}\varphi(x,z)dz\Big)dh\Big\}dx \quad \text{as } n\to\infty.
\end{aligned}$$

In the above calculations, we used the change of variables in the second equality and divided the integral into infinite sums over $\mathbb{R} = \bigcup_{\ell=-\infty}^{\infty}[\ell,\ell+1)$ in the third and, used the periodicity of $b(x,h)$ with respect to the h variable after shifting the variable $x \mapsto x-\ell$ in the fourth.

Noting that the infinite sum is a finite sum since φ has compact support and then we can change the sum and the integral at the sixth equality. According to the fact that $h \mapsto \varphi(x,h)$ is continuous with compact support, it is easy to see that, for any $x \in \mathbb{R}$,

$$\frac{1}{n}\sum_{\ell=-\infty}^{\infty}\varphi\Big(x,\frac{h+\ell}{n}\Big) \quad\longrightarrow\quad \int_{\mathbb{R}}\varphi(x,z)dz \quad \textit{uniformly in } x \quad \text{as } n\to\infty.$$

In the same way as above,

$$\iint_{\mathbb{R}\times\mathbb{R}} b_2(x/\delta_n, h/\delta_n)\varphi(x,h)dxdh = \iint_{(0,1)\times(0,1)} b_2(x,h)\Big\{\frac{1}{n^2}\sum_{k,\ell=-\infty}^{\infty}\varphi\Big(\frac{x+k}{n},\frac{h+\ell}{n}\Big)\Big\}dxdh$$

$$\to \iint_{(0,1)\times(0,1)} b_2(x,h)dxdh\Big(\iint_{\mathbb{R}\times\mathbb{R}}\varphi(\xi,\eta)d\xi d\eta\Big) \quad \text{as } n\to\infty.$$

Thus the proof is complete. □

Before considering the convergence, we prepare some notations and introduce some notions in the subsequent subsections.

4.1. Periodic Sobolev spaces

Recall that $Y=(0,1)^d$ is the open unit cube in $\mathbb{R}^d$. Define a periodic Sobolev space $H_\#(Y)$ as the closure of $C_\#^\infty(Y)$, the restrictions of smooth functions defined on $\mathbb{R}^d$ which are Y-periodic, with respect to the $H^1(Y)$-norm, and then it becomes a Hilbert space with the same $H^1(Y)$-inner product. Define also $C_\#(Y)$ the space of all continuous functions defined on $\mathbb{R}^d$ which are Y-periodic. Then the following result is known:

Lemma 4.2. ([9, Proposition 3.49]) *If $u\in H_\#(Y)$, then the trace of u on the "opposite faces of Y" are equal.*

We also define the quotient space $\mathcal{H}_\#(Y) := H_\#(Y)/\mathbb{R}$ of $H_\#(Y)$ relative to the following binomial relation "$\sim$": for $u,v\in H_\#(Y)$,

$$u\sim v \quad\Longleftrightarrow\quad u-v= \text{ constant a.e. on } Y.$$

In other words, the space $\mathcal{H}_\#(Y)$ is the collections of all equivalent classes $\tilde{u}$ of functions u in $H_\#(Y)$:

$$\tilde{u} := \big\{v\in H_\#(Y) : u\sim v\big\}, \quad u\in H_\#(Y).$$

Lemma 4.3. ([9, Proposition 3.52]) *The space $\mathcal{H}_\#(Y)$ is a Hilbert space under the following inner product:*

$$\langle\tilde{u},\tilde{v}\rangle := \int_Y \nabla f(x)\cdot\nabla g(x)dx, \quad f\in\tilde{u},\ g\in\tilde{v},\ \tilde{u},\tilde{v}\in\mathcal{H}_\#(Y).$$

Moreover the dual space $\mathcal{H}_\#^*(Y)$ of $\mathcal{H}_\#(Y)$ is identified with the set

$$\Big\{\ell\in H_\#^*(Y):\ \ell(c)=0,\ c\in\mathbb{R}\Big\},$$

where

$$\langle\ell,\tilde{u}\rangle_{\mathcal{H}_\#^*(Y),\,\mathcal{H}_\#(Y)} := \langle\ell,f\rangle_{H_\#^*(Y),\,H_\#(Y)}, \quad \text{for } f\in\tilde{u}.$$

4.2. Two-scale convergence

We introduce a *two-scale convergence* in this subsection. For this, we follow the notation in [21]. Let D be again a bounded domain of $\mathbb{R}^d$. By definition, $L^p(D;C_\#(Y))$

for $1 \le p < \infty$ is the set of functions $f : D \to C_\#(Y)$ which is measurable and satisfies $\int_D \|f(x)\|^p_{C_\#(Y)} dx < \infty$, where $\|f(x)\|_{C_\#(Y)} = \sup_{y\in Y} |f(x)(y)|$ for $x \in D$. Each function $f \in L^p(D; C_\#(Y))$ is identified with a function $f : D \times \mathbb{R}^d \to \mathbb{R}$ by considering $f(x)(y)$ as $f(x, y)$. Denote by $\mathcal{D}(D; C_\#(Y))$ the set of all functions $\phi(x, y)$ defined on $D \times Y$ so that (i) $x \mapsto \phi(x, y)$ is a smooth function compactly supported in D for each $y \in Y$ and (ii) $y \mapsto \phi(x, y)$ belongs to $C_\#(Y)$ for each $x \in D$. A function ϕ in $\mathcal{D}(D; C_\#(Y))$ is called *admissible*.

The following proposition is essential for defining the *two-scale convergence*:

Proposition 4.1. (see *e.g.* [9, Lemma 9.1] or [21, Theorem 2]) *Let $f \in L^p(D; C_\#(Y))$. Then for each $\delta > 0$, the map $x \mapsto f(x, x/\delta)$ is a measurable function on D such that*

$$\Big\| f\Big(\cdot, \frac{\cdot}{\delta}\Big)\Big\|_{L^p(D)} \le \|f\|_{L^p(D;C_\#(Y))} := \Big(\int_D \Big(\sup_{y\in Y} |f(x, y)|\Big)^p dx\Big)^{1/p} \tag{4.4}$$

and

$$f\Big(\cdot, \frac{\cdot}{\delta}\Big) \quad \textit{converges to} \quad \int_Y f(\cdot, y)dy \quad \textit{weakly in } L^p(D) \tag{4.5}$$

as $\delta \to 0$. In particular, if $f \in L^2(D; C_\#(Y))$,

$$\lim_{\delta\to 0} \int_D \Big| f\Big(x, \frac{x}{\delta}\Big)\Big|^2 dx = \int_D \Big(\int_Y |f(x, y)|^2 dy\Big) dx. \tag{4.6}$$

Let $\{\delta_n\}$ be any decreasing sequence with $\delta_n \to 0$ $(n \to \infty)$ and fix it.

Definition 4.1. *A sequence $\{u_n\}$ of $L^p(D)$ is said to two-scale converge to $u \in L^p(D \times Y)$ if*

$$\lim_{n\to\infty} \int_D u_n(x)\phi\Big(x, \frac{x}{\delta_n}\Big) dx = \int_D \Big(\int_Y u(x, y)\phi(x, y)dy\Big) dx$$

for any admissible function ϕ.

The following theorem is important to show the Γ-convergence in our setting. Recall that D is a bounded open set of $\mathbb{R}^d$.

Theorem 4.1. (see *e.g.* [9, Theorem 9.8] or [1, Theorem 2.2]) *Let $\{u_n\}$ be a bounded sequence of $H^1_0(D)$. Then there exist a subsequence $\{n_k\}$, functions u_0 in $H^1_0(D)$ and $v = v(x, y)$ in $L^2(D; \mathcal{H}_\#(Y))$ such that* (i) *u_{n_k} converges to u_0 weakly in $H^1_0(D)$;* (ii) *u_{n_k} converges to u_0 strongly in $L^2(D)$;* (iii) *u_{n_k} two-scale converges to u_0;* (iv) *∇u_{n_k} two-scale converges to $\nabla_x u_0(x) + \nabla_y v(x, y)$. Here u_0 is identified as a function in $L^2(D; C_\#(Y))$ via $u_0(x, y) := u_0(x)$.*

Proposition 4.2. *For each $i = 1, 2, \dots, d$, there exists an $\omega_i \in H_\#(Y)$ so that*

$$\int_Y A(y)\nabla\omega_i(y) \cdot \nabla\varphi(y)dy = -\int_Y A(y)\boldsymbol{e}_i \cdot \nabla\varphi(y)dy, \quad \text{for } \varphi \in H_\#(Y) \tag{4.7}$$

and such ω_i is unique up to an additive constant.

Proof: This is shown by using the Lax-Milgram theorem. In fact, set

$$\ell_i(v) := -\int_Y A(y)\boldsymbol{e}_i \cdot \nabla v(y)dy, \quad v \in H_\#(Y), \quad i = 1, 2, \ldots, d.$$

Then we find that $\ell_i(v) = \ell_i(v')$ if v and v' are in $H_\#(Y)$ with $v - v' =$ constant a.e. on Y. In other words, ℓ_i is a linear functional on $\mathcal{H}_\#(Y)$, not just on $H_\#(Y)$. According to **(A1)**, we see

$$\big|\ell_i(v)\big| \le \beta \int_Y \nabla v(y) \cdot \nabla v(y)dy = \beta\langle \tilde{v}, \tilde{v}\rangle, \quad v \in H_\#(Y) \text{ (or } \tilde{v} \in \mathcal{H}_\#(Y)).$$

That is, ℓ_i is continuous on $H_\#(Y)$ or $\mathcal{H}_\#(Y)$. On the other hand, it is easily seen that

$$\tilde{\mathcal{E}}^c(\tilde{u}, \tilde{v}) := \int_Y A(y)\nabla u(y) \cdot \nabla v(y)dy, \quad u, v \in H_\#(Y)$$

defines a coercive bilinear form on $\mathcal{H}_\#(Y)$ which is also continuous. Thus by the Lax-Milgram theorem, there exists an $\omega_i \in H_\#(Y)$ so that

$$\tilde{\mathcal{E}}^c(\tilde{\omega}_i, \tilde{v}) = \ell_i(v), \quad v \in H_\#(Y).$$

Because of the definition of $\mathcal{H}_\#(Y)$, ω_i is unique up to an additive constant. Thus

$$\tilde{\mathcal{E}}^c(\tilde{\omega}_i, \tilde{\varphi}) = \int_Y A(y)\nabla\omega_i(y) \cdot \nabla\varphi(y)dy = -\int_Y A(y)\boldsymbol{e}_i \cdot \nabla\varphi(y)dy = \ell_i(\varphi), \quad \varphi \in H_\#(Y).$$

□

We define the so-called *homogenized effective matrix* $A_{\mathsf{eff}} = (a^{\mathsf{eff}}_{ij})$ by

$$a^{\mathsf{eff}}_{ij} = \int_Y \Big(a_{ij}(y) + \sum_{k=1}^d a_{ik}(y)\frac{\partial\omega_j}{\partial y_k}(y)\Big)dy, \quad i, j = 1, 2, \ldots, d. \tag{4.8}$$

Plugging $\varphi = \omega_j \in H_\#(Y)$ as a test function in (4.7), we find

$$\int_Y A(y)\Big(\boldsymbol{e}_i + \nabla\omega_i(y)\Big) \cdot \nabla\omega_j(y)dy = 0 \tag{4.9}$$

and

$$\begin{aligned} a^{\mathsf{eff}}_{ij} &= \int_Y \Big(a_{ij}(y) + \sum_{k=1}^d a_{ik}(y)\frac{\partial\omega_j}{\partial y_k}(y)\Big)dy \\ &= \int_Y A(y)\big(\mathbf{e}_j + \nabla\omega_j(y)\big) \cdot \boldsymbol{e}_i\, dy \\ &= \int_Y A(y)\big(\mathbf{e}_j + \nabla\omega_j(y)\big) \cdot \big(\boldsymbol{e}_i + \nabla\omega_i(y)\big)\, dy \end{aligned}$$

for $i, j = 1, 2, \ldots, d$, and this implies that the effective matrix A_{eff} is symmetric and satisfies **(A1)** for some positive numbers α_0 and β_0 (see *e.g.* [2, §1] or [9, Chapter 6]).

Moreover, we see that

$$\int_Y A(y)\big(\xi + \nabla\phi_\xi(y)\big)\cdot\nabla\varphi(y)dy = 0$$

holds for any $\varphi \in H_\#(Y)$ and $\xi \in \mathbb{R}^d$, where $\phi_\xi := \sum_{i=1}^d \xi_i\omega_i \in H_\#(Y)$ for $\xi \in \mathbb{R}^d$.

Since $A(y)$ is symmetric for each $y \in Y$, we find that, for any $\xi \in \mathbb{R}^d$ and $\varphi \in H_\#(Y)$,

$$\begin{aligned}
&\int_Y A(y)\big(\xi + \nabla\varphi(y)\big)\cdot\big(\xi + \nabla\varphi(y)\big)dy \\
&= \int_Y A(y)\Big(\underline{\xi + \nabla\phi_\xi(y)} + \underset{\sim\sim\sim}{\nabla\varphi(y) - \nabla\phi_\xi(y)}\Big)\cdot\Big(\underline{\xi + \nabla\phi_\xi(y)} + \underset{\sim\sim\sim}{\nabla\varphi(y) - \nabla\phi_\xi(y)}\Big)dy \\
&= \int_Y A(y)\big(\underline{\xi + \nabla\phi_\xi(y)}\big)\cdot\big(\underline{\xi + \nabla\phi_\xi(y)}\big)dy \\
&\quad + 2\int_Y A(y)\big(\underline{\xi + \nabla\phi_\xi(y)}\big)\cdot\big(\underset{\sim\sim\sim}{\nabla\varphi(y) - \nabla\phi_\xi(y)}\big)dy \\
&\quad + \int_Y A(y)\big(\underset{\sim\sim\sim}{\nabla\varphi(y) - \nabla\phi_\xi(y)}\big)\cdot\big(\underset{\sim\sim\sim}{\nabla\varphi(y) - \nabla\phi_\xi(y)}\big)dy.
\end{aligned}$$

Noting that the second term of the right hand side vanishes because $\varphi - \phi_\xi \in H_\#(Y)$ and the third term is nonnegative, it follows that

$$\begin{aligned}
\int_Y A(y)\big(\xi + \nabla\varphi(y)\big)\cdot\big(\xi + \nabla\varphi(y)\big)dy &\geq \int_Y A(y)\big(\underline{\xi + \nabla\phi_\xi(y)}\big)\cdot\big(\underline{\xi + \nabla\phi_\xi(y)}\big)dy \\
&= A_{\mathsf{eff}}\,\xi\cdot\xi
\end{aligned} \tag{4.10}$$

for any $\xi \in \mathbb{R}^d$.

4.3. Convergence of the Dirichlet forms

The Γ-convergence is used to show a convergence of minimizing problems for functionals on a metric space. We apply this convergence to our homogenization problem. Before that, as a quadratic functional $u \mapsto \mathcal{E}(u,u)$ of a symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(D)$, we extend the domain $\mathcal{F}$ to the whole space $L^2(D)$ by defining $\mathcal{E}(u,u) = \infty$ for every $u \in L^2(D)\setminus\mathcal{F}$.

Definition 4.2. *Consider a sequence of symmetric Dirichlet forms $\{(\mathcal{E}^\delta,\mathcal{F}^\delta)\}_{\delta>0}$ on $L^2(D)$ and $(\mathcal{E}^0,\mathcal{F}^0)$ a (limiting) symmetric Dirichlet form on $L^2(D)$. We say that the family $\{\mathcal{E}^\delta\}$ is said to Γ-converge to $\mathcal{E}^0$ on $L^2(D)$ as $\delta \to 0$ if the following two conditions are satisfied for any $u \in L^2(D)$:*

(Γ1) *For every sequence $\{u_\delta\}$ that converges to u_0 in $L^2(D)$, one has*

$$\liminf_{\delta\to 0}\mathcal{E}^\delta(u_\delta,u_\delta) \geq \mathcal{E}^0(u_0,u_0).$$

(Γ2) *There exists a sequence $\{u_\delta\}$ that converges to u_0 in $L^2(D)$ such that*

$$\limsup_{\delta\to 0}\mathcal{E}^\delta(u_\delta,u_\delta) \leq \mathcal{E}^0(u_0,u_0).$$

We also introduce a stronger convergence, so-called the Mosco convergence, of the forms.

The family $\{\mathcal{E}^\delta\}$ *is said to converge to* $\mathcal{E}^0$ *on* $L^2(D)$ *in the sense of Mosco as* $\delta \to 0$ *if the following two conditions are satisfied for any* $u \in L^2(D)$:

(M1) *For every sequence* $\{u_\delta\}$ *that converges to* u_0 **weakly** *in* $L^2(D)$*, one has*

$$\liminf_{\delta \to 0} \mathcal{E}^\delta(u_\delta, u_\delta) \ge \mathcal{E}^0(u_0, u_0).$$

(M2) *There exists a sequence* $\{u_\delta\}$ *that converges to* u_0 *in* $L^2(D)$ *such that*

$$\limsup_{\delta \to 0} \mathcal{E}^\delta(u_\delta, u_\delta) \le \mathcal{E}^0(u_0, u_0).$$

Let $(\mathcal{E}^\delta, H_0^1(D))$ be the Dirichlet forms on $L^2(D)$ defined in (4.1) for $\delta > 0$. In the following, we furthermore assume:

(A4) the matrix-valued function $x \mapsto A(x)$ is continuous and

$$0 < \exists\, \gamma < 2 \quad \text{s.t.} \quad C_0 := \int_{0<|h|<1} |h|^\gamma \nu(h) dh < \infty \tag{4.11}$$

(*c.f.* (3.4)).

Proposition 4.3. (Γ-convergence) *Assume* **(A1)-(A4)** *hold. Then the family of symmetric Dirichlet forms* $\{\mathcal{E}^\delta\}$ *defined in* (4.1) Γ*-converges to the symmetric Dirichlet form* $\mathcal{E}^0$ *on* $L^2(D)$ *as* $\delta \to 0$*. Here*

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int_D A_{\mathsf{eff}} \nabla u(x) \cdot \nabla v(x) dx + \frac{1}{2} \iint_{\substack{D\times D \\ x\neq y}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\bar{b}(x)\nu(y-x) dxdy$$

for $u, v \in H_0^1(D)$ *and* $\bar{b}(x) := \int_Y b_1(x,h)dh + \iint_{Y\times Y} b_2(z,h)dzdh,\ x \in D$.

Proof. Take any decreasing sequence $\{\delta_n\}$ with $\delta_n \to 0$ as $n \to \infty$. Then it is enough to see the sequence $\{\mathcal{E}^{\delta_n}\}$ converges to the unique limit $\mathcal{E}^0$ on $L^2(D)$ in the sense of Γ as $n \to \infty$.

<u>(Γ1)</u>: Take any $\{u_n\} \subset L^2(D)$ and $u \in L^2(D)$ so that u_n converges to u in $L^2(D)$ *strongly.* We may assume $\liminf_{n\to\infty} \mathcal{E}^{\delta_n}(u_n, u_n) < \infty$, and then, taking a subsequence $\{n_k\}$, $\mathcal{E}^{\delta_{n_k}}(u_{n_k}, u_{n_k})$ converges to $B := \liminf_{n\to\infty} \mathcal{E}^{\delta_n}(u_n, u_n)$. We may also assume that $\{u_{n_k}\} \subset H_0^1(D)$. Then by **(A1)**, for any n_k,

$$\frac{\alpha}{2}\int_D \big|\nabla u_{n_k}(x)\big|^2 dx \le \frac{1}{2}\int_D A^{\delta_{n_k}}(x) \nabla u_{n_k}(x) \cdot \nabla u_{n_k}(x) dx \le \mathcal{E}^{\delta_{n_k}}(u_{n_k}, u_{n_k}).$$

This implies that $\{u_{n_k}\}_{n_k}$ is a bounded sequence in $H_0^1(D)$. Then by Theorem 4.1, for taking further subsequence if necessarily, there exists $v \in L^2(D; \mathcal{H}_\#(Y))$ so that

$$\begin{cases} \text{(i)} & u_{n_k} \text{ converges to } u \text{ \textit{weakly} in } H_0^1(D), \\ \text{(ii)} & u_{n_k} \text{ converges to } u \text{ \textit{strongly} in } L^2(D), \\ \text{(iii)} & u_{n_k} \text{ converges to } u \text{ \textit{two-scale}}, \\ \text{(iv)} & \nabla u_{n_k} \text{ converges to } \nabla_x u + \nabla_y v(x,y) \text{ \textit{two-scale}}. \end{cases}$$

Take an admissible function $\varphi \in \mathcal{D}(D; C_\#(Y))$. Then by the symmetry of A, we see

$$\begin{aligned}
0 \le &\int_Y A\Big(\frac{x}{\delta_{n_k}}\Big)\Big(\nabla_x u_{n_k}(x) - \nabla_x u(x) - \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) \\
&\qquad \cdot \Big(\nabla_x u_{n_k}(x) - \nabla_x u(x) - \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) dx \\
= &\int_Y A\Big(\frac{x}{\delta_{n_k}}\Big)\nabla_x u_{n_k}(x) \cdot \nabla_x u_{n_k}(x) dx \\
&- 2\int_Y A\Big(\frac{x}{\delta_{n_k}}\Big)\nabla_x u_{n_k}(x) \cdot \Big(\nabla_x u(x) + \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) dx \\
&+ \int_Y A\Big(\frac{x}{\delta_{n_k}}\Big)\Big(\nabla_x u(x) + \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) \cdot \Big(\nabla_x u(x) + \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) dx \\
= &\int_Y A\Big(\frac{x}{\delta_{n_k}}\Big)\nabla_x u_{n_k}(x) \cdot \nabla_x u_{n_k}(x) dx \\
&- 2\int_Y \nabla_x u_{n_k}(x) \cdot A\Big(\frac{x}{\delta_{n_k}}\Big)\Big(\nabla_x u(x) + \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) dx \\
&+ \int_Y A\Big(\frac{x}{\delta_{n_k}}\Big)\Big(\nabla_x u(x) + \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) \cdot \Big(\nabla_x u(x) + \nabla_y \varphi\Big(x, \frac{x}{\delta_{n_k}}\Big)\Big) dx \\
=: &\,(\mathrm{I})_{n_k} - 2(\mathrm{II})_{n_k} + (\mathrm{III})_{n_k}.
\end{aligned}$$

Since $(x, y) \mapsto A(y)\Phi(x, y)$ belongs to $\big(\mathcal{D}(D; C_\#(Y))\big)^d$, that is, it is admissible for any $\Phi(x, y) \in \big(\mathcal{D}(D; C_\#(Y))\big)^d$, we see

$$\begin{aligned}
\lim_{k\to\infty} (\mathrm{II})_{n_k} &= \iint_{D\times Y} \Big(\nabla_x u(x) + \nabla_y v(x, y)\Big) \cdot A(y)\Big(\nabla_x u(x) + \nabla_y \varphi(x, y)\Big) dxdy \\
&= \iint_{D\times Y} A(y)\Big(\nabla_x u(x) + \nabla_y v(x, y)\Big) \cdot \Big(\nabla_x u(x) + \nabla_y \varphi(x, y)\Big) dxdy
\end{aligned}$$

for any $\varphi \in \mathcal{D}(D; C_\#(Y))$. Since the space $\mathcal{D}(D; C_\#^\infty(Y))$ is dense in $L^2(D; \mathcal{H}_\#(Y))$, we take a sequence of functions in $\mathcal{D}(D; C_\#^\infty(Y))$ converging to v in $L^2(D; \mathcal{H}_\#(Y))$, we conclude here that

$$\begin{aligned}
&\lim_{k\to\infty} \mathcal{E}^{c,\delta_{n_k}}(u_{n_k}, u_{n_k}) \\
&\quad = \lim_{k\to\infty} \frac{1}{2}\int_D A(x/\delta_{n_k})\nabla u_{n_k}(x) \cdot \nabla u_{n_k}(x) dx \\
&\quad \ge \frac{1}{2}\iint_{D\times Y} A(y)\Big(\nabla_x u(x) + \nabla_y v(x, y)\Big) \cdot \Big(\nabla_x u(x) + \nabla_y v(x, y)\Big) dxdy \\
&\quad \ge \frac{1}{2}\int_D A_{\mathsf{eff}} \nabla u(x) \cdot \nabla u(x) dx
\end{aligned}$$

Here the last inequality of the right hand side follows from the estimate, by putting $\xi_x = \nabla u(x) \in \mathbb{R}^d$ in (4.10),

$$\inf_{\phi \in H_\#(Y)} \int_Y A(y)\big(\nabla u(x) + \nabla \phi(y)\big) \cdot \big(\nabla u(x) + \nabla \phi(y)\big) dy$$

$$\geq \int_Y A(y)\big(\xi(x)+\nabla\phi_{\xi(x)}(y)\big)\cdot\big(\xi(x)+\nabla\phi_{\xi(x)}(y)\big)dy \ = A_{\mathsf{eff}}\nabla u(x)\cdot\nabla u(x)$$

(see *e.g.* the proof of [2, Theorem 4.2]).

Now, we estimate the jump part $\mathcal{E}^{j,n_k}(u_{n_k},u_{n_k})$: for any compact set K of $D\times D\setminus\{(x,x):x\in D\}$, it follows that

$$\begin{aligned}\mathcal{E}^{j,\delta n_k}(u_{n_k},u_{n_k}) &= \frac{1}{2}\iint_{\substack{D\times D\\ x\neq y}}\big(u_{n_k}(x)-u_{n_k}(y)\big)^2 b^{\delta_{n_k}}(x,y-x)\nu(y-x)dydx\\ &\geq \frac{1}{2}\iint_K\big(u_{n_k}(x)-u_{n_k}(y)\big)^2 b^{\delta_{n_k}}(x,y-x)\nu(y-x)dydx.\end{aligned}$$

According to **(A4)** and the fact that $\{u_{n_k}\}$ converges to u *strongly* in $L^2(D)$, we see that

$$\varphi_{n_k}(x,y) := 1_K(x,y)\big(u_{n_k}(x)-u_{n_k}(y)\big)^2\nu(y-x)$$

converges to

$$\varphi(x,y) := 1_K(x,y)\big(u(x)-u(y)\big)^2\nu(y-x)$$

in $L^1(\mathbb{R}^d\times\mathbb{R}^d)$. In fact, we find

$$\begin{aligned}&\iint_{\mathbb{R}^d\times\mathbb{R}^d}\big|\varphi_{n_k}(x,y)-\varphi(x,y)\big|dydx\\ &= \iint_K\Big|\big(u_{n_k}(x)-u_{n_k}(y)\big)^2-\big(u(x)-u(y)\big)^2\Big|\nu(y-x)dydx\\ &\leq \Lambda\iint_K\Big|\big(u_{n_k}(x)-u_{n_k}(y)\big)^2-\big(u(x)-u(y)\big)^2\Big|dydx\\ &\leq \Lambda\sqrt{\iint_K\Big(\big(u_{n_k}(x)-u(x)\big)-\big(u_{n_k}(y)-u(y)\big)\Big)^2dydx}\\ &\quad\times\sqrt{\iint_K\Big(\big(u_{n_k}(x)+u(x)\big)-\big(u_{n_k}(y)+u(y)\big)\Big)^2dydx}\\ &\leq \Lambda\sqrt{2\iint_K\Big(\big(u_{n_k}(x)-u(x)\big)^2+\big(u_{n_k}(y)-u(y)\big)^2\Big)dydx}\\ &\quad\times\sqrt{4\iint_K\Big((u_{n_k}(x))^2+(u(x))^2+u_{n_k}(y)^2+(u(y))^2\Big)dydx}\\ &\leq \Lambda\sqrt{2\,\mathrm{vol}(D)\Big(\|u_{n_k}-u\|^2_{L^2(D)}+\|u_{n_k}-u\|^2_{L^2(D)}\Big)}\\ &\quad\times\sqrt{\mathrm{vol}(D)\Big(4\big(\|u_{n_k}\|^2_{L^2(D)}+\|u\|^2_{L^2(D)}\big)+\big(\|u_{n_k}\|^2_{L^2(D)}+\|u\|^2_{L^2(D)}\big)\Big)}\\ &\to 0 \quad \text{as} \quad n_k\to\infty.\end{aligned}$$

Here $\Lambda := \sup\{\nu(x-y):(x,y)\in K\}\in(0,\infty)$ by local boundedness of ν. Thus by

Lemma 4.1, we see that

$$\begin{aligned}
&\liminf_{n_k\to\infty} \mathcal{E}^{j,\delta_{n_k}}(u_{n_k}, u_{n_k}) \\
&\geq \liminf_{n_k\to\infty} \frac{1}{2}\iint_{x\neq y} 1_K(x,y)\big(u_{n_k}(x)-u_{n_k}(y)\big)^2 b^{\delta_{n_k}}(x,y-x)\nu(y-x)dydx \\
&= \liminf_{n_k\to\infty} \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_{n_k}(x,y) b^{\delta_{n_k}}(x,y)dydx \\
&= \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,y)\bar{b}(x)dydx \\
&= \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} 1_K(x,y)\big(u(x)-u(y)\big)^2\bar{b}(x)\nu(y-x)dydx \\
&= \frac{1}{2}\iint_{K} \big(u(x)-u(y)\big)^2\bar{b}(x)\nu(y-x)dydx.
\end{aligned}$$

So

$$\liminf_{n_k\to\infty} \mathcal{E}^{j,\delta_{n_k}}(u_{n_k}, u_{n_k}) \geq \frac{1}{2}\iint_{K} \big(u(x)-u(y)\big)^2\bar{b}(x)\nu(y-x)dydx$$

holds for any compact set K of $D\times D\setminus\{(x,x)\}$. Putting $K\nearrow D\times D\setminus\{(x,x)\}$, it follows that

$$\lim_{n_k\to\infty} \mathcal{E}^{j,\delta_{n_k}}(u_{n_k}, u_{n_k}) \geq \frac{1}{2}\iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2\bar{b}(x)\nu(x-y)dxdy.$$

Therefore we have

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{E}^{\delta_n}(u_n,u_n) &= \lim_{n_k\to\infty}\big(\mathcal{E}^{c,\delta_{n_k}}(u_{n_k}, u_{n_k}) + \mathcal{E}^{j,\delta_{n_k}}(u_{n_k}, u_{n_k})\big) \\
&\geq \frac{1}{2}\int_D A_{\mathsf{eff}}\nabla u(x)\cdot\nabla u(x)dx \\
&\quad + \frac{1}{2}\iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2\bar{b}(x)\nu(y-x)dxdy \;= \mathcal{E}^0(u,u).
\end{aligned}$$

<u>(Γ2)</u>: Take any $u\in C_0^\infty(D)$ and $\varphi\in\widetilde{\mathcal{D}}(D;C_\#^\infty(Y))$ and set for $n\in\mathbb{N}$,

$$u_n(x) = u(x) + \delta_n\varphi(x, x/\delta_n), \quad x\in D. \tag{4.12}$$

Here $\varphi\in\widetilde{\mathcal{D}}(D;C_\#^\infty(Y))$ if and only if there exist $\{f_i\}_{i=1}^\ell\subset C_0^\infty(D)$ and $\{g_i\}_{i=1}^\ell\subset C_\#^\infty(Y)$ for some $\ell\in\mathbb{N}$ such that $\varphi(x,y)=\sum_{i=1}^\ell f_i(x)g_i(y)$. Then we see that

$$\|u_n-u\|_{L^2(D)} = \delta_n\|\varphi(\cdot,\cdot/\delta_n)\|_{L^2(D)} \leq \delta_n\|\varphi\|_{L^2(D;C_\#(Y))}\to 0 \quad (n\to\infty).$$

This means that u_n converges to u *strongly* in $L^2(D)$.

We first estimate the local part $\mathcal{E}^{c,\delta_n}(u_n,u_n)$ as follows. Since

$$\nabla u_n(x) = \nabla u(x) + \delta_n(\nabla_x\varphi)(x,x/\delta_n) + (\nabla_y\varphi)(x,x/\delta_n),$$

we see that

$$
\begin{aligned}
&\mathcal{E}^{c,\delta_n}(u_n,u_n)\\
&= \frac{1}{2}\int_D A(x/\delta_n)\nabla u_n(x)\cdot\nabla u_n(x)dx\\
&= \frac{1}{2}\int_D A(x/\delta_n)\Big(\nabla u(x)+\delta_n(\nabla_x\varphi)(x,x/\delta_n)+(\nabla_y\varphi)(x,x/\delta_n)\Big)\\
&\qquad\cdot\Big(\nabla u(x)+\delta_n(\nabla_x\varphi)(x,x/\delta_n)+(\nabla_y\varphi)(x,x/\delta_n)\Big)dx\\
&= \frac{1}{2}\int_D A(x/\delta_n)\nabla u(x)\cdot\nabla u(x)dx+\int_D A(x/\delta_n)(\nabla_y\varphi)(x,x/\delta_n)\cdot\nabla u(x)dx\\
&\quad+\frac{\delta_n}{2}\int_D A(x/\delta_n)\Big(2\nabla u(x)+\delta_n(\nabla_x\varphi)(x,x/\delta_n)+2(\nabla_y\varphi)(x,x/\delta_n)\Big)\cdot(\nabla_x\varphi)(x,x/\delta_n)dx\\
&\quad+\frac{1}{2}\int_D A(x/\delta_n)(\nabla_y\varphi)(x,x/\delta_n)\cdot(\nabla_y\varphi)(x,x/\delta_n)dx.
\end{aligned}
$$

According to the two-scale convergence, we see that

$$
\lim_{n\to\infty}\frac{1}{2}\int_D A(x/\delta_n)\nabla u(x)\cdot\nabla u(x)dx=\frac{1}{2}\iint\limits_{D\times Y} A(y)\nabla u(x)\cdot\nabla u(x)dxdy,
$$

$$
\lim_{n\to\infty}\int_D A(x/\delta_n)(\nabla_y\varphi)(x,x/\delta_n)\cdot\nabla u(x)dx=\iint\limits_{D\times Y} A(y)(\nabla_y\varphi)(x,y)\cdot\nabla u(x)dxdy
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{2}\int_D A(x/\delta_n)(\nabla_y\varphi)(x,x/\delta_n)\cdot(\nabla_y\varphi)(x,x/\delta_n)dx\\
&\quad=\frac{1}{2}\iint_{D\times Y} A(y)(\nabla_y\varphi)(x,y)\cdot(\nabla_y\varphi)(x,y)dxdy.
\end{aligned}
$$

The third term goes to 0 as $n\to\infty$. So we find that

$$
\begin{aligned}
\lim_{n\to\infty}\mathcal{E}^{c,\delta_n}(u_n,u_n) &= \frac{1}{2}\iint\limits_{D\times Y} A(y)\nabla u(x)\cdot\nabla u(x)dxdy+\iint\limits_{D\times Y} A(y)(\nabla_y\varphi)(x,y)\cdot\nabla u(x)dxdy\\
&\quad+\frac{1}{2}\iint\limits_{D\times Y} A(y)(\nabla_y\varphi)(x,y)\cdot(\nabla_y\varphi)(x,y)dxdy\\
&= \frac{1}{2}\iint\limits_{D\times Y} A(y)\big(\nabla u(x)+(\nabla_y\varphi)(x,y)\big)\cdot\big(\nabla u(x)+(\nabla_y\varphi)(x,y)\big)dxdy. \qquad (4.13)
\end{aligned}
$$

We now consider the jump part $\mathcal{E}^{j,\delta_n}(u_n,u_n)$.

$$
\begin{aligned}
&\mathcal{E}^{j,\delta_n}(u_n,u_n)\\
&\quad=\iint\limits_{\substack{D\times D\\ x\neq y}}\big(u_n(x)-u_n(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx
\end{aligned}
$$

$$= \iint_{\substack{D\times D\\ x\neq y}} \Big\{ \big(u(x)-u(y)\big) + \delta_n\big(\varphi(x,x/\delta_n)-\varphi(y,y/\delta_n)\big)\Big\}^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$= \iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$+ 2\delta_n \iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)\big(\varphi(x,x/\delta_n)-\varphi(y,y/\delta_n)\big) b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$+ \delta_n^2 \iint_{\substack{D\times D\\ x\neq y}} \big(\varphi(x,x/\delta_n)-\varphi(y,y/\delta_n)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$=: (\mathrm{I})_n + 2(\mathrm{II})_n + (\mathrm{III})_n.$$

As for the first term $(\mathrm{I})_n$, take a relatively compact open set O with $\mathrm{supp}[u] \subset O \subset \overline{O} \subset D$ and set

$$\eta := d\big(\mathrm{supp}[u], D\setminus O\big) = \inf\big\{|x-y| :\ x\in \mathrm{supp}[u],\ y\in D\setminus O\big\} > 0.$$

Then divide the integral in $(\mathrm{I})_n$ into three parts as follows:

$$(\mathrm{I})_n = \iint_{O\times O\setminus \mathsf{diag}} \big(u(x)-u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$+ \iint_{O\times(D\setminus O)} \big(u(x)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$+ \iint_{(D\setminus O)\times O} \big(u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$= \iint_{x\neq y} 1_{O\times O}\big(u(x)-u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$+ \iint_{\mathrm{supp}[u]\times(D\setminus O)} \big(u(x)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$+ \iint_{(D\setminus O)\times \mathrm{supp}[u]} \big(u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$
$$=: (\text{I-i}) + (\text{I-ii}) + (\text{I-iii}).$$

Since $u\in C_0^\infty(D)$,

$$\big|u(x)-u(y)\big| \le \big\|\, |\nabla u|\, \big\|_\infty |x-y|, \quad x,y\in D$$

and

(4.14)
$$1_{O\times O}(x,y)\big(u(x)-u(y)\big)^2\nu(y-x) \le 1_{O\times O}(x,y)\Big(2\|u\|_\infty^2 \vee \big\|\,|\nabla u|\,\big\|_\infty^2\Big)\Big(1\wedge|x-y|^2\Big)\nu(y-x)$$

hold. Noting the right hand side of (4.14) is integrable on $\mathbb{R}^d \times \mathbb{R}^d$ by **(A2)**, we have from Lemma 4.1,

$$\begin{aligned}\lim_{n\to\infty} \text{(I-i)} &= \lim_{n\to\infty} \iint_{x\neq y} 1_{O\times O}(x,y)\big(u(x)-u(y)\big)^2 \nu(y-x) b^{\delta_n}(x,y-x)dydx \\ &= \iint_{x\neq y} 1_{O\times O}(x,y)\big(u(x)-u(y)\big)^2 \nu(y-x)\bar{b}(x)dxdy \\ &= \iint_{O\times O\setminus \mathsf{diag}} \big(u(x)-u(y)\big)^2 \bar{b}(x)\nu(y-x)dxdy.\end{aligned}$$

According to **(A2)**, the local boundedness of ν on $\mathbb{R}^d \setminus \{0\}$ and the estimate $|x-y|\geq \eta$ for $(x,y)\in O^c\times \mathrm{supp}[u]$ or $(x,y)\in \mathrm{supp}[u]\times O^c$, we find that

$$1_{\mathrm{supp}[u]\times (D\setminus O)}(x,y)u(x)^2\nu(y-x) \quad \text{and} \quad 1_{(D\setminus O)\times \mathrm{supp}[u]}(x,y)u(y)^2\nu(y-x)$$

are both integrable on $\mathbb{R}^d \times \mathbb{R}^d$. Then by Lemma 4.1 again, we see

$$\lim_{n\to\infty} \text{(I-ii)} = \iint_{\mathrm{supp}[u]\times (D\setminus O)} u(x)^2\bar{b}(x)\nu(y-x)dydx$$

and

$$\lim_{n\to\infty} \text{(I-iii)} = \iint_{(D\setminus O)\times \mathrm{supp}[u]} u(y)^2\bar{b}(x)\nu(y-x)dydx.$$

Hence

$$\begin{aligned}\lim_{n\to\infty} \text{(I)}_n &= \lim_{n\to\infty} \iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx \\ &= \iint_{O\times O\setminus \mathsf{diag}} \big(u(x)-u(y)\big)^2\bar{b}(x)\nu(y-x)dxdy \\ &\quad + \iint_{\mathrm{supp}[u]\times(D\setminus O)} u(x)^2\bar{b}(x)\nu(y-x)dydx + \iint_{(D\setminus O)\times\mathrm{supp}[u]} u(y)^2\bar{b}(x)\nu(y-x)dydx \\ &= \iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2\bar{b}(x)\nu(y-x)dydx.\end{aligned}$$

We now show $\lim_{n\to\infty}\text{(III)}_n = 0$ before considering the term (II)_n. Since $\varphi(x,y) = \sum_{i=1}^{\ell} f_i(x)g_i(y)$ for some $\{f_i\}\subset C_0^\infty(D)$ and $\{g_i\}\subset C_\#^\infty(Y)$ (see (4.12)), we see that

$$\text{(III)}_n = \delta_n^2 \iint_{\substack{D\times D\\ x\neq y}} \Big\{\sum_{i=1}^{\ell}\Big(f_i(x)g_i(x/\delta_n) - f_i(y)g_i(y/\delta_n)\Big)\Big\}^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx$$

$$
\begin{aligned}
&= \delta_n^2 \iint_{\substack{D\times D\\ x\neq y}} \Big\{ \sum_{i=1}^{\ell} \Big(\big(f_i(x)-f_i(y)\big) g_i(x/\delta_n) + f_i(y)\big(g_i(x/\delta_n)-g_i(y/\delta_n)\big)\Big)\Big\}^2 \\
&\qquad\qquad \times b^{\delta_n}(x,y-x)\nu(y-x)dydx \\
&\le 4\delta_n^2 \sum_{i=1}^{\ell} \iint_{\substack{D\times D\\ x\neq y}} \big(f_i(x)-f_i(y)\big)^2 g_i(x/\delta_n)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx \\
&\quad + 4\delta_n^2 \sum_{i=1}^{\ell} \iint_{\substack{D\times D\\ x\neq y}} f_i(y)^2 \big(g_i(x/\delta_n)-g_i(y/\delta_n)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx \\
&=: 4(\text{III-1}) + 4(\text{III-2}).
\end{aligned}
$$

The boundedness of the functions b, f_i and g_i implies that

$$
(\text{III-1}) \le \delta_n^2\, \beta \sum_{i=1}^{\ell} \|g_i\|_\infty^2 \iint_{\substack{D\times D\\ x\neq y}} \big(f_i(x)-f_i(y)\big)^2 \nu(y-x)dydx \to 0 \quad (n\to\infty).
$$

As for (III-2), divide the domain in the integral into three parts as follows:

$$
\begin{aligned}
(\text{III-2}) = \delta_n^2 \sum_{i=1}^{\ell} \Big(&\iint_{\substack{D\times D\\ 0<|x-y|<\delta_n}} + \iint_{\substack{D\times D\\ \delta_n\le|x-y|<1}} + \iint_{\substack{D\times D\\ 1\le|x-y|}} \Big) f_i(y)^2 \big(g_i(x/\delta_n)-g_i(y/\delta_n)\big)^2 \\
&\times b^{\delta_n}(x,y-x)\nu(y-x)dydx \\
=:\ &(\text{i}) + (\text{ii}) + (\text{iii}).
\end{aligned}
$$

By **(A2)**, we see

$$
(\text{iii}) \le 4\delta_n^2 \beta \sum_{i=1}^{\ell} \|g_i\|_\infty^2 \Big(\int_D f_i(y)^2 dy\Big)\Big(\int_{|h|\ge 1} \nu(h)dh\Big) \ \to\ 0 \quad (\text{as } \ n\to\infty).
$$

The condition (4.11) and the boundedness of g_i and b give us

$$
\begin{aligned}
(\text{ii}) &\le 4\beta \sum_{i=1}^{\ell} \|g_i\|_\infty^2 \int_D f_i(y)^2\ \delta_n^{2-\gamma} \Big(\int_{\delta_n\le|x-y|<1} |x-y|^\gamma \nu(y-x)dx\Big)dy \\
&\le 4\beta C_0 \delta_n^{2-\gamma} \sum_{i=1}^{\ell} \|g_i\|_\infty^2 \int_D f_i(y)^2 dy \ \to\ 0 \quad (\text{as } n\to\infty).
\end{aligned}
$$

The first term is estimated by

$$
(\text{i}) \le \beta \sum_{i=1}^{\ell} \|\,|\nabla g_i|\,\|_\infty^2 \int_D f_i(y)^2 \Big(\int_{0<|x-y|<1} 1_{B(\delta_n)}(x-y)|x-y|^2\nu(x-y)dx\Big)dy
$$

and the right hand side goes to 0 as $n\to\infty$ by using the dominated convergence theorem, where $B(\delta_n)$ is the open ball with radius δ_n at the origin.

As for the term $(\mathrm{II})_n$, using the Schwarz inequality, we see that

$$
\begin{aligned}
(\mathrm{II})_n &\le \sqrt{\iint\limits_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx} \\
&\quad \times \sqrt{\delta_n^2 \iint\limits_{\substack{D\times D\\ x\neq y}} \big(\varphi(x,x/\delta_n)-\varphi(y,y/\delta_n)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx} \\
&= \sqrt{(\mathrm{I})_n\cdot(\mathrm{III})_n} \quad \to 0 \quad (\text{as } n\to\infty).
\end{aligned}
$$

Hence we see that

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{E}^{j,\delta_n}(u_n,u_n) &= \lim_{n\to\infty} \frac{1}{2}\iint\limits_{\substack{D\times D\\ x\neq y}} \big(u_n(x)-u_n(y)\big)^2 b^{\delta_n}(x,y-x)\nu(y-x)dydx \\
&= \frac{1}{2}\iint\limits_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2 \bar{b}(x)\nu(y-x)dydx. \qquad (4.15)
\end{aligned}
$$

Summarizing the calculations done above, we have

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{E}^{\delta_n}(u_n,u_n) &= \frac{1}{2}\iint\limits_{D\times Y} A(y)\big(\nabla u(x)+(\nabla_y\varphi)(x,y)\big)\cdot\big(\nabla u(x)+(\nabla_y\varphi)(x,y)\big)dxdy \\
&\quad + \frac{1}{2}\iint\limits_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2 \bar{b}(x)\nu(y-x)dydx \qquad (4.16)
\end{aligned}
$$

for any $u\in C_0^\infty(D)$ and $\varphi\in\widetilde{\mathcal{D}}(D;C_\#^\infty(Y))$, where

$$u_n(x)=u(x)+\delta_n\varphi(x,x/\delta_n),\quad x\in D,\ n\in\mathbb{N}.$$

Since $\{\omega_i\}_{i=1}^d\subset H_\#(Y)$ and the fact that $C_\#^\infty(Y)$ is dense in $H_\#(Y)$, we can take sequences $\{\rho_i^n\}_{n\in\mathbb{N}}\subset C_\#^\infty(Y)$ for each $i=1,\dots,d$ so that ρ_i^n converges to ω_i not just in $H_\#(Y)$ but also in $\mathcal{H}_\#(Y)$. This implies that $\varphi_n(x,y):=\sum_{i=1}^d(\partial_i u/\partial x_i)(x)\rho_i^n(y)$ converges to $\varphi_u(x,y):=\sum_{i=1}^d(\partial_i u/\partial x_i)(x)\omega_i(y)$ in $L^2(D;\mathcal{H}_\#(Y))$. So, putting

$$u_n(x)=u(x)+\delta_n\varphi_{m_n}(x,x/\delta_n),\quad x\in D$$

for a subsequence $\{m_n\}$, we find that using (4.16),

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{E}^{\delta_n}(u_n,u_n) &= \frac{1}{2}\iint\limits_{D\times Y} A(y)\big(\nabla u(x)+(\nabla_y\varphi_u)(x,y)\big)\cdot\big(\nabla u(x)+(\nabla_y\varphi_u)(x,y)\big)dxdy \\
&\quad + \frac{1}{2}\iint\limits_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2 \bar{b}(x)\nu(y-x)dydx
\end{aligned}
$$

$$= \frac{1}{2}\int_D A_{\mathsf{eff}}\nabla u(x)\cdot\nabla u(x)dx + \frac{1}{2}\iint_{\substack{D\times D\\ x\neq y}} \big(u(x)-u(y)\big)^2\overline{b}(x)\nu(y-x)dydx$$
$$= \mathcal{E}^0(u,u)$$

holds, and the proof of (Γ2) is complete. □

The following lemma immediately follows from Sobolev's embedding theorem:

Lemma 4.4. (Asymptotic Compactness) *The sequence of Dirichlet forms $(\mathcal{E}^{\delta_n}, H_0^1(D))$ on $L^2(D)$ is* **asymptotically compact** *in the sense that, every sequence $\{u_n\}\subset L^2(D)$ with $\liminf_{n\to\infty}\mathcal{E}^{\delta_n}(u_n,u_n)<\infty$ has a subsequence that converges strongly in $L^2(D)$.*

Then by Lemma 2.3.2 in [24], we can conclude

Theorem 4.2. (Mosco Convergence) *The sequence of Dirichlet forms $(\mathcal{E}^{\delta}, H_0^1(D))$ on $L^2(D)$ converges to $(\mathcal{E}^0, H_0^1(D))$ in the sense of Mosco as $\delta\to 0$.*

5. Strong Feller property

We show a strong Feller property of L^2-resolvent associated with the Dirichlet form $(\mathcal{E}, H^1(\mathbb{R}^d))$ given in Section 3 and the Dirichlet form $(\mathcal{E}^\delta, H_0^1(D))$ given in Section 4. Throughout this section we assume **(A1)-(A3)**.

5.1. Strong Feller property of L^2-resolvent associated with $(\mathcal{E}, H^1(\mathbb{R}^d))$

Let $(\mathcal{E}, H^1(\mathbb{R}^d))$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ given in Section 3 and $\{G_\lambda:\lambda>0\}$ be the L^2-resolvent associated with $(\mathcal{E}, H^1(\mathbb{R}^d))$. We show Proposition 3.1. As in the proof of Lemma 3.1,

$$C_1\|u\|^2_{H^1(\mathbb{R}^d)} \le \mathcal{E}(u,u)+\|u\|^2_{L^2(\mathbb{R}^d)} \le C_2\|u\|^2_{H^1(\mathbb{R}^d)}, \qquad u\in H^1(\mathbb{R}^d),$$

for some positive constants C_i $(i=1,2)$. Therefore, for any $\lambda>0$ and any $T\in H^{-1}(\mathbb{R}^d)$, the dual space of $H^1(\mathbb{R}^d)$, there exists a unique $u\in H^1(\mathbb{R}^d)$ such that

$$\mathcal{E}_\lambda(u,\varphi)=\langle T,\varphi\rangle, \qquad \varphi\in H^1(\mathbb{R}^d),$$

where $\mathcal{E}_\lambda(u,\varphi)=\mathcal{E}(u,\varphi)+\lambda(u,\varphi)_{L^2(\mathbb{R}^d)}$. We denote this function u by $G_\lambda T$. Obviously $G_\lambda T$ coincides with $G_\lambda f$ where $\langle T,\varphi\rangle=(f,\varphi)_{L^2(\mathbb{R}^d)}$ for $f\in L^2(\mathbb{R}^d)$. For $T\in H^{-1}(\mathbb{R}^d)$, there are $f_i\in L^2(\mathbb{R}^d)$ $(i=0,1,\cdots,d)$ such that

$$\langle T,\varphi\rangle=\int_{\mathbb{R}^d} f_0\varphi\,dx+\sum_{i=1}^d\int_{\mathbb{R}^d} f_i\partial_i\varphi\,dx, \qquad \varphi\in H^1(\mathbb{R}^d), \tag{5.1}$$

where $\partial_i\varphi$ is the weak partial derivative $(i=1,2,\cdots,d)$. Proposition 3.1 is an immediate consequence of the following three propositions.

Proposition 5.1. *Let $T\in H^{-1}(\mathbb{R}^d)$ and assume $f_i\in L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)\,(i=$*

$0, 1, \cdots, d)$ *with* $p > d$ *for the representation* (5.1). *Then*

$$\|G_\lambda T\|_{L^\infty(\mathbb{R}^d)} \le C_3 \sum_{i=0}^{d} \left\{ \|f_i\|_{L^2(\mathbb{R}^d)} + \|f_i\|_{L^p(\mathbb{R}^d)} \right\}, \tag{5.2}$$

for some positive constant C_3 *depending on* $d,\ p,\ \lambda$ *and* α.

PROPOSITION 5.2. *Assume* (3.4). *Let* $T \in H^{-1}(\mathbb{R}^d)$ *and assume* $f_i \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $(i = 0, 1, \cdots, d)$ *with* $d < p \le \gamma/(\gamma - 1)$ *for the representation* (5.1). *Then* $G_\lambda T$ *is Hölder continuous on* $\mathbb{R}^d$.

PROPOSITION 5.3. *Assume* (3.4). *Then* $G_\lambda \left(C_0(\mathbb{R}^d)\right) \subset C_\infty(\mathbb{R}^d)$.

Proposition 5.1 is obtained by the same method as Theorem 4.2 of [28] (see also Corollary of [13]). So we omit the proof.

We now show Proposition 5.2. Let $E \subset \mathbb{R}^d$ be a bounded domain. We put

$$H^{1,p^*}(E) = \left\{ u \in L^{p^*}(E) :\ \partial_i u \in L^{p^*}(E), i = 1, 2, \ldots, d \right\}$$

and denote by $H_0^{1,p^*}(E)$ the closure of $C_0^\infty(E)$ in $H^{1,p^*}(E)$. We denote by $H^{-1,p}(E)$ the dual space of $H_0^{1,p^*}(E)$, where $1/p + 1/p^* = 1$. When $p = 2$, we write $H^{-1}(E)$ in place of $H^{-1,2}(E)$. Let us consider the following quadratic form.

$$\mathcal{E}_E^c(u, v) = \frac{1}{2} \int_E A(x) \nabla u(x) \cdot \nabla v(x)\, dx. \tag{5.3}$$

Then $(\mathcal{E}_E^c, H_0^1(E))$ is a regular Dirichlet form on $L^2(E)$. For $T \in H^{-1}(E)$, there exists a unique u, denoted by $G_E^c T$, belonging to $H_0^1(E)$ such that

$$\mathcal{E}_E^c(u, \varphi) = \langle T, \varphi \rangle, \qquad \varphi \in H_0^1(E).$$

Since E is bounded, $T \in H^{-1,p}(E)$ with $p > 2$ has a unique extension $\widetilde{T} \in H^{-1}(E)$ and satisfies $\langle T, \varphi \rangle = \langle \widetilde{T}, \varphi \rangle$ for $\varphi \in C_0^\infty(E)$. So we write $G_E^c T$ in place of $G_E^c \widetilde{T}$ for $T \in H^{-1,p}(E)$ with $p > 2$.

The following result is obtained by G.Stampacchia (see Theorem 4.2 of [28], also see Corollary of [13]). We set $B(a, r) = \{x \in \mathbb{R}^d : |x - a| < r\}$.

THEOREM 5.1. (Theorem 4.2 of [28]) *Let* $p > d$, $E = B(a, r)$ *and* $T \in H^{-1,p}(E)$. *Then*

$$\|G_E^c T\|_{L^\infty(E)} \le C_4 r^{d(1/2 - 1/p - 1/q)} \|T\|_{H^{-1,p}(E)}, \tag{5.4}$$

for some positive constant C_4 *depending on* $d,\ p$ *and* α, *where* $q = 2d/(d-2)$ *if* $d \ge 3$, $q = 4p/(p-2)$ *if* $d = 2$.

We next derive a bound for the jump part $\mathcal{E}^j(G_\lambda T, \varphi)$. We put $|E| = \mathrm{vol}(E)$.

LEMMA 5.1. *Under the same assumption as Proposition 5.2,*

$$|\mathcal{E}^j(G_\lambda T, \varphi)| \le C_5 \|\varphi\|_{H_0^{1,p^*}(B(a,r))}, \quad \varphi \in C_0^\infty(B(a, r)), \tag{5.5}$$

where $0 < r \le \rho < \infty$, $1/p + 1/p^* = 1$, *and* C_5 *is a positive constant independent of* a *and* r.

Proof: Let $\varphi \in C_0^\infty(B(a,r))$ and $0 < r \le \rho$. We set $u = G_\lambda T$. Then

$$\begin{aligned}
\mathcal{E}^j(u,\varphi) &= \frac{1}{2}\iint_{x\neq y} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big) b(x,y-x)\nu(y-x)\,dydx \\
&= \frac{1}{2}\iint_{\substack{B(a,2\rho)\times B(a,2\rho)\\ x\neq y}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big) b(x,y-x)\nu(y-x)\,dydx \\
&\quad + \frac{1}{2}\int_{B(a,2\rho)} \varphi(x)\,dx \int_{B(a,2\rho)^c} \big(u(x)-u(y)\big) b(x,y-x)\nu(y-x)\,dy \\
&\quad + \frac{1}{2}\int_{B(a,2\rho)} \big(-\varphi(y)\big)\,dy \int_{B(a,2\rho)^c} \big(u(x)-u(y)\big) b(x,y-x)\nu(y-x)\,dx \\
&=: \langle L_1,\varphi\rangle + \langle L_2,\varphi\rangle + \langle L_3,\varphi\rangle.
\end{aligned}$$

By means of **(A3)**,

$$\begin{aligned}
|\langle L_1,\varphi\rangle| &\le \frac{\beta}{2}\Big(\iint_{\substack{B(a,2\rho)\times B(a,2\rho)\\ x\neq y}} |u(x)-u(y)|^p \nu(y-x)\,dydx\Big)^{1/p} \\
&\quad \times \Big(\iint_{\substack{B(a,2\rho)\times B(a,2\rho)\\ x\neq y}} |\varphi(x)-\varphi(y)|^{p^*} \nu(y-x)\,dydx\Big)^{1/p^*} =: \frac{\beta}{2}(U)^{1/p}(V)^{1/p^*}.
\end{aligned}$$

Since $\varphi \in C_0^\infty(B(a,r))$, we get the following.

$$\begin{aligned}
V &\le \int_{B(a,2\rho)} dx \int_{0<|h|\le 4\rho} |\varphi(x)-\varphi(x+h)|^{p^*} \nu(h)\,dh \\
&\le \int_{B(a,2\rho)} dx \int_{0<|h|\le 4\rho} \Big(\int_0^1 \Big|\frac{d}{dt}\varphi(x+th)\Big|\,dt\Big)^{p^*} \nu(h)\,dh \\
&\le \int_{B(a,2\rho)} dx \int_{0<|h|\le 4\rho} \Big(\int_0^1 |\nabla\varphi(x+th)|\cdot|h|\,dt\Big)^{p^*} \nu(h)\,dh \\
&\le \int_{B(a,2\rho)} dx \int_{0<|h|\le 4\rho} |h|^{p^*}\nu(h)\,dh \int_0^1 |\nabla\varphi(x+th)|^{p^*}\,dt \\
&\le \int_{0<|h|\le 4\rho} |h|^{p^*}\nu(h)\,dh \int_0^1 dt \int_{B(a,6\rho)} |\nabla\varphi(z)|^{p^*}\,dz \\
&= N_{p^*}\int_{B(a,r)} |\nabla\varphi(z)|^{p^*}\,dz.
\end{aligned}$$

We note that $\gamma \le p^* < d/(d-1)$ and $N_{p^*} := \int_{0<|h|\le 4\rho} |h|^{p^*}\nu(h)\,dh < \infty$ by means of (3.4) and **(A2)**. Since $p > 2$ and $u \in L^\infty(\mathbb{R}^d)$ by Proposition 5.1, we also have the following.

$$U \leq \left(2\|u\|_{L^\infty(\mathbb{R}^d)}\right)^{p-2} \iint\limits_{\substack{B(a,2\rho)\times B(a,2\rho)\\ x\neq y}} |u(x)-u(y)|^2\nu(y-x)\,dydx$$
$$\leq \left(2\|u\|_{L^\infty(\mathbb{R}^d)}\right)^{p-2} N_2 \int\limits_{B(a,6\rho)} |\nabla u(z)|^2\,dz,$$

where $N_2 := \int_{0<|h|\leq 4\rho} |h|^2\nu(h)\,dh < \infty$ by **(A2)**. Therefore we get

$$\left|\langle L_1,\varphi\rangle\right| \leq \beta\,(N_{p^*})^{1/p^*}\,(N_2)^{1/p}\,\|u\|^{1-2/p}_{L^\infty(\mathbb{R}^d)}\|\,|\nabla u|\,\|^{2/p}_{L^2(\mathbb{R}^d)}\|\,|\nabla\varphi|\,\|_{L^{p^*}(B(a,r))}. \tag{5.6}$$

It is easy to see the following.

$$\left|\langle L_2,\varphi\rangle\right| + \left|\langle L_3\varphi\rangle\right| \leq 2\beta\|u\|_{L^\infty(\mathbb{R}^d)}\int_{B(a,r)}|\varphi(x)|\,dx\int_{|h|>\rho}\nu(h)\,dh$$
$$\leq 2\beta N_0\|u\|_{L^\infty(\mathbb{R}^d)}|B(a,\rho)|^{1/p}\|\varphi\|_{L^{p^*}(B(a,r))}, \tag{5.7}$$

where $N_0 := \int_{|h|>\rho}\nu(h)\,dh < \infty$ by **(A2)**. Putting

$$C_5 = \beta\,(N_{p^*})^{1/p^*}\,(N_2)^{1/p}\,\|G_\lambda T\|^{1-2/p}_{L^\infty(\mathbb{R}^d)}\||\nabla G_\lambda T|\|^{2/p}_{L^2(\mathbb{R}^d)}$$
$$+2\beta N_0\|G_\lambda T\|_{L^\infty(\mathbb{R}^d)}|B(a,\rho)|^{1/p}, \tag{5.8}$$

we get (5.5) from (5.6) and (5.7). □

Now we give

Proof (of Proposition 5.2): Fix $0<\rho<1$ and $a\in\mathbb{R}^d$ arbitrarily, and put $E = B(a,r)$ for $0<r\leq\rho$. We set $u = G_\lambda T$ and define the operator T^* by

$$\langle T^*,\varphi\rangle := \langle T,\varphi\rangle - \lambda\int_{\mathbb{R}^d} u\varphi\,dx - \mathcal{E}^j(u,\varphi),$$

for $\varphi\in C_0^\infty(\mathbb{R}^d)$. By means of (5.1) and Lemma 5.1,

$$|\langle T^*,\varphi\rangle| \leq c_1\|\varphi\|_{H_0^{1,p^*}(E)}, \qquad \varphi\in C_0^\infty(E), \tag{5.9}$$

where c_1 is a positive constant independent of a and r, and it is given by

$$c_1 = \sum_{i=0}^d \|f_i\|_{L^p(\mathbb{R}^d)} + \lambda\|u\|_{L^\infty(\mathbb{R}^d)}|B(a,\rho)|^{1/p} + C_5.$$

(5.9) implies $T^*\in H^{-1,p}(E)$. Therefore there is a unique $v := G_E^c T^*\in H_0^1(E)$. It follows from Theorem 5.1 and (5.9) that

$$\|v\|_{L^\infty(E)} \leq c_2 r^{d(1/2-1/p-1/q)}, \tag{5.10}$$

where c_2 is a positive constant independent of a and r, and q is a positive number given in Theorem 5.1.

We set $w = u|_E - v$, where $u|_E$ is the restriction of u to E. Since $\mathcal{E}_E^c(u|_E, \varphi) = \langle \widetilde{T}, \varphi \rangle$ for $\varphi \in C_0^\infty(E)$, we get

$$\begin{cases} w \in H^1(E), \\ \mathcal{E}_E^c(w, \varphi) = 0, \ \ \varphi \in C_0^\infty(E). \end{cases}$$

Employing results from elliptic differential equations (see *e.g.* [16, 25, 28] etc.), we obtain the following

$$\mathrm{Osc}\,(w, B(a, r/4)) \le c_3 \mathrm{Osc}\,(w, B(a, r))\,,$$

where c_3 is a constant satisfying $0 < c_3 < 1$ independent of a and r, and $\mathrm{Osc}(g, E) := \operatorname{ess\,sup}_E g - \operatorname{ess\,inf}_E g$ (cf. Section 5 of [25], Lemma 7.3 of [28]). Combining this with (5.10), we have

$$\begin{aligned} \mathrm{Osc}\,(u, B(a, r/4)) &\le \mathrm{Osc}\,(w, B(a, r/4)) + \mathrm{Osc}\,(v, B(a, r/4)) \\ &\le c_3 \mathrm{Osc}\,(w, B(a, r)) + 2\|v\|_{L^\infty(E)} \\ &\le c_3 \mathrm{Osc}\,(u, B(a, r)) + 4\|v\|_{L^\infty(E)} \\ &\le c_3 \mathrm{Osc}\,(u, B(a, r)) + 4c_2 r^{d(1/2-1/p-1/q)}. \end{aligned}$$

By Lemma 7.6 of [28], there are two constants $c_4 > 0$ and $\xi \in (0, 1)$ such that

$$\mathrm{Osc}(u, B(a, r)) \le c_4 r^\xi, \qquad 0 < r \le \rho/2.$$

Since c_4 and ξ are independent of r and a, this shows that u is Hölder continuous on $\mathbb{R}^d$. □

Proof (of Proposition 5.3): Let $g \in C_0(\mathbb{R}^d)$ and put $u = G_\lambda g$. By means of Propositions 5.1 and 5.2, u is bounded continuous on $\mathbb{R}^d$. Therefore it suffices to show that for any $\varepsilon > 0$ there is a compact set K such that $\|u\|_{L^\infty(K^c)} < \varepsilon$.

For R satisfying $\mathrm{supp}[g] \subset B(0, R)$, we take $a \in B(0, R+1)^c$ and $0 < r < 1$. We put $E = B(a, r)$. Then

$$\mathcal{E}_\lambda(u, \varphi) = (g, \varphi) = 0, \qquad \varphi \in C_0^\infty(E),$$

and hence

$$\mathcal{E}_E^c(u|_E, \varphi) = \langle S, \varphi \rangle, \qquad \varphi \in C_0^\infty(E),$$

where

$$\langle S, \varphi \rangle = -\lambda \int_E u\varphi\, dx - \mathcal{E}^j(u, \varphi).$$

By virtue of Lemma 5.1,

$$|\langle S, \varphi \rangle| \le c_1 \|\varphi\|_{H_0^{1,p^*}(E)}, \qquad \varphi \in C_0^\infty(E), \tag{5.11}$$

where $d < p \le \gamma/(\gamma - 1)$, $1/p + 1/p^* = 1$, $c_1 = \lambda \|u\|_{L^\infty(\mathbb{R}^d)} |B(a, 1)|^{1/p} + C_5$, and C_5 is given by (5.8) with $\rho = 1$. We set $v = G_E^c S$. By Theorem 5.1 and (5.11),

$$\|v\|_{L^\infty(E)} \le C_4 |E|^{1/2-1/p-1/q} \|S\|_{H^{-1,p}(E)} \le c_2 r^{d(1/2-1/p-1/q)},$$

where q is a positive number given in Theorem 5.1. Note that c_2 is a positive constant independent of a and r. $w := u|_E - v$ satisfies

$$\begin{cases} w \in H^1(E), \\ \mathcal{E}_E^c(w, \varphi) = 0, \quad \varphi \in C_0^\infty(E). \end{cases}$$

By using a local estimate for w (cf. Theorem 1 of [25], Corollary 5.2 of [28]),

$$\|w\|_{L^\infty(B(a,r/2))} \le c_3 r^{-d/2} \|w\|_{L^2(E)},$$

where c_3 is a positive constant depending on α, β, d. Therefore

$$\begin{aligned} \|u\|_{L^\infty(B(a,r/2))} &\le \|v\|_{L^\infty(B(a,r/2))} + \|w\|_{L^\infty(B(a,r/2))} \\ &\le \|v\|_{L^\infty(E)} + c_3 r^{-d/2} \|w\|_{L^2(E)} \\ &\le c_3 r^{-d/2} \|u\|_{L^2(B(0,R)^c)} + \{1 + c_3 r^{-d/2} |E|^{1/2}\} \|v\|_{L^\infty(E)}. \end{aligned}$$

Let $\varepsilon > 0$. Since $c_4 := c_3 r^{-d/2}|E|^{1/2} + 1$ is independent of r, there is an $r \in (0,1)$ such that $c_4\|v\|_{L^\infty(E)} \le c_2 c_4 r^{d(1/2-1/p-1/q)} < \varepsilon/2$. We then take an R such that $c_3 r^{-d/2}\|u\|_{L^2(B(0,R)^c)} < \varepsilon/2$. For such R and r, we get

$$\|u\|_{L^\infty(B(0,R+1)^c)} \le \sup_{a \in B(0,R+1)^c} \|u\|_{L^\infty(B(a,r/2))} < \varepsilon.$$

Thus we obtain $u \in C_\infty(\mathbb{R}^d)$. □

5.2. Strong Feller property of L^2-resolvent associated with $(\mathcal{E}^\delta, H_0^1(D))$

Let $(\mathcal{E}^\delta, H_0^1(D))$ be the Dirichlet form on $L^2(D)$ defined by (4.1). We set $\mathcal{E}_\lambda^\delta(u,v) := \mathcal{E}^\delta(u,v) + \lambda(u,v)_{L^2(D)}$ for $\lambda \ge 0$ and $\delta > 0$. As in (4.2), for each $\lambda \ge 0$ there are positive constants $C_i = C_i(\lambda)$ $(i = 6, 7)$ such that

$$C_6\|u\|^2_{H_0^1(D)} \le \mathcal{E}_\lambda^\delta(u,u) \le C_7\|u\|^2_{H_0^1(D)}, \quad u \in H_0^1(D), \quad \delta > 0. \tag{5.12}$$

We note that C_i $(i = 6, 7)$ are independent of $\delta > 0$.

By means of (5.12), for any $\lambda \ge 0$ and $T \in H^{-1}(D)$ there exists a unique $u \in H_0^1(D)$ such that

$$\mathcal{E}_\lambda^\delta(u, \varphi) = \langle T, \varphi \rangle, \quad \varphi \in H_0^1(D).$$

We denote this function u by $G_\lambda^\delta T$. Obviously $G_\lambda^\delta T$ coincides with $G_\lambda^\delta f$ when $\langle T, \varphi\rangle = (f, \varphi)_{L^2(D)}$ for $f \in L^2(D)$. Under additional conditions on a bounded domain D and the Lévy measure density $\nu(h)$, we show a Feller property of $G_\lambda^\delta f$.

We assume that the boundary ∂D is Lipschitz continuous with a finite number of outward or inward Hölder cusp boundary points. More precisely, we state condition **(D)** on a bounded domain D. For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we let $x' = (x_1, \cdots, x_{d-1})$ so that $x = (x', x_d)$.

(D) There is a finite open covering $\{U_k\}_{k\in I}$ of ∂D such that

$$U_k \cap D = \{\zeta = (\zeta', \zeta_d) \,:\, |\zeta| < \rho_k,\ F_k(\zeta') < \zeta_d\}, \tag{5.13}$$

for some Cartesian coordinate system $\zeta = (\zeta', \zeta_d)$, where $\rho_k > 0$ and F_k is a Lipschitz continuous function vanishing at the origin of $\mathbb{R}^{d-1}$, or $F_k(\zeta') = c_k|\zeta'|^{\kappa_k}$ with $c_k \neq 0$ and $0 < \kappa_k < 1$.

We set

$$I_C = \{k \in I \,:\, U_k \cap D \text{ is represented by (5.13) with } F_k(\zeta') = c_k|\zeta'|^{\kappa_k} \text{ and } c_k > 0\},$$
$$I_Q = \{k \in I \,:\, U_k \cap D \text{ is represented by (5.13) with } F_k(\zeta') = c_k|\zeta'|^{\kappa_k} \text{ and } c_k < 0\}.$$

For each $k \in I$, denote by a_k ($\in \partial D$) the origin of U_k with respect to the coordinate system ζ. a_k for $k \in I_C$ [resp. $k \in I_Q$] may be called an outward [resp. inward] cusp boundary point.

In [15], a strong Feller property of L^2-resolvent associated with the following Dirichlet form on $L^2(O)$ was investigated.

$$\begin{cases} \tilde{\mathcal{E}}(u,v) = \dfrac{1}{2}\displaystyle\int_O A(x)\nabla u(x) \cdot \nabla v(x) dx, \\ \tilde{\mathcal{F}} = H^1(O), \end{cases}$$

where $A(x)$ satisfies **(A1)**, and O is a Lipschitz domain with cusp boundary points and satisfies Condition (H) of [15]. Our condition **(D)** reduces to (H) with O replaced by D. Further it reduces to Condition (A) of [15]. Therefore propositions in this subsection hold true under more general condition.

In Section 3 of [15], some Sobolev inequalities are proved for functions on a Lipschitz domain with cusp boundary points. By using them we get the following Sobolev inequality. There is a positive constant C_8 such that

$$\left(\int_D |u|^q \, dx\right)^{1/q} \leq C_8 \|u\|_{H^1(D)}, \tag{5.14}$$

for $u \in H^1(D)$, $2 \leq q \leq 2(d-1+\kappa^*)/(d-1-\kappa^*)$, where $\kappa^* = \min\{\kappa_k \,:\, k \in I_C\}$. In the case $I_C = \varnothing$, (5.14) holds for $2 \leq q \leq 2d/(d-2)$ if $d \geq 3$, and for $2 \leq q < \infty$ if $d = 2$.

Since D is bounded, $T \in H^{-1,p}(D)$ with $p > 2$ has a unique extension $\widetilde{T} \in H^{-1}(D)$ and satisfies $\langle T, \varphi\rangle = \langle \widetilde{T}, \varphi\rangle$ for $\varphi \in C_0^\infty(D)$. So we write $G_\lambda^\delta T$ in place of $G_\lambda^\delta \widetilde{T}$ for $T \in H^{-1,p}(D)$ with $p > 2$.

Using (5.14) and following a standard argument as in Theorem 4.2 of [28] (see also Corollary of [13]), we can get the following L^p-estimate. From now on, we let $\kappa^* = 1$ in the case $I_C = \varnothing$.

Proposition 5.4. *Assume* **(D)**. *Let $p > 1 + (d-1)/\kappa^*$, $T \in H^{-1,p}(D)$, and $\lambda \geq 0$. Then there exists a positive constant C_9 independent of δ such that*

$$\|G_\lambda^\delta T\|_{L^\infty(D)} \leq C_9 \|T\|_{H^{-1,p}(G)}. \tag{5.15}$$

Let E be a subdomain of D and consider the following quadratic form.

$$\mathcal{E}_E^{c,\delta}(u,v) = \frac{1}{2}\int_E A(\frac{x}{\delta})\nabla u(x)\cdot\nabla v(x)\,dx,$$

where $A(x)$ satisfies **(A1)**. $\mathcal{E}_E^{c,\delta}$ is the same as $\mathcal{E}_E^c$ given by (5.3) with $A(x/\delta)$ in place of $A(x)$. Then $(\mathcal{E}_E^{c,\delta}, H_0^1(E))$ is a regular Dirichlet form on $L^2(E)$. For $T \in H^{-1}(E)$, there exists a unique u, denoted by $G_E^{c,\delta}T$, belonging to $H_0^1(E)$ such that

$$\mathcal{E}_E^{c,\delta}(u,\varphi) = \langle T,\varphi\rangle, \qquad \varphi \in H_0^1(E).$$

For $T \in H^{-1,p}(E)$ with $p > 2$, we also write $G_E^{c,\delta}T$ in place of $G_E^{c,\delta}\widetilde{T}$ with a unique extension $\widetilde{T} \in H^{-1}(E)$ satisfying $\langle T,\varphi\rangle = \langle\widetilde{T},\varphi\rangle$ for $\varphi \in C_0^\infty(E)$.

Theorem 5.1 holds true for $G_E^{c,\delta}T$ when $\overline{E}\cap\{a_k : k \in I_C\cup I_Q\} = \varnothing$, where $\overline{E} = E\cup\partial E$ and $E = B(a,r)\cap D$ for $a \in \overline{D} := D\cup\partial D$ and $0 < r < \infty$, that is, there is a positive constant C_{10} independent of a, r and δ such that

$$\|G_E^{c,\delta}T\|_{L^\infty(E)} \le C_{10} r^{d(1/2-1/p-1/q)}\|T\|_{H^{-1,p}(E)}, \tag{5.16}$$

for $p > d$ and $T \in H^{-1,p}(E)$.

When $E = B(a_k,r)\cap U_k\cap D$ for $k \in I_C\cup I_Q$, by the same argument as for Proposition 5.4, we can get the following.

Proposition 5.5. *Assume* **(D)**. *(i) Let $k \in I_C$ and $p > 1+(d-1)/\kappa_k$. Then there exists a positive constant C_{11} independent of k, ρ and δ such that*

$$\|G_E^{c,\delta}T\|_{L^\infty(E)} \le C_{11}\rho^{\{p-1-(d-1)/\kappa_k\}/p}\|T\|_{H^{-1,p}(E)},$$

where $E = B(a_k,\rho)\cap U_k\cap D$, $0<\rho\le\rho_k$, and $T \in H^{-1,p}(E)$.

(ii) Let $k \in I_Q$ and $p > d$. Then there exists a positive constant C_{12} independent of k, ρ and δ such that

$$\|G_E^{c,\delta}T\|_{L^\infty(E)} \le C_{12}\rho^{d(1/2-1/p-1/q)}\|T\|_{H^{-1,p}(E)},$$

where $E = B(a_k,\rho)\cap U_k\cap D$, $0<\rho<\rho_k$, $T \in H^{-1,p}(E)$, and q is a positive number given in Theorem 5.1.

Here we note the following (see Lemma 3.1 of [15]). Put $\rho^* = \min\{\rho_k : k \in I_C\cup I_Q\}$. We may assume $\rho^* \le 1$ without loss of generality.

Lemma 5.2. *For any $\rho \in (0,\rho^*]$, there exists a positive number r_ρ such that $B(a,r_\rho)\cap\{a_k : k \in I_C\cup I_Q\} = \varnothing$ for every $a \in \partial D\setminus\bigcup_{k\in I_C\cup I_Q}B(a_k,\rho)$. Furthermore, for any $r \in (0,r_\rho]$ there exist a positive number η_r and a subset $\Lambda_r \subset \partial D\setminus\bigcup_{k\in I_C\cup I_Q}B(a_k,\rho)$ such that for every $b \in \partial D$, the set $B(b,\eta_r)$ is contained in one of the following sets : $B(a_k,\rho)$ for $k \in I_C\cup I_Q$, $B(a,r)$ for $a \in \Lambda_r$.*

We next consider a Harnack inequality for solutions $u \in H^1(E)$ of

$$\mathcal{E}_E^{c,\delta}(u,\varphi) = 0, \quad \varphi \in H_0^1(E). \tag{5.17}$$

In the same way as Theorem 3.3 of [15], we can get the following Harnack inequality.

Proposition 5.6. *Assume* **(D)**. *(i) Let $k \in I_C \cup I_Q$, $0 < \rho \le \rho^*$, $0 < \eta < 1$ and set $E_\rho = B(a_k, \rho) \cap U_k \cap D$. If $u \in H^1(E_\rho)$ is a nonnegative solution of (5.17) with $E = E_\rho$ and satisfies $|\{x : u(x) \ge 1\} \cap E_{\rho/2}| \ge \eta |E_{\rho/2}|$, then there exists a positive constant C_{13} independent of k, ρ and δ such that*

$$\operatorname*{ess\,inf}_{E_{\rho/4}} u \ge C_{13}.$$

(ii) Let $0 < \rho \le \rho^*$, $0 < r \le r_\rho$, $a \in \partial D \setminus \bigcup_{k \in I_C \cup I_Q} B(a_k, \rho)$, $0 < \eta < 1$ and set $E_r = B(a, r) \cap D$. If $u \in H^1(E_r)$ is a nonnegative solution of (5.17) with $E = E_r$ and satisfies $|\{x : u(x) \ge 1\} \cap E_{r/2}| \ge \eta |E_{r/2}|$, then there exists a positive constant C_{14} independent of a, r and δ such that

$$\operatorname*{ess\,inf}_{E_{r/4}} u \ge C_{14}.$$

We now turn to the uniform continuity of $G_\lambda^\delta T$.

Proposition 5.7. *Assume* **(D)** *and the following.*

$$1 < \exists \gamma < 1 + \kappa^*/(d-1) \quad \text{s.t.} \quad \int_{0<|h|<1} |h|^\gamma \nu(h) dh < \infty. \tag{5.18}$$

Let $1 + (d-1)/\kappa^* < p \le \gamma/(\gamma - 1)$, $T \in H^{-1,p}(D)$ and $\lambda \ge 0$. Then $G_\lambda^\delta T$ is uniformly continuous on D, and hence $G_\lambda^\delta T$ can be extended as a continuous function on $\overline{D}$. Furthermore $\{G_\lambda^\delta T\}_{\delta > 0}$ is equi-continuous on $\overline{D}$.

In order to show this proposition, we derive a bound for the jump part $\mathcal{E}^{j,\delta}(G_\lambda^\delta T, \varphi)$.

Lemma 5.3. *Under the same assumption as Proposition 5.7,*

$$|\mathcal{E}^{j,\delta}(G_\lambda^\delta T, \varphi)| \le C_{15} \|\varphi\|_{H_0^{1,p^*}(B(a,r)\cap D)}, \quad \varphi \in C_0^\infty(B(a,r) \cap D), \tag{5.19}$$

where $0 < r \le \rho < \infty$, $1/p + 1/p^* = 1$, and C_{15} is a positive constant independent of a, r and δ.

This lemma is obtained in the same way as Lemma 5.1. So we omit the proof.

Proposition 5.7 is obtained essentially by the same argument as in Theorem 4.2 of [15]. So we only give an outline of the proof.

Proof (of Proposition 5.7): Fix a $k \in I_C$ and $s \in (0, \rho^*/2]$ arbitrarily. Set $E_s = B(a_k, s) \cap U_k \cap D$. We put $u = G_\lambda^\delta T$ and define the operator T^* by

$$\langle T^*, \varphi \rangle := \langle T, \varphi \rangle - \lambda \int_D u \varphi \, dx - \mathcal{E}^{j,\delta}(u, \varphi), \quad \varphi \in C_0^\infty(D).$$

By means of Proposition 5.1 and Lemma 5.3,

$$|\langle T^*, \varphi \rangle| \le c_1 \|\varphi\|_{H_0^{1,p^*}(E_s)}, \qquad \varphi \in C_0^\infty(E_s), \tag{5.20}$$

where c_1 is a positive constant independent of $k,\ s$ and δ. (5.20) implies $T^* \in H^{-1,p}(E_s)$. Therefore there is a unique $v := G^{c,\delta}_{E_s} T^* \in H^1_0(E_s)$. By means of Proposition 5.5 (i),

$$\|v\|_{L^\infty(E_s)} \le c_2 s^{\{p-1-(d-1)/\kappa_k\}/p} \le c_2 s^{\{p-1-(d-1)/\kappa^*\}/p}, \tag{5.21}$$

where c_2 is a positive constant independent of $k,\ s$ and δ.

We set $w = u|_{E_s} - v$. Since $\mathcal{E}^{c,\delta}_{E_s}(u|_{E_s}, \varphi) = \langle T^*, \varphi \rangle$ for $\varphi \in C_0^\infty(E_s)$, we get

$$\begin{cases} w \in H^1(E_s), \\ \mathcal{E}^{c,\delta}_{E_s}(w, \varphi) = 0,\ \varphi \in C_0^\infty(E_s). \end{cases}$$

Following the same argument as in Section 5 of [25] (see also Lemma 7.3 of [28]), we obtain the following

$$\mathrm{Osc}\left(w, E_{s/4}\right) \le c_3 \mathrm{Osc}\left(w, E_s\right),$$

where c_3 is a constant satisfying $0 < c_3 < 1$ independent of $k,\ s$ and δ. Therefore we have

$$\begin{aligned} \mathrm{Osc}\left(u, E_{s/4}\right) &\le \mathrm{Osc}\left(v, E_{s/4}\right) + \mathrm{Osc}\left(w, E_{s/4}\right) \\ &\le 2\|v\|_{L^\infty(E_s)} + c_3 \mathrm{Osc}\left(w, E_s\right) \\ &\le 4\|v\|_{L^\infty(E_s)} + c_3 \mathrm{Osc}\left(u, E_s\right). \end{aligned}$$

Combining this with (5.21) and using Lemma 7.6 of [28], we get

$$\mathrm{Osc}(u, E_s) \le c_4 s^{\xi_1}, \qquad 0 < s \le \rho^*/4,\ k \in I_C,$$

for some positive constant c_4 and $\xi_1 \in (0,1)$, where c_4 and ξ_1 are independent of $k,\ s$ and δ. In the same way we also get

$$\mathrm{Osc}(u, E_s) \le c_5 s^{\xi_2}, \qquad 0 < s \le \rho^*/4,\ k \in I_Q,$$

for some c_5 (> 0) and $\xi_2 \in (0,1)$ independent of $k,\ s$ and δ.

By means of Lemma 5.2, for any $\rho \in (0, \rho^*/2]$, there is a positive number r_ρ such that $B(a, r_\rho) \cap \{a_k : k \in I_C \cup I_Q\} = \varnothing$ for $a \in \partial D^\#_\rho := \partial D \setminus \bigcup_{k \in I_C \cup I_Q} B(a_k, \rho)$. Putting $E_{a,s} = B(a,s) \cap D$, we get in the same way as above,

$$\mathrm{Osc}(u, E_{a,s}) \le c_6 s^{\xi_3}, \qquad 0 < s \le r_\rho/4,\ a \in \partial D^\#_\rho,$$

for some c_6 (> 0) and $\xi_3 \in (0,1)$ independent of $a,\ s$ and δ.

Let η be a positive number fixed arbitrarily and set $D_\eta = \{x \in D : \mathrm{dist}(x, \partial D) < \eta\}$. Since $\overline{B(a,s)} \subset D$ for $0 < s \le \eta/4$ and $a \in D \setminus D_\eta$, combining results from elliptic differential equations (see *e.g.* [16, 25, 28] etc.), we obtain the following

$$\mathrm{Osc}(u, B(a,s)) \le c_7 s^{\xi_4}, \qquad 0 < s \le \eta/4,\ a \in D \setminus D_\eta,$$

for some c_7 (> 0) and $\xi_4 \in (0,1)$ independent of $a,\ s$ and δ. Following the same argument as in the last half of the proof of Theorem 4.2 of [15], we get the uniform continuity of u on D. Since c_i and ξ_i are independent of δ, $\{G^\delta_\lambda T\}_{\delta > 0}$ is equi-continuous on $\overline{D}$. □

Let $(\mathcal{E}^0, H_0^1(D))$ be the Dirichlet form on $L^2(D)$ given in Proposition 4.3 and $\{G_\lambda^0 : \lambda \geq 0\}$ be the L^2-resolvent associated with $(\mathcal{E}^0, H_0^1(D))$. Since the effective matrix A_{eff} is symmetric and satisfies **(A1)** for some positive numbers α_0 and β_0, we immediately obtain the following.

Corollary 5.1. *Assume* **(D)** *and (5.18). Let* $1 + (d-1)/\kappa^* < p \leq \gamma/(\gamma-1)$, $T \in H^{-1,p}(D)$ *and* $\lambda \geq 0$. *Then* $G_\lambda^0 T$ *is uniformly continuous on* D, *and hence* $G_\lambda^0 T$ *can be extended as a continuous function on* $\overline{D}$.

We also note the following.

Corollary 5.2. *Assume* **(D)** *and (5.18). Let* $1 + (d-1)/\kappa^* < p \leq \gamma/(\gamma-1)$, $f \in L^p(D)$ *and* $\lambda \geq 0$. *For any decreasing sequence* $\{\delta_n\}$ *with* $\delta_n \to 0$ *as* $n \to \infty$, $G_\lambda^{\delta_n} f$ *uniformly converges to* $G_\lambda^0 f$ *on* $\overline{D}$.

Proof. By virtue of Theorem 4.2, the sequence $(\mathcal{E}^{\delta_n}, H_0^1(D))$ Mosco-converges to $(\mathcal{E}^0, H_0^1(D))$ on $L^2(D)$. Therefore, for $g \in L^2(D)$, $G_\lambda^{\delta_n} g$ converges to $G_\lambda^0 g$ in $L^2(D)$ as $n \to \infty$. Let $f \in L^p(D)$ where $1 + (d-1)/\kappa^* < p < \gamma/(\gamma-1)$. Since $\{G_\lambda^{\delta_n} f\}_n$ is eqi-bounded and eqi-continuous on $\overline{D}$ by means of Propositions 5.4 and 5.7, there exist a subsequence $\{G_\lambda^{\delta_{n'}} f\}_{n'}$ and a continuous function h on $\overline{D}$ such that $G_\lambda^{\delta_{n'}} f$ converges to h uniformly on $\overline{D}$. Noting $f \in L^2(D)$, we get $h = G_\lambda^0 f$. Since the limit $h = G_\lambda^0 f$ is unique, we obtain the conclusion. □

Let $\mathsf{M}^\delta = (X^\delta(t),\ \mathbb{P}_x^\delta)$ be the symmetric Hunt processes on D associated with $(\mathcal{E}^\delta, H_0^1(D))$ for $\delta > 0$. Then $X^\delta(\cdot) \in \mathbb{D}_{D_\partial}$, where D_∂ is the one-point compactification of D and

$$\mathbb{D}_{D_\partial} := \Big\{ f : [0,\infty) \to D_\partial \,\Big|\, f \text{ is right continuous having left limits} \Big\}.$$

The following is obtained in the same way as Proposition 3.3. Let $\tau_D^\delta = \inf\{t > 0 : X^\delta(t) \notin D\}$.

Proposition 5.8. *Assume* **(D)** *and* (5.18). *Then the law of* $\{X_t^\delta,\ t \in [0,T]\}$ *on* $\{T < \tau_D^\delta\}$ *is tight in the space* $\mathbb{D}_{D_\partial}[0,T]$ *for any* $T > 0$, *where*

$$\mathbb{D}_{D_\partial}[0,T] := \Big\{ f : [0,T] \to D_\partial \,\Big|\, f \textit{ is right continuous having left limits} \Big\}.$$

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References

[1] G. Allaire, A brief introduction to homogenization and miscellaneous applications, ESAIM: Proceedings and Surveys – Mathematical and numerical approaches for multiscale problem–, **37** (2012), 1-49

[2] F. Alouges, Introduction to periodic homogenization, Interdisciplinary Information Sciences, **22** (2016), 147-186

[3] H. Attouch, Variational Convergence for Functions and Operators, –Applicable Mathematics Series, Pitman, 1984

[4] R. Bass, T. Kumagai and T. Uemura, Convergence of symmetric Markov chains on $\mathbb{Z}^d$, Probab. Th. Rel. Fields, **148** (2010), 107-140

[5] A. Bensoussan, J.-L. Lions and G. Papanicolaou, Asymptotic Analysis For Periodic Structures, AMS Chelsea Publishing, 2011 (Originally published from North-Holland Pub. Co., 1978)

[6] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang, Stochastic calculus for symmetric Markov processes, Annal. Prob., **36** (2008), 931-970

[7] Z.-Q. Chen, P. Kim and T. Kumagai, Discrete approximation of symmetric jump processes on metric measure spaces, Probab. Theory Relat. Fields, **155** (2013), 703-749

[8] Z.-Q. Chen and T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps, Rev. Mat. Iberoamericana, **26** (2010), 551-589

[9] D. Cioranescu and P. Donato, An Introduction to Homogenization, Oxford University Press, 1999 (Reprinted, 2010)

[10] S. N. Ethier and T. G. Kurt, Markov Processes : Characterization and Convergence, John Wiley & Sons, 1986

[11] W. Farkas, N. Jacob and R.L. Schilling, Function spaces related to continuous negative definite functions: ψ-Bessel potential spaces, Dissertationes Mathematicae **CCCXCIII** (2001), 1-62

[12] W. Farkas, N. Jacob and R.L. Schilling: Feller semigroups, L^p-sub-Markovian semigroups, and applications to pseudo-differential operators with negative definite symbols, Forum Math., **13** (2001), 51-90

[13] M. Fukushima, On an L^p-estimate of resolvents of Markov Processes, Publ. RIMS, Kyoto Univ., **13** (1977), 277-284

[14] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, 2nd rev. and ext. ed., Walter de Gruyter, 2011

[15] M. Fukushima and M. Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps, Probab. Theory Relat. Fields, **106** (1996), 521-557

[16] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl, Sci. Fis. Mat. Nat., **3** (1957), 25-43

[17] M. Hino, Convergence of non-symmetric forms, J. Math. Kyoto Univ., **38** (1998), 329-341

[18] M. Horie, T. Inuzuka and H. Tanaka, Homogenization of certain one-dimensional discontinuous Markov processes, Hiroshima Math. J., **7** (1977), 629-641

[19] N. Jacob, Pseudo-Differential Operators and Markov Processes (3 vols), Imperial College Press, 2001-2005

[20] J. Jacod and A. N. Shiryaev, Limit Theorems for Stochastic Processes, Springer-Verlag, 1987

[21] D. Lukkassen, G. Nguetseng and P. Wall, Two-scale convergence, Inter. J. Pure and Applied Mathematics, **2** (2002), 33-81

[22] J. Masamune, T. Uemura and J. Wang, On the conservativeness and the recurrence of symmetric jump-diffusions, J. Func. Anal., **263** (2012), 3984-4008

[23] V.G. Maz'ja, Sobolev Spaces – with Applications to Elliptic Partial Differential Equations, 2nd ed., Springer-Verlag, 2011

[24] U. Mosco, Composite media and asymptotic Dirichlet forms, J. Func. Anal., **123** (1994), 368-421

[25] J. Moser, A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations, Comm. Pure Appl. Math., **XIII** (1960), 457-468

[26] R.L. Schilling and T. Uemura, On the Feller property of Dirichlet forms generated by pseudo differential operators, Tohoku Math. J., **59** (2007), 401-422

[27] R.L. Schilling and T. Uemura, Homogenization of symmetric Lévy processes on $\mathbb{R}^d$, to appear in J. Revue Roumaine Math. Pures Appl.

[28] G. Stampacchia, Le problèm de Dirichlet pour les equations elliptiques du second order à coefficient discontinus, *Ann. Inst. Fourier, Grenoble*, **15** (1965), 189-258

[29] K. Suzuki and T. Uemura, On instability of global path properties of symmetric Dirichlet forms under Mosco-convergence, Osaka J. Math., **53** (2016), 567-590

[30] M. Tomisaki, Homogenization of cadlag processes, J. Math. Soc. Japan, **44** (1992), 281-305

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