THE SCHRÖDINGER EQUATION IN L^p SPACES FOR OPERATORS WITH HEAT KERNEL SATISFYING POISSON TYPE BOUNDS

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Abstract. Let L be a non-negative self-adjoint operator acting on $L^2(X)$ where X is a space of homogeneous type with a dimension n. In this paper, we study sharp endpoint L^p-Sobolev estimates for the solution of the initial value problem for the Schrödinger equation $i\partial_t u + Lu = 0$ and show that for all $f \in L^p(X)$, $1 < p < \infty$,

$$\left\|e^{itL}(I+L)^{-\sigma n} f\right\|_p \leq C(1+|t|)^{\sigma n}\|f\|_p, \quad t \in \mathbb{R}, \quad \sigma \geq \left|\frac{1}{2}-\frac{1}{p}\right|,$$

where the semigroup e^{-tL} generated by L satisfies a Poisson type upper bound.

1. Introduction

1.1. **Background and main result.** We will consider the initial value problem for the Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) + Lu(x,t) = 0 & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = f(x), \end{cases} \tag{1.1}$$

where L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ with the heat kernel $h_t(x,y)$ of the semigroup e^{-tL} satisfying the Poisson type upper bound: there exist positive constants C and $m, a > 0$ such that for all $x, y \in \mathbb{R}^n$, $t > 0$,

$$|h_t(x,y)| \leq C t^{-\frac{n}{m}} \left(1 + \frac{|x-y|}{t^{1/m}}\right)^{-n-a}. \tag{1.2}$$

The aim of this paper is to focus on the L^p estimate for the solution of this Schrödinger equation with the sharp index. Our main result in this paper is the following.

Theorem 1.1. *Suppose that L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the above heat kernel upper bound (1.2) with $a > [\frac{n}{2}] + 1$. Then for any $p \in (1, \infty)$, there exist constants $C, C_p > 0$, independent of t and f, such that*

$$\|e^{itL}(I+L)^{-\sigma n} f\|_{L^p(\mathbb{R}^n)} \leq C_p (1+|t|)^{\sigma n} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for any } \sigma \geq \sigma_p := \left|\frac{1}{2}-\frac{1}{p}\right|. \tag{1.3}$$

We point out that when L is the standard Laplacian on $\mathbb{R}^n$, the sharp endpoint L^p-Sobolev estimate was first studied by Miyachi [30, 31]. Later it has been extensively studied in different types

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of Schrödinger equation in $\mathbb{R}^n$, see for example [3, 20, 21, 22], where the key tool is Fourier transform. Recently, Jensen and Nakamura [25, 26] developed an idea on obtaining the L^p estimates of the Schödinger equation by using the commutator method. D'Ancola and Nicola [11] applied this method to prove the uniform local L^p estimates for the solution of this Schrödinger problem, that is, they obtained the sharp estimate of $\|e^{itL}\varphi(L)f\|_{L^p}$ with optimal growth in time and optimal regularity loss, where φ is a compactly supported function in $C^\infty(\mathbb{R})$ and the heat kernel upper bound of the operator L can be relaxed from exponential decay to polynomial decay. See also [4] for the extension of [11] to space of homogeneous type. However, their results cannot be applied to proving (1.3). In [6], the first, second, fourth and fifth authors of this paper studied the Schrödinger equation for nonnegative self-adjoint operator L whose heat kernel satisfies the Gaussian upper bound, and obtained the sharp estimate (1.3). The method in [6] is to use the sharp maximal function estimate, which depends heavily on the important tool of function calculus studied by Christ, Hebisch, McIntosh, Duong et., as well as Blunck and Kunstmann and so on, where the exponential decay of the heat kernel plays an essential role.

Our main result Theorem 1.1 reduces the kernel upper bound to pointwise Poisson bound, and gives the full range of the sharp L^p estimate for the solution to the initial value problem for a Schrödinger equation. To obtain this, we develop several new techniques comparing to the previous closely related results [6, 11, 30, 31]. We now explain these in the next subsection in details.

1.2. **Assumptions, framework and new techniques of the proof.** To show Theorem 1.1, we will work on a more general setting in order to cover many important examples that are in the scope of $\mathbb{R}^n$. Throughout this paper, we assume that X is a metric space, with a distance d and a nonnegative, Borel, doubling measure μ on X satisfying $\mu(X) = +\infty$.

To be more precise, we first recall the basic setup for the metric space X. Let $B(x,r) = \{y \in X : d(x,y) < r\}$ be the open ball with center $x \in X$ and radius $r > 0$ and let $V(x,r) = \mu(B(x,r))$, the volume of $B(x,r)$. We say that (X,d,μ) satisfies the doubling property (see Chapter 3, [10]) if there exists a constant $C > 0$ such that

$$V(x,2r) \le CV(x,r), \quad \forall r > 0,\ x \in X. \tag{1.4}$$

Then the doubling property implies the following strong homogeneity inequality,

$$V(x,\lambda r) \le C\lambda^n V(x,r) \tag{1.5}$$

for some $C, n > 0$ uniformly for all $\lambda \ge 1$ and $x \in X$. In the Euclidean space with Lebesgue measure, n is the dimension of the space. In our results, the critical index is always expressed in terms of the homogeneous dimension n. Note also that there exist c and D, $0 \le D \le n$ so that

$$V(y,r) \le c\Big(1 + \frac{d(x,y)}{r}\Big)^D V(x,r) \tag{1.6}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, the property (1.6) with $D = n$ is a direct consequence of the triangle inequality with respect to the metric d and the strong homogeneity property. In the cases of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, D can be chosen to be 0.

Suppose that L is a non-negative self-adjoint operator on $L^2(X)$, one can formally define an Schrödinger group e^{itL}, using the spectral theory for L. Assume that L has a spectral resolution:

$$Lf = \int_0^\infty \lambda dE_L(\lambda) f, \quad f \in L^2(X),$$

where $E_L(\lambda)$ is the projection-valued measure supported on the spectrum of L. Then the operator e^{itL} is defined by

$$e^{itL} f = \int_0^\infty e^{it\lambda} dE_L(\lambda) f \tag{1.7}$$

for $f \in L^2(X)$, and forms the Schrödinger group. By the spectral theorem ([28]), the operator e^{itL} is continuous on $L^2(X)$. Assuming $f \in L^2(X)$, $u(x,t) = e^{itL} f$ solves the following initial value problem for the Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) + Lu(x,t) = 0 & x \in X, \quad t > 0, \\ u(x,0) = f(x). \end{cases}$$

It is interesting to investigate L^p-mapping properties for the Schrödinger group e^{itL} on $L^p(X)$ for some $p, 1 \le p \le \infty$.

We now introduce the polynomial off-diagonal estimate $(\mathrm{PEV}^{a,m}_{p_0,2})$ for L, which, when moving back to $\mathbb{R}^n$, is weaker than the Poisson type decay (1.2).

Definition 1.2. *Let L be a non-negative self-adjoint operator on $L^2(X)$ and $m > 0$, $a > 0$. We say that the semigroup e^{-tL} generated by L, satisfies the property $(\mathrm{PEV}^{a,m}_{p_0,2})$ if there exists a constant $C > 0$ such that for all $x, y \in X$, $t > 0$,*

$$\|\chi_{B(x,t^{1/m})} e^{-tL} V^{\sigma_{p_0}}_{t^{1/m}} \chi_{B(y,t^{1/m})}\|_{p_0 \to 2} \le C \left(1 + \frac{d(x,y)}{t^{1/m}}\right)^{-n-a}, \tag{$\mathrm{PEV}^{a,m}_{p_0,2}$}$$

where $V^{\sigma_{p_0}}_{t^{1/m}}$ is a pointwise multiplier operator defined by $V^{\sigma_{p_0}}_{t^{1/m}} f(x) := V(x, t^{1/m})^{\sigma_{p_0}} f(x)$.

Note that if the semigroup e^{-tL} has integral kernel $h_t(x,y)$ satisfying the following Poisson type upper bound:

$$|h_t(x,y)| \le C V(x, t^{1/m})^{-1} \left(1 + \frac{d(x,y)}{t^{1/m}}\right)^{-n-a} \tag{1.8}$$

for all $t > 0$ and all $x, y \in X$, then L satisfies the property $(\mathrm{PEV}^{a,m}_{p,2})$ with $p = 1$. Based on the condition $(\mathrm{PEV}^{a,m}_{p_0,2})$, our main result is the following, which covers Theorem 1.1 when we restrict our (X, d, μ) to the setting of $\mathbb{R}^n$.

Theorem 1.3. *Suppose that (X, d, μ) is a space of homogeneous type with a dimension n and that L satisfies the property $(\mathrm{PEV}^{\kappa,m}_{p_0,2})$ for some $1 \le p_0 < 2$, $m > 0$ and $\kappa > \kappa_0 := [\frac{n}{2}] + 1$. Then for any $p \in (p_0, p_0')$, there exist a constant $C_p > 0$, independent of t and f, such that*

$$\|e^{itL}(I + L)^{-\sigma_p n} f\|_{L^p(X)} \le C_p (1 + |t|)^{\sigma_p n} \|f\|_{L^p(X)}. \tag{1.9}$$

As a consequence, this estimate (1.9) holds for all $1 < p < \infty$ when the heat kernel of L satisfies a Poisson type upper bound (1.8) for $a > \kappa_0$.

Remark 1.4. *(i). One can also consider the L^p boundedness of Schrödinger groups for the self-adjoint operator L having a lower bound such that $L + M_0 \geq 0$ as in* [4, 35]*, where $M_0 \geq 0$. In this setting, if L satisfies the off-diagonal estimate*

$$\|\chi_{B(x,t^{1/m})} e^{-tL} V^{\sigma_{p_0}}_{t^{1/m}} \chi_{B(y,t^{1/m})}\|_{p_0\to 2} \leq C e^{tM_0} \left(1 + \frac{d(x,y)}{t^{1/m}}\right)^{-n-\kappa},$$

for some $1 \leq p_0 < 2$, $m > 0$ and $\kappa > \kappa_0 := [\frac{n}{2}] + 1$, then for any $M > M_0$ and $p \in (p_0, p_0')$, by considering the non-negative self-adjoint operator $L_{M_0} := L + M_0$ as in [4]*, we have*

$$\|e^{itL}(M + L)^{-\sigma_p n} f\|_{L^p(X)} \leq C_p (1 + |t|)^{\sigma_p n} \|f\|_{L^p(X)}.$$

(ii). Under the assumption of $(\mathrm{PEV}^{\kappa,m}_{p_0,2})$*, we can see by a similar argument in the proof of Theorem 1.3 (see also* [6, Theorem 4.1]*) that for any $p \in (p_0, p_0')$, $\|e^{itL}(I+tL)^{-\sigma_p}\|_{p\to p} \leq C$. This, together with Theorem 5.5, implies the L^p boundedness of local Schrödinger flow $\|e^{itL}\varphi(L)f\|_{L^p}$, where φ is a compactly supported function in $C^\infty(\mathbb{R}_+)$. That is,*

$$\|e^{itL}\varphi(2^{-k}L)\|_{p\to p} \leq C_p (1 + 2^k|t|)^{\sigma_p}.$$

This result is already proved in [11] *on $\mathbb{R}^n$ and in* [4] *on homogeneous spaces.*

To show Theorem 1.3, we first study a Hardy space $H^1_L(X)$ defined via a suitable Littlewood–Paley area function and then we show that such a Hardy space allows molecular decomposition and complex interpolation. Hence, this theorem can be reduced to proving the endpoint estimate of $e^{itL}(I + L)^{-\sigma_p n}$, that is,

Proposition 1.5. *Suppose that (X, d, μ) is a space of homogeneous type with a dimension n and that L satisfies the property* $(\mathrm{PEV}^{\kappa,m}_{2,2})$ *for some $m > 0$ and $\kappa > \kappa_0 := [\frac{n}{2}] + 1$. Then there exists a constant $C > 0$, independent of t and f, such that*

$$\|e^{itL}(I + L)^{-n/2} f\|_{H^1_L(X)} \leq C(1 + |t|)^{n/2} \|f\|_{H^1_L(X)}.$$

This result extends the main theorem in [7] to the boundedness from H^1_L to H^1_L. To obtain the above proposition, our main approach is to establish the following new off-diagonal estimates for the oscillatory spectral multiplier $e^{itL}F(L)$.

Proposition 1.6. *There exist constants $C, c_0 > 0$ such that for any ball $B \subset X$ with radius r_B and for any $\lambda > 0$, $j \geq 6$,*

$$\left\|\chi_{U_j(B)} e^{itL} F(L) \chi_B f\right\|_2 \leq C 2^{-j\kappa_0} (\sqrt[m]{\lambda} r_B)^{-\kappa_0} (1 + \lambda|t|)^{\kappa_0} (\sqrt[m]{\lambda} r)^{-c_0} \|\delta_\lambda F\|_{C^{\kappa_0+1}} \|f\|_2$$

for all Borel functions F such that supp$F \subseteq [-\lambda, \lambda]$, where $r = \min\{r_B, \lambda^{-1/m}\}$.

We would like to point out that:

(1) Under the assumption of the Gaussian upper bounds (GE_m) of an operator L, the Phragmén-Lindelöf Theorem is the central tool for the optimal extension of the $L^2 - L^2$ off-diagonal estimates of the real semigroup $e^{-\tau L}$ for real values $\tau \in \mathbb{R}_+$ to that for complex values $z \in \mathbb{C}_+$. By using complex semigroup $e^{-(i\tau-1)\lambda^{-1}L}$ to represent spectral multiplier $F(L)$, the authors

in [7] (see also [18]) showed that for any $s \geq 0$, there exists a constant $C > 0$ such that for any $j \geq 2$,

$$\|\chi_{U_j(B)} F(L) \chi_B\|_{2\to 2} \leq C(\sqrt[m]{\lambda} 2^j r_B)^{-s} \|F(\lambda \cdot)\|_{B^s}$$

for all balls $B \subset X$, and all Borel functions F such that supp$F \subset [-\lambda, \lambda]$. It should be noted that due to the appearance of the Besov norm $\|\cdot\|_{B^s}$, this inequality can be used to obtain a sharp $L^2 - L^2$ off-diagonal estimate of compactly supported spectral multiplier with an oscillatory term e^{itL}, which plays a crucial role in showing the boundedness on H^1_L for Schrödinger groups.

However, under a mild decay assumption ($\mathrm{PEV}^{\kappa,m}_{p_0,2}$) or the Gaussian upper bounds (GE_m) for $m < 2$, Phragmén-Lindelöf Theorem cannot be applied to obtaining a suitable substitution for e^{-zL} in a similar way.

(2) Without (GE_m), inspired by Davies's work ([12]), Duong and Robinson ([15]) used Poisson formula for subharmonic function to obtain the off-diagonal decay of the heat kernel $K_{e^{-zL}}(x, y)$ for the special case $X = \mathbb{R}^n$ and $p_0 = 1$.

However, such an estimate is not enough to obtain the required off-diagonal estimates of $F(L)$, since the off-diagonal decay becomes slower and disappears gradually, as the angle $\arg z$ increases from 0 to $\frac{\pi}{2}$.

Therefore, the previous methods cannot be expected to obtain off-diagonal estimates of $F(L)$.

To overcome this main difficulty, we develop a completely different method to obtain a suitable replacement by means of amalgam blocks and commutators. To illustrate that, we split the whole process into three steps:

• **Step 1:** Inspired by [8], by representing the spectral multiplier $F(L)$ as $E(e^{-\frac{L}{\lambda}})e^{-\frac{L}{\lambda}}$ and then studying the off-diagonal properties of the operators $E(e^{-\frac{L}{\lambda}})$ and $e^{-\frac{L}{\lambda}}$ separately, we obtain $L^2 - L^2$ amalgam-type off-diagonal estimates of $F(L)$ (see Lemma 3.5);

• **Step 2:** Define $R_\lambda := (I + \lambda^{-1}L)^{-1}$. Inspired by [11], by representing the oscillatory spectral multiplier $e^{itL}F(L)$ as $R_\lambda^{2^{\kappa+1}-2} e^{itL} \widetilde{F}(L)$, where $\widetilde{F}$ is a compactly supported Borel function satisfying supp$\widetilde{F} \subset [-\lambda, \lambda]$, and then studying the off-diagonal property of the operator $R_\lambda^{2^{\kappa+1}-2} e^{itL}$, we obtain $L^2 - L^2$ amalgam-type off-diagonal estimates of $e^{itL}F(L)$ (see Lemma 3.10);

• **Step 3:** By embedding the set $U_{j-1}(B)$ into a countable union of amalgam block with a suitable size (which is different from the one chosen in [8] and [11], since there are two parameters that we need to consider: the radius of B and the size of supp$F$), we obtain $L^2 - L^2$ off-diagonal estimates of $e^{itL}F(L)$ (see Proposition 1.6).

1.3. **Applications.** Our results, Theorems 1.1 and 1.3 can be applied to all examples which are discussed in [5, 6, 8, 11, 14].

We now provide two more particular examples:

1. Fractional Schrödinger operator with potentials on $\mathbb{R}^n$.

Let $n \geq 1$ and W_1, W_2 be locally integrable non-negative functions on $\mathbb{R}^n$. Consider the fractional Schrödinger operator with potentials W_1 and W_2:

$$L = (-\Delta + W_1)^{\beta} + W_2(x), \quad \beta \in (0, 1].$$

The particular case $\beta = 1/2$ is often referred to the relativistic Schrödinger operator. The operator L is self-adjoint as an operator associated with a well defined closed quadratic form. By the classical subordination formula (see for example, [19, Section 5.4]) together with the Feynman-Kac formula it follows that the semigroup kernel $h_t(x, y)$ associated to e^{-tL} satisfies the estimate

$$0 \leq h_t(x, y) \leq Ct^{-n/2\beta} \left(1 + t^{-\frac{1}{2\beta}}|x - y|\right)^{-(n+2\beta)}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. Hence, estimate (1.2) holds for $m = 2\beta$ and $\alpha = 2\beta$. If $n = 1$ and $\beta > 1/2$, then we apply Theorem 1.1 to obtain the sharp L^p-Sobolev estimate (1.3) for the Schrödinger equation (1.1) for the operator L.

2. Sub-Laplacian on certain Carnot–Carathédory spaces developed by Nagel and Stein [33].

We first recall the background of this setting [33]. Let M be a connected smooth manifold and $\{\mathbb{X}_1, \cdots, \mathbb{X}_k\}$ are k given smooth real vector fields on M satisfying Hörmander condition of order m, i. e., these vector fields together with their commutators of order $\leq m$ span the tangent space to M at each point.

It was shown in [33] that there is a pseudo-metric d on M such that $d(x, y)$ is C^∞ on $M \times M \backslash \{\text{diagonal}\}$, and for $x \neq y$

$$|\partial_X^K \partial_Y^L d(x, y)| \lesssim d(x, y)^{1-K-L}.$$

Here ∂_X^K are products of K vector fields $\{X_1, \cdots X_k\}$ acting as derivatives on the x variable, and ∂_Y^L are corresponding L vector fields acting on the y variable. There is also a doubling measure on M, which was given in each specific example of M.

Consider the sub-Laplacian $\mathcal{L}$ on M in self-adjoint form, given by

$$\mathcal{L} = \sum_{j=1}^{k} \mathbb{X}_j^* \mathbb{X}_j.$$

Here $(\mathbb{X}_j^* \varphi, \psi) = (\varphi, \mathbb{X}_j \psi)$, where $(\varphi, \psi) = \int\limits_M \varphi(x)\bar{\psi}(x)dx$, and $\varphi, \psi \in C_0^\infty(M)$, the space of C^∞ functions on M with compact support. In general, $\mathbb{X}_j^* = -\mathbb{X}_j + a_j$, where $a_j \in C^\infty(M)$. The solution of the following initial value problem for the heat equation

$$\frac{\partial u}{\partial s}(x, s) + \mathcal{L}_x u(x, s) = 0$$

with $u(x, 0) = f(x)$ is given by $u(x, s) = H_s(f)(x)$, where H_s is the operator given via the spectral theorem by $H_s = e^{-s\mathcal{L}}$, and an appropriate self-adjoint extension of the non-negative operator $\mathcal{L}$ initially defined on $C_0^\infty(M)$. Nagel and Stein proved that for $f \in L^2(X)$,

$$H_s(f)(x) = \int_M H(s, x, y) f(y) d\mu(y)$$

and the heat kernel $H(s,x,y)$ satisfy the following property (see Proposition 2.3.1 in [33] and Theorem 2.3.1 in [32]):

For every integer $N \geq 0$,

$$|H(s,x,y)| \lesssim \frac{1}{V(x,d(x,y)) + V(x,\sqrt{s}) + V(y,\sqrt{s})} \Big(\frac{\sqrt{s}}{d(x,y)+\sqrt{s}} \Big)^{\frac{N}{2}},$$

which implies (1.8) obviously. Hence, we can apply Theorem 1.1 to obtain the sharp L^p-Sobolev estimate (1.3) for the Schrödinger equation (1.1) for the operator $\mathcal{L}$.

1.4. **Notation and structure of the paper.** For $1 \leq p \leq +\infty$, we denote the norm of a function $f \in L^p(X,d\mu)$ by $\|f\|_p$. If T is a bounded linear operator from $L^p(X,d\mu)$ to $L^q(X,d\mu)$, $1 \leq p,q \leq +\infty$, we write $\|T\|_{p\to q}$ for the operator norm of T. The indicator function of a subset $E \subseteq X$ is denoted by χ_E. Besides, let $\mathcal{D}(T)$ be the domain of an operator T. Throughout the paper, V_s is a pointwise multiplier operator defined by $V_s f(x) := V(x,s)f(x)$ and $\delta_r F$ is the dilation of a function F, defined by $\delta_r F(x) := F(rx)$. Recall that n is the dimension of the space X, we will write

$$\kappa_0 = \left[\frac{n}{2}\right] + 1, \quad \text{and} \quad \sigma_p = \left|\frac{1}{p} - \frac{1}{2}\right|, \quad 1 \leq p \leq \infty. \tag{1.10}$$

Also, for any given ball B in X we set

$$U_0(B) := B \quad \text{and} \quad U_j(B) := 2^j B \backslash 2^{j-1} B, \quad j = 1,2,\ldots. \tag{1.11}$$

This paper is organised as follows. In Section 2 we provide the preliminaries, including the fundamental properties of the off-diagonal estimate ($\mathrm{PEV}^{a,m}_{p_0,2}$) and the Hardy space associated with the operator L satisfying ($\mathrm{PEV}^{a,m}_{p_0,2}$). In Section 3, we develop new techniques on the off-diagonal estimates for the compactly supported spectral multipliers (Proposition 3.1) and the oscillatory compactly supported spectral multipliers (Proposition 1.6). In Section 4 we prove our main result Theorem 1.3. In the last section we provide some results on the molecular decomposition and interpolation of the Hardy space $H^1_L(X)$.

2. Preliminaries

2.1. **Basic properties of** ($\mathrm{PEV}^{a,m}_{p_0,2}$)**.** In this subsection, we recall some basic properties of ($\mathrm{PEV}^{a,m}_{p_0,2}$), which were essentially discussed in [8]. Assume that L satisfies the property ($\mathrm{PEV}^{a,m}_{p_0,2}$) for some $1 \leq p_0 < 2$ and $m > 0$. By Hölder's inequality, the property ($\mathrm{PEV}^{a,m}_{p_0,2}$) implies that for any $p_0 \leq p \leq 2$,

$$\begin{aligned} \|\chi_{B(x,\lambda^{1/m})} e^{-\lambda L} V^{\sigma_p}_{\lambda^{1/m}} \chi_{B(y,\lambda^{1/m})} f\|_2 &\leq C\left(1 + \frac{d(x,y)}{\lambda^{1/m}}\right)^{-n-a} \|V^{\frac{1}{p}-\frac{1}{p_0}}_{\lambda^{1/m}} \chi_{B(y,\lambda^{1/m})} f\|_{p_0} \\ &\leq C\left(1 + \frac{d(x,y)}{\lambda^{1/m}}\right)^{-n-a} \|f\|_p. \end{aligned} \tag{$\mathrm{PEV}^{a,m}_{p,2}$}$$

In particular, we have

$$\|\chi_{B(x,\lambda^{1/m})} e^{-\lambda L} \chi_{B(y,\lambda^{1/m})}\|_{2\to 2} \leq C\left(1 + \frac{d(x,y)}{\lambda^{1/m}}\right)^{-n-a}.$$

Next, we divide X into countable partitions with different size parameters. For every $r > 0$, we choose a sequence $\{x_i\}_{i=1}^{\infty} \in X$ such that $d(x_i, x_j) > \frac{r}{2}$ for $i \neq j$ and $\sup_{x\in X} \inf_i d(x, x_i) \leq \frac{r}{2}$. Such sequence exists since X is separable. Set

$$D = \bigcup_{i\in\mathbb{N}} B(x_i, r/4).$$

Then we define the amalgam block $Q_i(r)$ by the formula

$$Q_i(r) = B(x_i, r/4) \bigcup \left[B\left(x_i, r/2\right) \setminus \left(\bigcup_{j<i} B\left(x_j, r/2\right) \setminus D \right) \right],$$

so that $\{Q_i(r)\}_i$ is a countable partition of X. Namely,

$$X = \bigcup_{i\in\mathbb{N}} Q_i(r), \tag{2.1}$$

where $Q_i(r) \cap Q_j(r) = \emptyset$ if $i \neq j$. We say that x_i is the center of $Q_i(r)$ and r is the diameter of $Q_i(r)$. Such a partition of X is not unique. For a fixed partition, let $\mathcal{I}_r$ be a index set consisting of all $i \in \mathbb{N}$ such that

$$i \in \mathcal{I}_r \Leftrightarrow x_i \text{ is the center of } Q_i(r).$$

Observe that $Q_i(r) \subset B(x_i, r)$ and there exists a uniform constant $C > 0$ depending only on the doubling constant such that $\mu(Q_i(r)) \geq C\mu(B_i)$.

The following lemma is a simple consequence of the estimate $(\mathrm{PEV}_{p,2}^{a,m})$.

Lemma 2.1. *Let $p_0 \leq p \leq 2$, then there exists a constant $C > 0$ such that for any $\lambda > 0$,*

$$\left\| e^{-\frac{L}{\lambda}} V_{\lambda^{-1/m}}^{\sigma_p} f \right\|_2 \leq C\|f\|_p.$$

Proof. The definition of amalgam block allows us to decompose $X = \cup_{\alpha\in\mathbb{N}} Q_\alpha(\lambda^{-1/m})$, and then

$$\begin{aligned}
\left\| e^{-\frac{L}{\lambda}} V_{\lambda^{-1/m}}^{\sigma_p} f \right\|_2 &= \left(\sum_{\beta\in\mathcal{I}_{\lambda^{-1/m}}} \sum_{\alpha\in\mathcal{I}_{\lambda^{-1/m}}} \left\| \chi_{Q_\alpha(\lambda^{-1/m})} e^{-\frac{L}{\lambda}} V_{\lambda^{-1/m}}^{\sigma_p} \chi_{Q_\beta(\lambda^{-1/m})} f \right\|_2^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{\beta\in\mathcal{I}_{\lambda^{-1/m}}} \sum_{\alpha\in\mathcal{I}_{\lambda^{-1/m}}} \left(1 + \frac{d(x_\alpha, x_\beta)}{\lambda^{1/m}} \right)^{-2n-2a} \|\chi_{Q_\beta(\lambda^{-1/m})} f\|_p^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{\beta\in\mathcal{I}_{\lambda^{-1/m}}} \|\chi_{Q_\beta(\lambda^{-1/m})} f\|_p^2 \right)^{\frac{1}{2}} \\
&\leq C\|f\|_p,
\end{aligned}$$

where in the last inequality we used the embedding $\ell^p \subset \ell^2$. □

As a corollary, we can show the following $L^{p_0} - L^2$ spectral multiplier theorem.

Lemma 2.2. *Let $p_0 \leq p \leq 2$, then there exist constants $C, c_0 > 0$ such that*

$$\|F(L) V_{\lambda^{-1/m}}^{\sigma_p} f\|_2 \leq C\|\delta_\lambda F\|_{C^1}\|f\|_p$$

for all Borel functions F such that supp$F \subseteq [-\lambda, \lambda]$.

Proof. Let $E(\tau) := F(-\lambda \log\tau)\tau^{-1}$ so that $F(L) = E(e^{-\frac{L}{\lambda}})e^{-\frac{L}{\lambda}}$. Then it follows from the Fourier inversion formula that

$$E(e^{-\frac{L}{\lambda}}) = \int_{-\infty}^{+\infty} e^{i\xi e^{-\frac{L}{\lambda}}} \hat{E}(\xi) d\xi.$$

This, together with the spectral theorem and Lemma 2.1, yields that

$$\begin{aligned}
\|F(L)V^{\sigma_p}_{\lambda^{-1/m}} f\|_2 &= \Big\| \int_{-\infty}^{+\infty} e^{i\xi e^{-\frac{L}{\lambda}}} e^{-\frac{L}{\lambda}} V^{\sigma_p}_{\lambda^{-1/m}} f \hat{E}(\xi) d\xi \Big\|_2 \\
&\le \int_{-\infty}^{+\infty} \Big\| e^{i\xi e^{-\frac{L}{\lambda}}} e^{-\frac{L}{\lambda}} V^{\sigma_p}_{\lambda^{-1/m}} f \Big\|_2 |\hat{E}(\xi)| d\xi \\
&\le \|\hat{E}\|_1 \Big\| e^{-\frac{L}{\lambda}} V^{\sigma_p}_{\lambda^{-1/m}} f \Big\|_2 \\
&\le C\|\delta_\lambda F\|_{C^1} \|f\|_p,
\end{aligned}$$

where in the last inequality we used the fact that supp$\delta_\lambda F \subset [-1, 1]$ implies that

$$\int_{-\infty}^{+\infty} |\hat{E}(\xi)| d\xi \le C\|E\|_{H^1} \le C\|\delta_\lambda F\|_{H^1} \le C\|\delta_\lambda F\|_{C^1}. \tag{2.2}$$

This finishes the proof of Lemma 2.2. □

2.2. **Preliminaries on Hardy space $H^p_L(X)$.** There were numerous number of references (see for example, [2, 13, 16, 17, 23, 27]) studying the theory of the Hardy spaces associated with certain operators, especially those ones satisfying the Gaussian upper bounds ($\mathrm{GE_m}$). At the beginning of this section, for any $1 \le p \le 2$, we will extend some basic definitions to the Hardy space H^p_L associated with operators which only satisfy the estimate ($\mathrm{PEV}^{\kappa,m}_{2,2}$) for some $m > 0$ and $\kappa > \kappa_0$.

In this article, we will define the Hardy space $H^p_L(X)$ associated with operators in terms of Littlewood-Paley type area function instead of the semigroup factor $t^m L e^{-t^m L}$. Thanks to the compactly supported property of the Littlewood-Paley function and the off-diagonal estimate (3.1), we can see below that in our setting, this definition is much more convenient to obtain the L^p boundedness for Schrödinger groups.

Usually, we should define the L^2 adapted Hardy space $\mathcal{H}^2(X) := \overline{R(L)}$, that is, the closure of the range of L in $L^2(X)$. Then $L^2(X)$ is the orthogonal sum of $\mathcal{H}^2(X)$ and the null space $N(L)$. However in our setting, that is, if L satisfies the property ($\mathrm{PEV}^{\kappa,m}_{p_0,2}$), then $N(L) = \{0\}$. Indeed, it can be verified that for any $f \in N(L)$ and $t > 0$, we have $e^{-tL} f = f$ and therefore, by the condition ($\mathrm{PEV}^{\kappa,m}_{p_0,2}$),

$$\|f\|_{p_0'} \le \lim_{t\to+\infty} \sum_{i \in \mathcal{I}_{t^{1/m}}} \|\chi_{B(x,t^{1/m})} e^{-tL} \chi_{B(x_i,t^{1/m})} f\|_{p_0'} \le C \lim_{t\to+\infty} V(x, t^{1/m})^{-\sigma_{p_0}} \|f\|_2 = 0.$$

So in our setting, $\mathcal{H}^2(X) = L^2(X)$.

Next, let ϕ be a non-negative cut-off function on $C^\infty_c(\mathbb{R})$ such that supp$\phi \subset (1/4, 1)$ and define

$$S_{L,\phi}(f) := \left(\int_0^\infty \int_{d(x,y)<\tau^{1/m}} |\phi(\tau L) f(y)|^2 \frac{d\mu(y)}{V(x, \tau^{1/m})} \frac{d\tau}{\tau} \right)^{\frac{1}{2}}.$$

Definition 2.3. Suppose that L satisfies the estimate ($\mathrm{PEV}_{2,2}^{\kappa,m}$) for some $m > 0$ and $\kappa > \kappa_0$. For any $1 \le p \le 2$, we define the Hardy space $H_L^p(X)$ associated with L be the completion of the space

$$\{f \in \mathcal{H}^2(X) : \ S_{L,\phi}(f) \in L^p(X)\}$$

endowed with the norm

$$\|f\|_{H_L^p(X)} = \|S_{L,\phi} f\|_{L^p(X)}.$$

Remark 2.4. It can be seen from Lemmas 2.7, 2.8, 2.9 and [13, Theorem 3.15, Proposition 4.4] that under the assumption that L satisfies Davies-Gaffney estimates (DG_m) for $m \ge 2$, that is, there exist constants $C, c > 0$ such that for all $\lambda > 0$, and all $x, y \in X$,

$$(\mathrm{DG}_m) \qquad \|\chi_{B(x,\lambda^{1/m})} e^{-\lambda L} \chi_{B(y,\lambda^{1/m})}\|_{2\to 2} \le C\exp\left(-c\left(\frac{d(x,y)}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right),$$

then the Hardy space $H_L^p(X)$ coincides with the Hardy space in some previous references (see for example, [2, 13, 16, 17, 23, 27]).

It is easy to show, by combining Lemmas 2.7 and 2.8, that this definition is independent of the choice of ϕ.

An important tool to study the endpoint Hardy space $H_L^1(X)$ is the molecule decomposition. To illustrate that, we need the following definition of $(1, 2, M, \epsilon)$-molecule associated with the operator L.

Definition 2.5. We say that a function $a(x) \in L^2(X)$ is called a $(1, 2, M, \epsilon)$-molecule associated with L for some $\epsilon > 0$ and $m \in \mathbb{N}$ if there exists a function $b \in \mathcal{D}(L^M)$ and a ball $B = B(x_B, r_B)$ such that

(i) $a = L^M b$;

(ii) For every $k = 0, 1, 2, \ldots, M$ and $j = 0, 1, 2, \ldots,$

$$\|(r_B^m L)^k b\|_{L^2(U_j(B))} \le 2^{-j\epsilon} r_B^{mM} \mu(2^j B)^{-\frac{1}{2}},$$

where we denote $\mathcal{D}(T)$ be the domain of an operator T.

To continue, we define the molecular Hardy spaces associated with L as follows.

Definition 2.6. Let $\{\lambda_j\}_{j=0}^{\infty} \in \ell^1$, $\{a_j\}$ is a sequence of $(1, 2, M, \epsilon)$-molecule, we say that $f = \sum_j \lambda_j a_j$, where the sum converges in $L^2(X)$, is a molecular $(1, 2, M, \epsilon)$-representation of f. Set

$$\mathbb{H}^1_{L,mol,M,\epsilon}(X) = \{f : f \text{ has a molecular } (1, 2, M, \epsilon) - \text{representation}\},$$

endowed with the norm

$$\|f\|_{H^1_{L,mol,M,\epsilon}(X)} = \inf\left\{\sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j m_j \text{ is a molecular } (1, 2, M, \epsilon) - \text{representation}\right\}.$$

We define the space $H^1_{L,mol,M,\epsilon}(X)$ as the completion of $\mathbb{H}^1_{L,mol,M,\epsilon}(X)$ with respect to this norm.

Now, we point out that the Hardy space $H_L^1(X)$ allows molecular decomposition and complex interpolation. To be precise, we prove the following three lemmas.

Lemma 2.7. *Suppose that (X, d, μ) is a space of homogeneous type with a dimension n and that L satisfies the property $(\mathrm{PEV}^{\kappa,m}_{2,2})$ for some $m > 0$ and $\kappa > \kappa_0$. Let M be a sufficient large constant, then we have $H^1_{L,mol,M,\epsilon}(X) = H^1_L(X)$ with equivalent norm, that is*

$$\|f\|_{H^1_{L,mol,M,\epsilon}(X)} \approx \|f\|_{H^1_L(X)}.$$

Lemma 2.8. *Suppose that (X, d, μ) is a space of homogeneous type with a dimension n and that L satisfies the property $(\mathrm{PEV}^{\kappa,m}_{2,2})$ for some $m > 0$ and $\kappa > \kappa_0$. Suppose $1 \le p_1 < p_2 < \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_1 + \theta/p_1$. Then*

$$[H^{p_1}_L(X), H^{p_2}_L(X)]_\theta = H^p_L(X),$$

where we recall that $[\cdot,\cdot]$ denotes the complex interpolation bracket.

Lemma 2.9. *Suppose that (X, d, μ) is a space of homogeneous type with a dimension n and that L satisfies the property $(\mathrm{PEV}^{\kappa,m}_{p_0,2})$ for some $1 \le p_0 < 2$, $m > 0$ and $\kappa > \kappa_0$. Then for any $p \in (p_0, 2]$, we have $H^p_L(X) = L^p(X)$ with equivalent norm, that is*

$$\|f\|_{H^p_L(X)} \approx \|f\|_{L^p(X)}.$$

Proof of Lemmas 2.7, 2.8 and 2.9 will be given in Section 5.

3. Off-diagonal estimates

This section is devoted to presenting some off-diagonal estimates for different kinds of spectral multipliers.

3.1. Off-diagonal estimates for compactly supported spectral multipliers.

In the following two subsections, we will develop a new method to obtain off-diagonal estimates for compactly supported spectral multipliers with oscillatory terms by means of the theory of amalgam block and some techniques related to commutators (see Lemma 3.5 and Lemma 3.10). These estimates play crucial roles in obtaining the sharp boundedness for Schrödinger groups. In the first step, we show off-diagonal estimates for compactly supported spectral multipliers without oscillatory term. That is, we will show the following proposition.

Proposition 3.1. *Let $p_0 \le p \le 2$, then there exist constants $C, c_0 > 0$ such that for any ball $B \subset X$ with radius r_B and for any $\lambda > 0$, $j \ge 5$,*

$$\|\chi_{U_j(B)} F(L) \chi_B\|_{p\to 2} \le C 2^{-j\kappa_0} \mu(B)^{-\sigma_p} (\sqrt[m]{\lambda} r_B)^{-\kappa_0 + n\sigma_p} (\sqrt[m]{\lambda} r)^{-c_0} \|\delta_\lambda F\|_{C^{\kappa_0+1}} \tag{3.1}$$

for all Borel functions F such that $\mathrm{supp} F \subseteq [-\lambda, \lambda]$, where $r = \min\{r_B, \lambda^{-1/m}\}$.

To begin with, we set $\Gamma(j, 0) = 1$ for $j \ge 1$, and we define $\Gamma(j, k)$ inductively by $\Gamma(j, k+1) := \sum_{\ell=k}^{j-1} \Gamma(\ell, k)$ for $1 \le k \le j-1$. For a given $r > 0$, we denote commutator inductively by

$$\mathrm{Ad}^0_{\ell,r}(T) := T;$$

$$\mathrm{Ad}^k_{\ell,r}(T) := \mathrm{Ad}^{k-1}_{\ell,r}\left(\frac{d(x_\ell, \cdot)}{r} T - T \frac{d(x_\ell, \cdot)}{r}\right), k \ge 1.$$

To continue, we recall a known formula for commutator of a Lipschitz function and an operator T on $L^2(X)$.

Lemma 3.2. *Let T be a self-adjoint operator on $L^2(X)$. Assume that for some $\eta \in \mathrm{Lip}(X)$, the commutator $[\eta, T]$, defined by $[\eta, T]f := \eta T f - T(\eta f)$, satisfies that for any $f \in \mathcal{D}(T)$, $\eta f \in \mathcal{D}(T)$ and that $[\eta, T]$ is bounded on $L^2(X)$. Then the following formula holds:*

$$[\eta, e^{itT}]f = it \int_0^1 e^{istT}[\eta, T]e^{i(1-s)tT} f ds, \quad \forall t \in \mathbb{R}, \ \forall f \in L^2(X).$$

Proof. The proof was given in [29]. □

Next, we recall a criterion for $L^p - L^q$ boundedness for linear operators.

Lemma 3.3. *Let T be a linear operator and $1 \le p \le q \le \infty$. For every $r > 0$,*

$$\|T\|_{p\to q} \le \sup_j \sum_i \|\chi_{Q_i(r)} T \chi_{Q_j(r)}\|_{p\to q} + \sup_i \sum_j \|\chi_{Q_i(r)} T \chi_{Q_j(r)}\|_{p\to q},$$

where $\{Q_i(r)\}_i$ is a countable partition of X.

Proof. The proof was given in [8, Lemma 2.1]. □

A direct consequence of this criterion is the following estimate.

Lemma 3.4. *There exists a constant $C > 0$, such that for any $\ell, \lambda > 0$, $\xi \in \mathbb{R}$, $r \le \lambda^{-1/m}$ and $0 \le k \le \kappa_0$,*

$$\left\|\mathrm{Ad}^k_{\ell,r}(e^{i\xi e^{-\frac{L}{\lambda}}})\right\|_{2\to 2} \le C(1 + |\xi|)^k(\sqrt[m]{\lambda} r)^{-k}.$$

Proof. By Lemma 3.2, for all $\ell > 0$,

$$\mathrm{Ad}_{\ell,r}(e^{i\xi e^{-\frac{L}{\lambda}}})f = i\xi \int_0^1 e^{is\xi e^{-\frac{L}{\lambda}}} \mathrm{Ad}_{\ell,r}(e^{-\frac{L}{\lambda}}) e^{i(1-s)\xi e^{-\frac{L}{\lambda}}} f ds.$$

Note that $e^{-\frac{L}{\lambda}}$ is a bounded operator on $L^2(X)$. Repeatedly, it can be reduced to showing that

$$\left\|\mathrm{Ad}^k_{\ell,r}(e^{-\frac{L}{\lambda}})\right\|_{2\to 2} \le C(\sqrt[m]{\lambda} r)^{-k}, 0 \le k \le \kappa_0. \tag{3.2}$$

By Lemma 3.3, it suffices to show that for every $0 \le k \le \kappa_0$, $\ell > 0$, $\lambda > 0$, $r \le \lambda^{-1/m}$,

$$\sup_{\alpha \in \mathcal{I}_{\lambda^{-1/m}}} \sum_{\beta \in \mathcal{I}_{\lambda^{-1/m}}} \left\|\chi_{Q_\beta(\lambda^{-1/m})} \mathrm{Ad}^k_{\ell,r}(e^{-\frac{L}{\lambda}}) \chi_{Q_\alpha(\lambda^{-1/m})}\right\|_{2\to 2} \le C(\sqrt[m]{\lambda} r)^{-k}. \tag{3.3}$$

To show (3.3), we note that

$$\chi_{Q_\beta(\lambda^{-1/m})} \mathrm{Ad}^k_{\ell,r}(e^{-\frac{L}{\lambda}}) \chi_{Q_\alpha(\lambda^{-1/m})} = \sum_{\gamma_1+\gamma_2+\gamma_3=k} \frac{k!}{\gamma_1!\gamma_2!\gamma_3!} \left(\frac{d(x_\beta, x_\ell)}{r} - \frac{d(x_\alpha, x_\ell)}{r}\right)^{\gamma_1} \left(\frac{d(\cdot, x_\ell)}{r} - \frac{d(x_\beta, x_\ell)}{r}\right)^{\gamma_2} \times$$
$$\chi_{Q_\beta(\lambda^{-1/m})} e^{-\frac{L}{\lambda}} \chi_{Q_\alpha(\lambda^{-1/m})} \left(\frac{d(x_\alpha, x_\ell)}{r} - \frac{d(\cdot, x_\ell)}{r}\right)^{\gamma_3}.$$

Observe that

- $\left|\frac{d(x_\beta,x_\ell)}{r} - \frac{d(x_\alpha,x_\ell)}{r}\right| \le \frac{d(x_\beta,x_\alpha)}{r}$;
- $\left|\frac{d(x,x_\ell)}{r} - \frac{d(x_\beta,x_\ell)}{r}\right| \chi_{Q_\beta(\lambda^{-1/m})}(x) \le (\sqrt[m]{\lambda} r)^{-1}$;

- $\left|\frac{d(x_\alpha,x_\ell)}{r}-\frac{d(y,x_\ell)}{r}\right|\chi_{Q_\alpha(\lambda^{-1/m})}(y)\leq(\sqrt[m]{\lambda}r)^{-1}$.

These, in combination with the estimate ($\mathrm{PEV}^{a,m}_{p,2}$) with $a=\kappa$, yields

$$
\begin{aligned}
&\sum_{\beta\in\mathcal{I}_{\lambda^{-1/m}}}\left\|\chi_{Q_\beta(\lambda^{-1/m})}\mathrm{Ad}^k_{\ell,r}(e^{-\frac{L}{\lambda}})\chi_{Q_\alpha(\lambda^{-1/m})}\right\|_{2\to2}\\
\leq& C\sum_{\gamma_1+\gamma_2+\gamma_3=k}\sum_{\beta\in\mathcal{I}_{\lambda^{-1/m}}}\Big(\frac{d(x_\beta,x_\alpha)}{r}\Big)^{\gamma_1}(\sqrt[m]{\lambda}r)^{-\gamma_2-\gamma_3}\left\|\chi_{Q_\beta(\lambda^{-1/m})}e^{-\frac{L}{\lambda}}\chi_{Q_\alpha(\lambda^{-1/m})}\right\|_{2\to2}\\
\leq& C(\sqrt[m]{\lambda}r)^{-k}\sum_{\beta\in\mathcal{I}_{\lambda^{-1/m}}}(1+\sqrt[m]{\lambda}d(x_\beta,x_\alpha))^{\kappa_0}\left\|\chi_{Q_\beta(\lambda^{-1/m})}e^{-\frac{L}{\lambda}}\chi_{Q_\alpha(\lambda^{-1/m})}\right\|_{2\to2}\\
\leq& C(\sqrt[m]{\lambda}r)^{-k}
\end{aligned}
$$

for some constant $C>0$ independent of α.

Hence, (3.3) is proved. □

The most technical lemma in this subsection is the following amalgam type off-diagonal estimate.

Lemma 3.5. *Let $p_0\leq p\leq 2$, then there exist constants $C,c_0>0$ such that for any ball $B\subset X$ with radius r_B and for any $\lambda>0$, $N\geq 4$, $r=\min\{r_B,\lambda^{-1/m}\}$,*

$$
\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\geq Nr_B}}\|\chi_{Q_\alpha(r)}F(L)\chi_{Q_\beta(r)\cap B}f\|_2^2\right)^{\frac12}\leq C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p
$$

for all Borel functions F such that supp$F\subseteq[-\lambda,\lambda]$.

Proof. Let $E(\tau):=F(-\lambda\log\tau)\tau^{-1}$ so that $F(L)=E(e^{-\frac{L}{\lambda}})e^{-\frac{L}{\lambda}}$. Then

$$
\begin{aligned}
&\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\geq Nr_B}}\|\chi_{Q_\alpha(r)}F(L)\chi_{Q_\beta(r)\cap B}f\|_2^2\right)^{\frac12}\\
\leq&\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\geq Nr_B}}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\alpha)\geq\frac12 d(x_\alpha,x_\beta)}}\left\|\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}\\
+&\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\geq Nr_B}}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\alpha)\leq\frac12 d(x_\alpha,x_\beta)}}\left\|\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}\\
=:&\mathrm{I}+\mathrm{II}.
\end{aligned}
$$

To estimate the first term I, we note that if $x\in Q_\alpha(r)$, then

$$
\frac{d(x,x_\gamma)}{r}\geq\frac{d(x_\gamma,x_\alpha)}{r}-\frac{d(x,x_\alpha)}{r}\geq\frac{d(x_\alpha,x_\gamma)}{r}-1\geq\frac12\frac{d(x_\alpha,x_\gamma)}{r}.
$$

Hence,

$$\begin{aligned}
\mathrm{I} &\le C\Bigg(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\alpha)\ge\frac12 d(x_\alpha,x_\beta)}}\Big(\frac{d(x_\alpha,x_\gamma)}{r}\Big)^{-2\kappa_0}\Big\|\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\Big\|_2^2\Bigg)^{\frac12}\\
&\le C\Bigg(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\gamma)\ge\frac{Nr_B}{2}}}\Big(\frac{d(x_\alpha,x_\gamma)}{r}\Big)^{-2\kappa_0}\Big\|\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\Big\|_2^2\Bigg)^{\frac12}\\
&\le C\Big(\frac{Nr_B}{r}\Big)^{-\kappa_0}\Bigg(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\sum_{\alpha\in\mathcal{I}_r}\Big\|\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\Big\|_2^2\Bigg)^{\frac12}\\
&= C\Big(\frac{Nr_B}{r}\Big)^{-\kappa_0}\Bigg(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\Big\|\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\Big\|_2^2\Bigg)^{\frac12}.
\end{aligned}$$

Combining this estimate with the Fourier inversion formula

$$E(e^{-\frac{L}{\lambda}}) = \int_{-\infty}^{+\infty} e^{i\xi e^{-\frac{L}{\lambda}}}\hat{E}(\xi)d\xi, \tag{3.4}$$

we obtain that

$$\begin{aligned}
\mathrm{I} &\le C\Big(\frac{Nr_B}{r}\Big)^{-\kappa_0}\Bigg(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\Big\|\int_{-\infty}^{+\infty}\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}e^{i\xi e^{-\frac{L}{\lambda}}}\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\hat{E}(\xi)d\xi\Big\|_2^2\Bigg)^{\frac12}\\
&\le C\Big(\frac{Nr_B}{r}\Big)^{-\kappa_0}\Bigg(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\Big(\int_{-\infty}^{+\infty}\Big\|\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}e^{i\xi e^{-\frac{L}{\lambda}}}\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\Big\|_2|\hat{E}(\xi)|d\xi\Big)^2\Bigg)^{\frac12}.
\end{aligned}$$

To continue, applying the following formula for commutators(see Lemma 3.1, [26]):

$$\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}e^{i\xi e^{-\frac{L}{\lambda}}} = \sum_{k=0}^{\kappa_0}\Gamma(\kappa_0,k)\mathrm{Ad}^k_{\gamma,r}(e^{i\xi e^{-\frac{L}{\lambda}}})\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0-k},$$

we have

$$\begin{aligned}
&\Big\|\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0}e^{i\xi e^{-\frac{L}{\lambda}}}\Big(1+\frac{d(\cdot,x_\gamma)}{r}\Big)^{-\kappa_0}\Big\|_{2\to2}\\
&\le C\sum_{k=0}^{\kappa_0}\Big\|\mathrm{Ad}^k_{\gamma,r}(e^{i\xi e^{-\frac{L}{\lambda}}})\Big(\frac{d(\cdot,x_\gamma)}{r}\Big)^{\kappa_0-k}\Big(1+\frac{d(\cdot,x_\gamma)}{r}\Big)^{-\kappa_0}\Big\|_{2\to2}\\
&\le C\sum_{k=0}^{\kappa_0}\big\|\mathrm{Ad}^k_{\gamma,r}(e^{i\xi e^{-\frac{L}{\lambda}}})\big\|_{2\to2}\\
&\le C(1+|\xi|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-\kappa_0},
\end{aligned}\tag{3.5}$$

where in the last inequality we applied Lemma 3.4. This, in combination with the Lemma 2.1 and the doubling condition (1.5), yields

$$\begin{aligned}
\mathrm{I} &\le C(N\sqrt[m]{\lambda}r_B)^{-\kappa_0}\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\left\|\left(1+\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}\int_{-\infty}^{+\infty}|\hat{E}(\xi)|(1+|\xi|)^{\kappa_0}d\xi \\
&\le C(N\sqrt[m]{\lambda}r_B)^{-\kappa_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\left(\sum_{\beta\in\mathcal{I}_r}\left\|e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12} \\
&\le C(N\sqrt[m]{\lambda}r_B)^{-\kappa_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\left(\sum_{\beta\in\mathcal{I}_r}\left\|\chi_{Q_\beta(r)\cap B}V_{\lambda^{-1/m}}^{-\sigma_p}f\right\|_p^2\right)^{\frac12} \\
&\le C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-n\sigma_p}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p, \qquad (3.6)
\end{aligned}$$

where we used the embedding $\ell^p\subset\ell^2$ and the fact that $\mathrm{supp}\delta_\lambda F\subset[-1,1]$ implies that

$$\int_{-\infty}^{+\infty}|\hat{E}(\xi)|(1+|\xi|)^{\kappa_0}d\xi\le C\|E\|_{H^{\kappa_0+1}}\le C\|\delta_\lambda F\|_{H^{\kappa_0+1}}\le C\|\delta_\lambda F\|_{C^{\kappa_0+1}}. \qquad (3.7)$$

To estimate the term II, we first note that $d(x_\gamma,x_\alpha)\le\frac12 d(x_\alpha,x_\beta)$ implies

$$d(x_\gamma,x_\beta)\ge d(x_\alpha,x_\beta)-d(x_\gamma,x_\alpha)\ge\frac12 d(x_\alpha,x_\beta).$$

Then

$$\begin{aligned}
\mathrm{II} &\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac12 d(x_\alpha,x_\beta)}}\left\|\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12} \\
&\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac{Nr_B}{2}}}\sum_{\alpha\in\mathcal{I}_r}\left\|\chi_{Q_\alpha(r)}E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12} \\
&=\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac{Nr_B}{2}}}\left\|E(e^{-\frac{L}{\lambda}})\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}.
\end{aligned}$$

This, in combination with (3.4) and the spectral theorem, yields

$$\mathrm{II}\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac{Nr_B}{2}}}\left(\int_{-\infty}^{+\infty}\left\|e^{i\xi e^{-\frac{L}{\lambda}}}\chi_{Q_\gamma(r)}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2|\hat{E}(\xi)|d\xi\right)^2\right)^{\frac12}$$

$$
\leq \left(\sum_{\beta \in \mathcal{I}_r} \sum_{\substack{\gamma \in \mathcal{I}_r \\ d(x_\gamma, x_\beta) \geq \frac{N r_B}{2}}} \left\| \chi_{Q_\gamma(r)} e^{-\frac{L}{\lambda}} \chi_{Q_\beta(r) \cap B} f \right\|_2^2 \right)^{\frac{1}{2}} \|\hat{E}\|_1. \tag{3.8}
$$

It follows from the estimate ($\mathrm{PEV}_{p,2}^{\kappa,m}$) and the doubling condition (1.5) that

$$
\begin{aligned}
&\left\| \chi_{Q_\gamma(r)} e^{-\frac{L}{\lambda}} \chi_{Q_\beta(r) \cap B} f \right\|_2 \\
\leq & C(1 + \sqrt[m]{\lambda} d(x_\gamma, x_\beta))^{-n-\kappa} \|\chi_{Q_\beta(r) \cap B} V_{\lambda^{-1/m}}^{-\sigma_p} f\|_p \\
\leq & C\mu(B)^{-\sigma_p} (1 + \sqrt[m]{\lambda} d(x_\gamma, x_\beta))^{-n-\kappa} (\sqrt[m]{\lambda} r_B)^{n\sigma_p} (\sqrt[m]{\lambda} r)^{-n\sigma_p} \|\chi_{Q_\beta(r) \cap B} f\|_p.
\end{aligned}
$$

This means that

$$
\begin{aligned}
&\left(\sum_{\beta \in \mathcal{I}_r} \sum_{\substack{\gamma \in \mathcal{I}_r \\ d(x_\gamma, x_\beta) \geq \frac{N r_B}{2}}} \left\| \chi_{Q_\gamma(r)} e^{-\frac{L}{\lambda}} \chi_{Q_\beta(r) \cap B} f \right\|_2^2 \right)^{\frac{1}{2}} \\
\leq & C\mu(B)^{-\sigma_p} (\sqrt[m]{\lambda} r_B)^{n\sigma_p} (\sqrt[m]{\lambda} r)^{-n\sigma_p} \left(\sum_{\beta \in \mathcal{I}_r} \sum_{\substack{\gamma \in \mathcal{I}_r \\ d(x_\gamma, x_\beta) \geq \frac{N r_B}{2}}} (1 + \sqrt[m]{\lambda} d(x_\gamma, x_\beta))^{-2n-2\kappa} \|\chi_{Q_\beta(r) \cap B} f\|_p^2 \right)^{\frac{1}{2}} \\
\leq & C\mu(B)^{-\sigma_p} (\sqrt[m]{\lambda} r_B)^{n\sigma_p} (\sqrt[m]{\lambda} r)^{-n\sigma_p} \left(\sum_{\beta \in \mathcal{I}_r} \sum_{k=0}^{\infty} \sum_{\substack{\gamma \in \mathcal{I}_r \\ 2^{k-1} N r_B \leq d(x_\gamma, x_\beta) < 2^k N r_B}} (1 + \sqrt[m]{\lambda} d(x_\gamma, x_\beta))^{-2n-2\kappa} \|\chi_{Q_\beta(r) \cap B} f\|_p^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

To continue, we point out that under the doubling condition (1.5), there are at most $(2^k N r_B / r)^n$ terms (up to multiplication by an absolute constant) in the last summation. Indeed,

$$
\begin{aligned}
\#\{\gamma \in \mathcal{I}_r,\ 2^{k-1} N r_B \leq d(x_\gamma, x_\beta) < 2^k N r_B\} &\leq \sum_{\substack{\gamma \in \mathcal{I}_r \\ d(x_\gamma, x_\beta) < 2^k N r_B}} \frac{V(x_\gamma, 2^k N r_B)}{V(x_\beta, 2^k N r_B)} \\
&\leq C \left(\frac{2^k N r_B}{r} \right)^n \sum_{\substack{\gamma \in \mathcal{I}_r \\ d(x_\gamma, x_\beta) < 2^k N r_B}} \frac{V(x_\gamma, r)}{V(x_\beta, 2^k N r_B)} \\
&\leq C \left(\frac{2^k N r_B}{r} \right)^n,
\end{aligned}
$$

where we used the notation # to denote the countable measure. Hence,

$$
\left(\sum_{\beta \in \mathcal{I}_r} \sum_{k=0}^{\infty} \sum_{\substack{\gamma \in \mathcal{I}_r \\ 2^{k-1} N r_B \leq d(x_\gamma, x_\beta) < 2^k N r_B}} (1 + \sqrt[m]{\lambda} d(x_\gamma, x_\beta))^{-2n-2\kappa} \|\chi_{Q_\beta(r) \cap B} f\|_p^2 \right)^{\frac{1}{2}}
$$

$$
\begin{aligned}
&\le C(\sqrt[m]{\lambda}r)^{-n/2}\left(\sum_{\beta\in\mathcal{I}_r}\sum_{k=0}^{\infty}(1+2^kN\sqrt[m]{\lambda}r_B)^{-2n-2\kappa}(2^kN\sqrt[m]{\lambda}r_B)^n\|\chi_{Q_\beta(r)\cap B}f\|_p^2\right)^{\frac12}\\
&\le C(\sqrt[m]{\lambda}r)^{-n/2}(1+N\sqrt[m]{\lambda}r_B)^{-\frac n2-\kappa}\left(\sum_{\beta\in\mathcal{I}_r}\|\chi_{Q_\beta(r)\cap B}f\|_p^2\right)^{\frac12}\\
&\le C(\sqrt[m]{\lambda}r)^{-n/2}(1+N\sqrt[m]{\lambda}r_B)^{-\frac n2-\kappa}\|f\|_p,
\end{aligned}
$$

where in the last inequality we used the embedding $\ell^p\subset\ell^2$. Thus,

$$
\begin{aligned}
\mathrm{II}&\le C\mu(B)^{-\sigma_p}(\sqrt[m]{\lambda}r_B)^{n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_0}(1+N\sqrt[m]{\lambda}r_B)^{-\frac n2-\kappa}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p\\
&\le C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p
\end{aligned}\tag{3.9}
$$

for some constant $c_0>0$ independent of λ, r, N, t.

Combining the estimates (3.6) and (3.9), we conclude that

$$
\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\|\chi_{Q_\alpha(r)}F(L)\chi_{Q_\beta(r)\cap B}f\|_2^2\right)^{\frac12}\le C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p.
$$

This finishes the proof of Lemma 3.5. □

Proof of Proposition 3.1. Now we embed the set $(2^{j-1}B)^c$ into a countable union of amalgam block $\{Q_\alpha(r)\}$, that is,

$$
(2^{j-1}B)^c\subseteq\bigcup_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_B)\ge 2^{j-2}r_B}}Q_\alpha(r).
$$

We observe that the condition $j\ge5$ implies that if $a,\beta\in\mathcal{I}_r$, $d(x_\alpha,x_B)\ge 2^{j-2}r_B$ and $Q_\beta(r)\cap B\neq\emptyset$, then $d(x_\alpha,x_\beta)\ge d(x_\alpha,x_B)-d(x_B,x_\beta)\ge 2^{j-2}r_B-r-r_B\ge 2^{j-3}r_B$. Hence, it follows from Lemma 3.5 with $N=2^{j-3}$ that

$$
\begin{aligned}
\|\chi_{U_j(B)}F(L)\chi_Bf\|_2&\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_B)\ge 2^{j-2}r_B}}\|\chi_{Q_\alpha(r)}F(L)\chi_{Q_\beta(r)\cap B}f\|_2^2\right)^{\frac12}\\
&\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge 2^{j-3}r_B}}\|\chi_{Q_\alpha(r)}F(L)\chi_{Q_\beta(r)\cap B}f\|_2^2\right)^{\frac12}\\
&\le C2^{-j\kappa_0}\mu(B)^{-\sigma_p}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p.
\end{aligned}
$$

This implies the estimate (3.1). □

3.2. **Off-diagonal estimates for oscillatory compactly supported spectral multipliers.** In the previous subsection, we obtain off-diagonal estimates for compactly supported spectral multipliers with sufficient smoothness, but this estimate is not suitable for those multiplier function with oscillatory term. Inspired by [11], to overcome this difficulty, we will use commutators techniques again to obtain much more subtle estimates for oscillatory compactly supported spectral multipliers. That is, we will the following Proposition 3.6, which is a general version of Proposition 1.6.

Proposition 3.6. *Let $p_0 \le p \le 2$, then there exist constants $C, c_0 > 0$ such that for any ball $B \subset X$ with radius r_B and for any $\lambda > 0$, $j \ge 6$,*

$$\left\|\chi_{U_j(B)} e^{itL} F(L) \chi_B f\right\|_2 \le C 2^{-j\kappa_0} \mu(B)^{-\sigma_p} (\sqrt[m]{\lambda} r_B)^{-\kappa_0 + n\sigma_p} (1+\lambda|t|)^{\kappa_0} (\sqrt[m]{\lambda} r)^{-c_0} \|\delta_\lambda F\|_{C^{\kappa_0+1}} \|f\|_p$$

for all Borel functions F such that supp$F \subseteq [-\lambda, \lambda]$, where $r = \min\{r_B, \lambda^{-1/m}\}$.

To begin with, for every $\lambda > 0$, we denote

$$R_\lambda := \Big(I + \frac{L}{\lambda}\Big)^{-1}.$$

The key observation is the following lemma.

Lemma 3.7. *For every $1 \le k \le \kappa_0$, $r, \lambda, \ell > 0$, the operator $\mathrm{Ad}_{\ell,r}(R_\lambda^{2^{k+1}-2} e^{itL})$ is given by a finite combination of operators of the following type:*

$$R_\lambda^{\mu_1} R_\lambda^{2^k-2} e^{itL} \mathrm{Ad}_{\ell,r}(R_\lambda^{\mu_2}) R_\lambda^{\mu_3}, \quad \mu_1, \mu_2, \mu_3 \in \mathbb{N},$$
$$R_\lambda^{\mu_1} \mathrm{Ad}_{\ell,r}(R_\lambda^{\mu_2}) R_\lambda^{2^k-2} e^{itL} R_\lambda^{\mu_3}, \quad \mu_1, \mu_2, \mu_3 \in \mathbb{N},$$
$$\lambda t \int_0^1 R_\lambda^{2^k-2} e^{i\rho tL} \mathrm{Ad}_{\ell,r}(R_\lambda) R_\lambda^{2^k-2} e^{i(1-\rho)tL} d\rho.$$

Proof. We deduce this result by induction on k. First of all, we point out that it is true for $k = 1$. Indeed, by the commutator formula and Lemma 3.2,

$$\begin{aligned}\mathrm{Ad}_{\ell,r}(R_\lambda e^{itL} R_\lambda) &= \mathrm{Ad}_{\ell,r}(R_\lambda) e^{itL} R_\lambda + R_\lambda e^{itL} \mathrm{Ad}_{\ell,r}(R_\lambda) + R_\lambda \mathrm{Ad}_{\ell,r}(e^{itL}) R_\lambda \\ &= \mathrm{Ad}_{\ell,r}(R_\lambda) e^{itL} R_\lambda + R_\lambda e^{itL} \mathrm{Ad}_{\ell,r}(R_\lambda) + i\lambda t \int_0^1 e^{i\rho tL} R_\lambda \mathrm{Ad}_{\ell,r}(L/\lambda) R_\lambda e^{i(1-\rho)tL} d\rho.\end{aligned}$$

Observe that

$$R_\lambda \mathrm{Ad}_{\ell,r}(L/\lambda) R_\lambda = R_\lambda \frac{d(x_\ell, \cdot)}{r} (R_\lambda^{-1} - I) R_\lambda - R_\lambda (R_\lambda^{-1} - I) \frac{d(x_\ell, \cdot)}{r} R_\lambda = -\mathrm{Ad}_{\ell,r}(R_\lambda).$$

Hence, the result for $k = 1$ is verified.

Now, let us assume this lemma holds for $k - 1$ and compute

$$\begin{aligned}\mathrm{Ad}_{\ell,r}(R_\lambda^{2^{k+1}-2} e^{itL}) &= \mathrm{Ad}_{\ell,r}(R_\lambda^{2^{k-1}} (R_\lambda^{2^k-2} e^{itL}) R_\lambda^{2^{k-1}}) \\ &= \mathrm{Ad}_{\ell,r}(R_\lambda^{2^{k-1}}) R_\lambda^{2^k-2} e^{itL} R_\lambda^{2^{k-1}} + R_\lambda^{2^{k-1}} \mathrm{Ad}_{\ell,r}(R_\lambda^{2^k-2} e^{itL}) R_\lambda^{2^{k-1}} + R_\lambda^{2^{k-1}} R_\lambda^{2^k-2} e^{itL} \mathrm{Ad}_{\ell,r}(R_\lambda^{2^{k-1}}).\end{aligned}$$

The first and the last term are of the desired form. The second one is also of the desired form by the inductive hypothesis, since

$$R_\lambda^{2^{k-1}} R_\lambda^{2^{k-1}-2} = R_\lambda^{2^k-2}.$$

This finishes the proof of Lemma 3.7. □

Lemma 3.8. *There exists a constant $C > 0$, such that for any $\ell, \lambda, r > 0$ and $0 \le k \le \kappa_0$,*

$$\|\mathrm{Ad}^k_{\ell,r}(R_\lambda)f\|_2 \le C(\sqrt[m]{\lambda}r)^{-k}\|f\|_2.$$

Proof. Denote $K_T(x,y)$ be the distribution kernel of the operator T. Then by induction on $k \in [0,\kappa_0]$, we have

$$\begin{aligned}\mathrm{Ad}^k_{\ell,r}(R_\lambda)f(x) &= \int_X \left(\frac{d(x_\ell,x)}{r} - \frac{d(x_\ell,y)}{r}\right)^k K_{R_\lambda}(x,y)f(y)d\mu(y)\\ &= (\sqrt[m]{\lambda}r)^{-k}\int_X \left(\sqrt[m]{\lambda}d(x_\ell,x) - \sqrt[m]{\lambda}d(x_\ell,y)\right)^k K_{R_\lambda}(x,y)f(y)d\mu(y).\end{aligned}$$

Applying the following representation formula

$$R_\lambda = \int_0^\infty e^{-\tau L/\lambda}e^{-\tau}d\tau,$$

we see that

$$\begin{aligned}\mathrm{Ad}^k_{\ell,r}(R_\lambda)f(x) &= (\sqrt[m]{\lambda}r)^{-k}\int_0^\infty\int_X \left(\sqrt[m]{\lambda}d(x_\ell,x) - \sqrt[m]{\lambda}d(x_\ell,y)\right)^k p_{\tau/\lambda}(x,y)f(y)d\mu(y)e^{-\tau}d\tau\\ &= (\sqrt[m]{\lambda}r)^{-k}\int_0^\infty \mathrm{Ad}^k_{\ell,(\tau/\lambda)^{1/m}}(e^{-\frac{\tau}{\lambda}L})f(x)\tau^{\frac{k}{m}}e^{-\tau}d\tau.\end{aligned}$$

It follows from (3.2) with r replaced by $(\tau/\lambda)^{1/m}$ and λ replaced by λ/τ that

$$\|\mathrm{Ad}^k_{\ell,(\tau/\lambda)^{1/m}}(e^{-\frac{\tau}{\lambda}L})f\|_2 \le C\|f\|_2,$$

which means that

$$\|\mathrm{Ad}^k_{\ell,r}(R_\lambda)f\|_2 \le (\sqrt[m]{\lambda}r)^{-k}\int_0^\infty \|\mathrm{Ad}^k_{\ell,(\tau/\lambda)^{1/m}}(e^{-\frac{\tau}{\lambda}L})f(x)\|_2\tau^{\frac{k}{m}}e^{-\tau}d\tau \le C(\sqrt[m]{\lambda}r)^{-k}\|f\|_2.$$

This ends the proof of Lemma 3.8. □

Now we apply Lemmas 3.7 and 3.8 to obtain the following crucial commutator estimate for Schrödinger group.

Lemma 3.9. *There exists a constant $C > 0$, such that for any $\ell, \lambda > 0$, $r \le \lambda^{-1/m}$ and $0 \le k \le j \le \kappa_0$,*

$$\|\mathrm{Ad}^k_{\ell,r}(R_\lambda^{2^{j+1}-2}e^{itL})\|_{2\to2} \le C(1+\lambda|t|)^k(\sqrt[m]{\lambda}r)^{-\mu k}.$$

Proof. The result will be shown by induction on $j = 0, \cdots, \kappa_0$.

By the spectral theorem, the result is true for $j = 0$. Now we assume it holds for $j-1$, and write, for $k \le j$,

$$\mathrm{Ad}^k_{\ell,r}(R_\lambda^{2^{j+1}-2}e^{itL}) = \mathrm{Ad}^{k-1}_{\ell,r}(\mathrm{Ad}_{\ell,r}(R_\lambda^{2^{j+1}-2}e^{itL})).$$

By Lemma 3.7 and the formula

$$\mathrm{Ad}^{k-1}_{\ell,r}(T_1\cdots T_n) = \sum_{\alpha_1+\cdots+\alpha_n=k-1}\frac{(k-1)!}{\alpha_1!\cdots\alpha_n!}\mathrm{Ad}^{\alpha_1}_{\ell,r}(T_1)\cdots\mathrm{Ad}^{\alpha_n}_{\ell,r}(T_n)$$

as well as the inductive hypothesis and Lemma 3.8, we obtain the inequality. □

Now, the result in the previous subsection can be extended to oscillatory compactly supported spectral multipliers.

Lemma 3.10. *Let $p_0 \le p \le 2$, then there exist constants $C, c_0 > 0$, such that for any ball $B \subset X$ with radius r_B and for any $\lambda > 0$, $N \ge 8$, $r = \min\{r_B, \lambda^{-1/m}\}$,*

$$\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\left\|\chi_{Q_\alpha(r)}e^{itL}F(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}$$
$$\le C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(1+\lambda|t|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-c_0}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p$$

for all Borel functions F such that supp$F \subseteq [-\lambda, \lambda]$.

Proof. To begin with, we note that

$$e^{itL}F(L) = R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\delta_{\lambda^{-1}}G(L),$$

where $G(L) = R_1^{-2^{\kappa_0+1}+2}\delta_\lambda F(L)$.

Hence,

$$\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\left\|\chi_{Q_\alpha(r)}e^{itL}F(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}$$
$$\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac12 d(x_\alpha,x_\beta)}}\left\|\chi_{Q_\alpha(r)}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}$$
$$+\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\ge Nr_B}}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\le\frac12 d(x_\alpha,x_\beta)}}\left\|\chi_{Q_\alpha(r)}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}$$
$$=:\mathrm{I}+\mathrm{II}.$$

To estimate the term I, we apply the spectral theorem to conclude that

$$\mathrm{I}\le\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac{Nr_B}{2}}}\sum_{\alpha\in\mathcal{I}_r}\left\|\chi_{Q_\alpha(r)}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}$$
$$=\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\gamma\in\mathcal{I}_r\\ d(x_\gamma,x_\beta)\ge\frac{Nr_B}{2}}}\left\|R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}$$

$$\leq \left(\sum_{\beta\in\mathcal{I}_r} \sum_{\substack{\gamma\in\mathcal{I}_r \\ d(x_\gamma,x_\beta)\geq \frac{Nr_B}{2}}} \|\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\|_2^2 \right)^{\frac{1}{2}}.$$

Since supp$\delta_{\lambda^{-1}}G \subset [-\lambda,\lambda]$, we can apply Lemma 3.5 to obtain that

$$\begin{aligned} \mathrm{I} &\leq C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_1}\|G\|_{C^{\kappa_0+1}}\|f\|_p \\ &\leq C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(\sqrt[m]{\lambda}r)^{-c_1}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p \end{aligned} \tag{3.10}$$

for some constant $c_1 > 0$ independent of λ, r, N, t.

As for the second term II, we first note that $d(x_\gamma, x_\beta) \leq \frac{1}{2}d(x_\alpha, x_\beta)$ implies

$$d(x_\gamma, x_\alpha) \geq d(x_\alpha, x_\beta) - d(x_\gamma, x_\beta) \geq \frac{1}{2}d(x_\alpha, x_\beta).$$

This means that if $x \in Q_\alpha(r)$, then

$$\frac{d(x,x_\gamma)}{r} \geq \frac{d(x_\alpha,x_\gamma)}{r} - \frac{d(x_\alpha,x)}{r} \geq \frac{d(x_\alpha,x_\gamma)}{r} - 1 \geq \frac{1}{2}\frac{d(x_\alpha,x_\gamma)}{r}.$$

Hence,

$$\begin{aligned} \mathrm{II} \leq& \left(\sum_{\beta\in\mathcal{I}_r} \sum_{\substack{\alpha\in\mathcal{I}_r \\ d(x_\alpha,x_\beta)\geq Nr_B}} \sum_{\substack{\gamma\in\mathcal{I}_r \\ d(x_\gamma,x_\alpha)\geq \frac{1}{2}d(x_\alpha,x_\beta)}} \left\|\chi_{Q_\alpha(r)}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2 \right)^{\frac{1}{2}} \\ \leq& \left(\sum_{\beta\in\mathcal{I}_r} \sum_{\gamma\in\mathcal{I}_r} \sum_{\substack{\alpha\in\mathcal{I}_r \\ d(x_\alpha,x_\gamma)\geq \frac{Nr_B}{2}}} \left(\frac{d(x_\alpha,x_\gamma)}{r}\right)^{-2\kappa_0} \left\|\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}\chi_{Q_\alpha(r)}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2 \right)^{\frac{1}{2}} \\ \leq& C\left(\frac{Nr_B}{r}\right)^{-\kappa_0} \left(\sum_{\beta\in\mathcal{I}_r} \sum_{\gamma\in\mathcal{I}_r} \sum_{\substack{\alpha\in\mathcal{I}_r \\ d(x_\alpha,x_\gamma)\geq \frac{Nr_B}{2}}} \left\|\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}\chi_{Q_\alpha(r)}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2 \right)^{\frac{1}{2}} \\ \leq& C\left(\frac{Nr_B}{r}\right)^{-\kappa_0} \left(\sum_{\beta\in\mathcal{I}_r} \sum_{\gamma\in\mathcal{I}_r} \left\|\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2 \right)^{\frac{1}{2}}. \end{aligned}$$

To continue, applying the following formula for commutators(see Lemma 3.1, [26]):

$$\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}R_\lambda^{2^{\kappa_0+1}-2}e^{itL} = \sum_{k=0}^{\kappa_0}\Gamma(\kappa_0,k)\mathrm{Ad}_{\gamma,r}^k(R_\lambda^{2^{\kappa_0+1}-2}e^{itL})\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0-k},$$

we obtain that there exists a constant $c_2 > 0$ such that

$$\begin{aligned} &\left\|\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}R_\lambda^{2^{\kappa_0+1}-2}e^{itL}\left(1+\frac{d(\cdot,x_\gamma)}{r}\right)^{-\kappa_0}\right\|_{2\to2} \\ &\leq C\sum_{k=0}^{\kappa_0}\left\|\mathrm{Ad}_{\gamma,r}^k(R_\lambda^{2^{\kappa_0+1}-2}e^{itL})\left(\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0-k}\left(1+\frac{d(\cdot,x_\gamma)}{r}\right)^{-\kappa_0}\right\|_{2\to2} \end{aligned}$$

$$\begin{aligned}&\leq C\sum_{k=0}^{\kappa_0}\left\|\mathrm{Ad}^k_{\gamma,r}(R_\lambda^{2^{\kappa_0+1}-2}e^{itL})\right\|_{2\to 2}\\ &\leq C(1+\lambda|t|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-c_2}, \qquad (3.11)\end{aligned}$$

where in the last inequality we applied Lemma 3.9. Applying the estimate (3.11), we conclude that there exists a constant $c_3>0$ such that

$$\begin{aligned}\mathrm{II}\leq& C(N\sqrt[m]{\lambda}r_B)^{-\kappa_0}(1+\lambda|t|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-c_3}\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\gamma\in\mathcal{I}_r}\left\|\left(1+\frac{d(\cdot,x_\gamma)}{r}\right)^{\kappa_0}\chi_{Q_\gamma(r)}\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}\\ \leq& C(N\sqrt[m]{\lambda}r_B)^{-\kappa_0}(1+\lambda|t|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-c_3}\left(\sum_{\beta\in\mathcal{I}_r}\left\|\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}. \qquad (3.12)\end{aligned}$$

Next we write $H(\tau)=(\delta_{\lambda^{-1}}G)(-\lambda\log\tau)\tau^{-1}$, then $\delta_{\lambda^{-1}}G(L)=H(e^{-\frac{L}{\lambda}})e^{-\frac{L}{\lambda}}$. In virtue of the Fourier inversion formula and the Lemma 2.1, we conclude that

$$\begin{aligned}\left(\sum_{\beta\in\mathcal{I}_r}\left\|\delta_{\lambda^{-1}}G(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}&\leq\left(\sum_{\beta\in\mathcal{I}_r}\left(\int_{-\infty}^{+\infty}\left\|e^{i\xi e^{-\frac{L}{\lambda}}}e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2|\hat{H}(\xi)|d\xi\right)^2\right)^{\frac12}\\ &\leq\|\hat{H}\|_1\left(\sum_{\beta\in\mathcal{I}_r}\left\|e^{-\frac{L}{\lambda}}\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}\\ &\leq C\|\delta_\lambda F\|_{C^{\kappa_0+1}}\left(\sum_{\beta\in\mathcal{I}_r}\left\|\chi_{Q_\beta(r)\cap B}V^{-\sigma_p}_{\lambda^{-1/m}}f\right\|_p^2\right)^{\frac12}\\ &\leq C\mu(B)^{-\sigma_p}(\sqrt[m]{\lambda}r_B)^{n\sigma_p}(\sqrt[m]{\lambda}r)^{-n\sigma_p}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p.\end{aligned}$$

This, in combination with (3.12), yields

$$\mathrm{II}\leq C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(1+\lambda|t|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-c_4}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p \qquad (3.13)$$

for some constant $c_4>0$ independent of λ, r, N, t.

Combining (3.10) and (3.13), we obtain that

$$\begin{aligned}&\left(\sum_{\beta\in\mathcal{I}_r}\sum_{\substack{\alpha\in\mathcal{I}_r\\ d(x_\alpha,x_\beta)\geq Nr_B}}\left\|\chi_{Q_\alpha(r)}e^{itL}F(L)\chi_{Q_\beta(r)\cap B}f\right\|_2^2\right)^{\frac12}\\ \leq& C\mu(B)^{-\sigma_p}N^{-\kappa_0}(\sqrt[m]{\lambda}r_B)^{-\kappa_0+n\sigma_p}(1+\lambda|t|)^{\kappa_0}(\sqrt[m]{\lambda}r)^{-c_5}\|\delta_\lambda F\|_{C^{\kappa_0+1}}\|f\|_p\end{aligned}$$

for some constant $c_5>0$ independent of λ, r, N, t.

This ends the proof of Lemma 3.10. □

Proof of Proposition 3.6. The proof of this proposition can be shown by a similar argument as in the proof of Proposition 3.1. We omit the details and leave it to the readers. □

4. Proof of Theorem 1.3

4.1. **Proof of boundedness on $H^1_L(X)$.** In this subsection, with the help of Proposition 3.6 and Lemma 2.7, we will borrow the ideas from [7, 24] to show Proposition 1.5.

Proof of Proposition 1.5. Choose M be a sufficient large constant and assume that $a(x)$ is a $(1,2,M,\epsilon)$-molecule associated to a ball $B = B(x_B, r_B)$ and $a = L^M b$ such that for every $k = 0,1,2,\dots,M$ and $j = 0,1,2,\dots,$

$$\|(r_B^m L)^k b\|_{L^2(U_j(B))} \le 2^{-j\epsilon} r_B^{mM} \mu(2^j B)^{-\frac{1}{2}}. \tag{4.1}$$

By Lemma 2.7 and a standard argument (see for example, [17, 23, 24, 27]), it suffices to show that

$$\left\| \left(\int_0^{+\infty} \int_{d(x,y)<\tau^{1/m}} \left|\phi(\tau L) e^{itL} F(L) a(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})} \frac{d\tau}{\tau} \right)^{\frac{1}{2}} \right\|_{L^1(X)} \le C(1+|t|)^{n/2}, \tag{4.2}$$

where, for simplicity, we denote $F(\lambda) = (1+\lambda)^{-n/2}$.

Let us show (4.2). Following [24, Lemma 8.1], we write

$$\begin{aligned} I &= m r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} ds \cdot I \\ &= m r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} (I - e^{-s^m L})^M ds + \sum_{\nu=1}^{M} C_{\nu,M} r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} e^{-\nu s^m L} ds \end{aligned}$$

for some constant $C_{\nu,M}$ depending on ν and M only. Besides, it follows by $\partial_s e^{-\nu s^m L} = -m\nu s^{m-1} L e^{-\nu s^m L}$ that

$$m\nu L \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} e^{-\nu s^m L} ds = e^{-\nu r_B^m L} - e^{-2\nu r_B^m L} = e^{-\nu r_B^m L}(I - e^{-r_B^m L}) \sum_{\mu=0}^{\nu-1} e^{-\mu r_B^m L}. \tag{4.3}$$

Then we iterate the procedure above M times to conclude that for every $x \in X$,

$$\begin{aligned} \phi(\tau L) F(L) a(x) &= \sum_{k=0}^{M-1} r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} (I - e^{-s^m L})^M G_{k,r_B,M}(L) \phi(\tau L) F(L) (r_B^{-mk} L^{M-k} b) ds \\ &\quad + \sum_{\nu=1}^{(2M-1)M} C(\nu,k,M) e^{-\nu r_B^m L} \phi(\tau L) F(L) (I - e^{-r_B^m L})^M (r_B^{-mM} b)(x) \\ &=: \sum_{k=0}^{M-1} E_k(x) + E_M(x), \end{aligned} \tag{4.4}$$

where

$$G_{0,r_B,M}(\lambda) := m^M \left(r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} (I - e^{-s^m \lambda})^M ds \right)^{M-1},$$

and for $k = 1,2,\cdots,M-1$,

$$G_{k,r_B,M}(\lambda) := (1 - e^{-r_B^m \lambda})^k \left(r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} (I - e^{-s^m \lambda})^M ds \right)^{M-k-1} \sum_{\nu=1}^{(2M-1)k} C(\nu,k,M) e^{-\nu r_B^m \lambda}.$$

To continue, we consider two cases: $k = 0,1,\cdots,M-1$ and $k = M$.

Case 1. $k = 0, 1, \cdots, M-1$. In this case, we see that

$$\begin{aligned}
&\left\|\left(\int_0^\infty\!\!\int_{d(x,y)<\tau^{1/m}} \left|e^{itL}E_k(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^1}\\
&\leq C \sup_{s\in[r_B,\sqrt[m]{2}r_B]} \left\|\left(\int_0^\infty\!\!\int_{d(x,y)<\tau^{1/m}} \left|e^{itL}F_{\tau,s}(L)(r_B^{-mk}L^{M-k}b)(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^1}\\
&\leq C\sum_{j\geq 0} \sup_{s\in[r_B,\sqrt[m]{2}r_B]} \left\|\left(\int_0^\infty\!\!\int_{d(x,y)<\tau^{1/m}} \left|e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^1}\\
&=: C\sum_{j\geq 0} \sup_{s\in[r_B,\sqrt[m]{2}r_B]} \|E(k,j,s)\|_{L^1(X)}, \qquad (4.5)
\end{aligned}$$

where $F_{\tau,s}(\lambda) := \phi_\tau(\lambda)F(\lambda)(1-e^{-s^m\lambda})^M G_{k,r_B,M}(\lambda)$ and

$$E(k,j,s) = \left(\int_0^\infty\!\!\int_{d(x,y)<\tau^{1/m}} \left|e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}.$$

Let us estimate the term $\|E(k,j,s)\|_{L^1(X)}$. Note that $\|F\|_\infty + \|G_{k,r_B,M}\|_{L^\infty} \leq C$. We apply the estimate (4.1) and the L^2-boundedness of the square function to see that

$$\begin{aligned}
&\|E(k,j,s)\|_{L^2(64(1+|t|)2^jB)}\\
&\leq C\left(\int_0^\infty \int_X \left|e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)\right|^2 \int_{d(x,y)<\tau^{1/m}} d\mu(x)\frac{d\mu(y)}{V(y,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{\frac12}\\
&\leq C\left(\int_0^\infty \left\|e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)\right\|_2^2\frac{d\tau}{\tau}\right)^{\frac12}\\
&\leq C\left\|(1-e^{-s^mL})^M G_{k,r_B,M}(L)e^{itL}F(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)\right\|_2\\
&\leq C\|r_B^{-mk}L^{M-k}b\|_{L^2(U_j(B))} \leq C2^{-j\epsilon}\mu(2^jB)^{-\frac12}.
\end{aligned}$$

Hence, it follows from the Cauchy-Schwarz inequality and the doubling condition (1.5) that

$$\|E(k,j,s)\|_{L^1(64(1+|t|)2^jB)} \leq \|E(k,j,s)\|_{L^2(64(1+|t|)2^jB)}\left(\frac{\mu(64(1+|t|)2^jB)}{\mu(2^jB)}\right)^{\frac12} \leq C2^{-j\epsilon}(1+|t|)^{n/2}.$$

Next we show that for some $\varepsilon' > 0$,

$$\|E(k,j,s)\|_{L^1((64(1+|t|)2^jB)^c)} \leq C2^{-j\varepsilon'}(1+|t|)^{n/2}. \qquad (4.6)$$

To prove (4.6), we write

$$\begin{aligned}
E(k,j,s) &= \left(\int_0^\infty\!\!\int_{d(x,y)<\tau^{1/m}} \left|e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\\
&\leq \sum_{\ell\in\mathbb{Z}}\left(\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<\tau^{1/m}} \left|e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)\right|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\\
&=: \sum_{\ell\in\mathbb{Z}} E(k,j,s,\ell).
\end{aligned}$$

If $\ell > \frac{1}{m}$, then let $\nu_0^+ \in \mathbb{Z}_+$ be a positive integer such that

$$8 < 2^{\nu_0^+ + j - \ell(m-1)/m} r_B \le 16, \ \text{if } 2^{j-\ell(m-1)/m} r_B \le \frac{1}{8};$$

$$\nu_0^+ = 7, \ \text{if } 2^{j-\ell(m-1)/m} r_B > \frac{1}{8}. \tag{4.7}$$

If $\ell \le \frac{1}{m}$, then let $\nu_0^- \in \mathbb{Z}_+$ be a positive integer such that

$$8 < 2^{\nu_0^- + j + (\ell-1)/m} r_B \le 16, \ \text{if } 2^{(\ell-1)/m + j} r_B \le \frac{1}{8};$$

$$\nu_0^- = 7, \ \text{if } 2^{(\ell-1)/m + j} r_B > \frac{1}{8}. \tag{4.8}$$

Then

$$\begin{aligned} \|E(k,j,s)\|_{L^1((64(1+|t|)2^j B)^c)} &\le \sum_{\ell > 1/m} \|E(k,j,s,\ell)\|_{L^1(B(x_B, 8(1+|t|)2^{\ell(m-1)/m}))} \\ &\quad + \sum_{\ell > 1/m} \sum_{\nu \ge \nu_0^+} \|E(k,j,s,\ell)\|_{L^1(U_{\nu+j}((1+|t|)B))} \\ &\quad + \sum_{\ell \le 1/m} \|E(k,j,s,\ell)\|_{L^1(B(x_B, 8(1+|t|)2^{-(\ell-1)/m}))} \\ &\quad + \sum_{\ell \le 1/m} \sum_{\nu \ge \nu_0^-} \|E(k,j,s,\ell)\|_{L^1(U_{\nu+j}((1+|t|)B))} \\ &=: \mathrm{I}(k,j,s) + \mathrm{II}(k,j,s) + \mathrm{III}(k,j,s) + \mathrm{IV}(k,j,s). \end{aligned} \tag{4.9}$$

Let us first estimate the terms $\mathrm{I}(k,j,s)$ and $\mathrm{II}(k,j,s)$. Note that there is no term $\mathrm{I}(k,j,s)$ if $2^{j-\ell(m-1)/m} r_B > \frac{1}{8}$ and $\ell > \frac{1}{m}$. Besides, when $2^{j-\ell(m-1)/m} r_B \le \frac{1}{8}$ and $\ell > \frac{1}{m}$, we apply the estimate (4.1) and the doubling condition (1.5) to get that

$$\begin{aligned} \|E(k,j,s,\ell)\|^2_{L^2(X)} &\le C \int_{2^{-\ell}}^{2^{-\ell+1}} \left\| e^{itL} F_{\tau,s}(L) \chi_{U_j(B)} (r_B^{-mk} L^{M-k} b) \right\|_2^2 \frac{d\tau}{\tau} \\ &\le C \int_{2^{-\ell}}^{2^{-\ell+1}} \|e^{it(\cdot)} F_{\tau,s}\|^2_{L^\infty} \|r_B^{-mk} L^{M-k} b\|^2_{L^2(U_j(B))} \frac{d\tau}{\tau} \\ &\le C \min\{1, (2^{\ell/m} r_B)^{2mM}\} 2^{-n\ell} 2^{-2j\epsilon} V(x_B, 2^j r_B)^{-1}, \end{aligned} \tag{4.10}$$

which, in combination with the doubling condition (1.5), yields that

$$\begin{aligned} \mathrm{I}(k,j,s) &\le \sum_{\ell \in \mathbb{Z}} \|E(k,j,s,\ell)\|_{L^2(B(x_B, 8(1+|t|)2^{\ell(m-1)/m}))} V(x_B, 8(1+|t|)2^{\ell(m-1)/m})^{\frac{1}{2}} \\ &\le C \sum_{\ell \in \mathbb{Z}} \min\{1, (2^{\ell/m} r_B)^{mM}\} 2^{-n\ell/2} 2^{-j\epsilon} \left(\frac{V(x_B, 8(1+|t|)2^{\ell(m-1)/m})}{V(x_B, 2^j r_B)} \right)^{\frac{1}{2}} \\ &\le C 2^{-j(\epsilon + \frac{n}{2})} \sum_{\ell \in \mathbb{Z}} \min\{1, (2^{\ell/m} r_B)^{mM}\} (2^{\ell/m} r_B)^{-n/2} (1+|t|)^{n/2} \\ &\le C 2^{-j(\epsilon + \frac{n}{2})} (1+|t|)^{n/2}. \end{aligned}$$

Next we estimate the term $\mathrm{II}(k,j,s)$. It follows from (4.7) that for $\tau \in [2^{-\ell}, 2^{-\ell+1}]$ and $\ell > \frac{1}{m}$, we have $\tau^{1/m} \le 2^{1/m} 2^{-\ell/m} \le 2^{\ell(m-1)/m} \le 2^{\nu+j-3}(1+|t|) r_B$. Therefore, if $d(x,y) < \tau^{1/m}$ and $x \in$

$U_{\nu+j}((1+|t|)B)$, then $y \in U'_{\nu+j}((1+|t|)B)$, where

$$U'_{\nu+j}((1+|t|)B) = U_{\nu+j+1}((1+|t|)B) \cup U_{\nu+j}((1+|t|)B) \cup U_{\nu+j-1}((1+|t|)B).$$

Then we have

$$\begin{aligned} &\|E(k,j,s,\ell)\|^2_{L^2(U_{\nu+j}((1+|t|)B))} \\ &\quad\le C\sum_{w=-1}^{1}\int_{2^{-\ell}}^{2^{-\ell+1}} \left\|\chi_{U_{\nu+j+w}((1+|t|)B)}e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)\right\|_2^2\frac{d\tau}{\tau} \\ &\quad\le C\sum_{w=-1}^{1}\int_{2^{-\ell}}^{2^{-\ell+1}} \left\|\chi_{U_{\nu+j+w}((1+|t|)B)}e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}\right\|_{2\to2}^2\|r_B^{-mk}L^{M-k}b\|^2_{L^2(U_j(B))}\frac{d\tau}{\tau}. \end{aligned} \tag{4.11}$$

To deal with the L^2-L^2 off-diagonal term involving the oscillatory semigroup e^{itL}, we apply Proposition 3.6, with r_B replaced by 2^jr_B and 2^j replaced by $2^{\nu+j+w+\nu_t}$, where $\nu_t \in \mathbb{N}$ and $2^{\nu_t} \le (1+|t|) < 2^{\nu_t+1}$, to see that there exist constants $C, c_0 > 0$ such that for $w \in \{-1,0,1\}$ and $\nu \ge 7$, $2^{-\ell} \le \tau \le 2^{-l+1}$,

$$\begin{aligned} &\left\|\chi_{U_{\nu+j+w}((1+|t|)B)}e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}\right\|_{2\to2} \\ &\quad\le C(2^\nu(1+|t|))^{-\kappa_0}(2^{\ell/m}2^jr_B)^{-\kappa_0}(1+2^\ell|t|)^{\kappa_0}(2^{\ell/m}r_{j,\ell})^{-c_0}\|\delta_{\tau^{-1}}F_{\tau,s}\|_{C^{\kappa_0+1}}, \end{aligned} \tag{4.12}$$

where $r_{j,\ell} = \min\{2^jr_B, 2^{-\ell/m}\}$. Note that the conditions $\mathrm{supp}\phi \subset (1/4,1)$ and $r_B \le s \le \sqrt[m]{2}r_B$ imply that if $2^{-\ell} \le \tau \le 2^{-l+1}$, then

$$\|\delta_{\tau^{-1}}F_{\tau,s}\|_{C^{\kappa_0+1}} \le C\min\{1,(2^{\ell/m}r_B)^{mM}\}2^{-\ell n/2}. \tag{4.13}$$

By a simple calculation, we can see that

$$\min\{1,(2^{\ell/m}r_B)^{mM}\}(2^{\ell/m}r_{j,\ell})^{-c_0} \le C\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}. \tag{4.14}$$

This, in combination with the estimates (4.1), (4.11), (4.12) and (4.13), yields

$$\begin{aligned} &\|E(k,j,s,\ell)\|_{L^2(U_{\nu+j}((1+|t|)B))} \\ &\quad\le C(2^\nu(1+|t|))^{-\kappa_0}(2^{\ell/m}2^jr_B)^{-\kappa_0}(1+2^\ell|t|)^{\kappa_0}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}2^{-\ell n/2}2^{-j\epsilon}V(x_B,2^jr_B)^{-\frac12}. \end{aligned}$$

This, together the doubling condition (1.5) and the definition of ν_0^+, indicates that

$$\begin{aligned} \mathrm{II}(k,j,s) &\le \sum_{\ell>1/m}\sum_{\nu\ge\nu_0^+}\|E(k,j,s,\ell)\|_{L^2(U_{\nu+j}((1+|t|)B))}V(x_B,2^{\nu+j}(1+|t|)r_B)^{\frac12} \\ &\le C2^{-j(\kappa_0+\epsilon)}(1+|t|)^{\frac n2-\kappa_0}\sum_{\ell>1/m}\sum_{\nu\ge\nu_0^+}2^{-\nu(\kappa_0-\frac n2)}(2^{\ell/m}r_B)^{-\kappa_0}(1+2^\ell|t|)^{\kappa_0}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}2^{-\ell n/2} \\ &\le C2^{-j(\frac n2+\epsilon)}(1+|t|)^{n/2}. \end{aligned}$$

Consider the terms $\mathrm{III}(k,j,s)$ and $\mathrm{IV}(k,j,s)$. Note that there is no term $\mathrm{III}(k,j,s)$ if $2^{(\ell-1)/m+j}r_B > \frac18$ and $\ell \le 1/m$. Therefore, when $2^{(\ell-1)/m+j}r_B \le \frac18$ and $\ell \le 1/m$, similar to the proof of (4.10), we obtain that

$$\begin{aligned} \|E(k,j,s,\ell)\|^2_{L^2(X)} &\le C\int_{2^{-\ell}}^{2^{-\ell+1}}\|e^{it(\cdot)}F_{\tau,s}\|^2_{L^\infty}\|r_B^{-mk}L^{M-k}b\|^2_{L^2(U_j(B))}\frac{d\tau}{\tau} \\ &\le C\int_{2^{-\ell}}^{2^{-\ell+1}}\min\{1,(\tau^{-1/m}r_B)^{2mM}\}2^{-2j\epsilon}V(x_B,2^jr_B)^{-1}\frac{d\tau}{\tau} \end{aligned}$$

$$\leq \quad C\min\{1, (2^{\ell/m}r_B)^{2mM}\}2^{-2j\epsilon}V(x_B, 2^j r_B)^{-1},$$

which gives

$$\begin{aligned}
\mathrm{III}(k,j,s) &\leq \sum_{\ell\leq 1/m}\|E(k,j,s,\ell)\|_{L^2(B(x_B,8(1+|t|)2^{-(\ell-1)/m}))}V(x_B, 8(1+|t|)2^{-(\ell-1)/m})^{\frac12}\\
&\leq C2^{-j(\epsilon+\frac{n}{2})}\sum_{\ell\leq 1/m}\min\{1,(2^{\ell/m}r_B)^{mM}\}(2^{\ell/m}r_B)^{-\frac{n}{2}}(1+|t|)^{n/2}\\
&\leq C2^{-j(\epsilon+\frac{n}{2})}(1+|t|)^{n/2}.
\end{aligned}$$

To estimate the term $\mathrm{IV}(k,j,s)$, we first note that it follows from (4.8) that for $\tau\in[2^{-\ell},2^{-\ell+1}]$, we have $\tau^{1/m}\leq 2^{1/m}2^{-\ell/m}\leq 2^{\nu+j-2}(1+|t|)r_B$. Hence, if $d(x,y)<\tau^{1/m}$ and $x\in U_{\nu+j}((1+|t|)B)$, then $y\in U'_{\nu+j}((1+|t|)B)$, where

$$U'_{\nu+j}((1+|t|)B) := U_{\nu+j+1}((1+|t|)B)\cup U_{\nu+j}((1+|t|)B)\cup U_{\nu+j-1}((1+|t|)B).$$

We write

$$\begin{aligned}
&\|E(k,j,s,\ell)\|^2_{L^2(U_{\nu+j}((1+|t|)B))}\\
&\quad\leq C\sum_{w=-1}^{1}\int_{2^{-\ell}}^{2^{-\ell+1}}\left\|\chi_{U_{\nu+j+w}((1+|t|)B)}e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)\right\|_2^2\frac{d\tau}{\tau}\\
&\quad\leq C\sum_{w=-1}^{1}\int_{2^{-\ell}}^{2^{-\ell+1}}\left\|\chi_{U_{\nu+j+w}((1+|t|)B)}e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}\right\|^2_{2\to2}\|r_B^{-mk}L^{M-k}b\|^2_{L^2(U_j(B))}\frac{d\tau}{\tau}.
\end{aligned}$$

Note that the conditions $\mathrm{supp}\phi\subset(1/4,1)$ and $r_B\leq s\leq\sqrt[m]{2}r_B$ implies if $2^{-\ell}\leq\tau\leq 2^{-l+1}$, then

(4.15) $$\|\delta_{\tau^{-1}}F_{\tau,s}\|_{C^{\kappa_0+1}}\leq C\min\{1,(2^{\ell/m}r_B)^{mM}\}.$$

This, together with the estimate (4.14), implies

$$\left\|\chi_{U_{\nu+j+w}((1+|t|)B)}e^{itL}F_{\tau,s}(L)\chi_{U_j(B)}\right\|_{2\to2}\leq C(2^\nu(1+|t|))^{-\kappa_0}(2^{\ell/m}2^jr_B)^{-\kappa_0}(1+2^\ell|t|)^{\kappa_0}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\},$$

and thus

$$\begin{aligned}
&\|E(k,j,s,\ell)\|_{L^2(U_{\nu+j}((1+|t|)B))}\\
&\quad\leq C(2^\nu(1+|t|))^{-\kappa_0}(2^{\ell/m}2^jr_B)^{-\kappa_0}(1+2^\ell|t|)^{\kappa_0}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}2^{-j\epsilon}V(x_B,2^jr_B)^{-\frac12}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{IV}(k,j,s) &\leq \sum_{\ell\leq 1/m}\sum_{\nu\geq\nu_0^-}\|E(k,j,s,\ell)\|_{L^2(U_{\nu+j}((1+|t|)B))}V(x_B,2^{\nu+j}((1+|t|)B))^{\frac12}\\
&\leq C2^{-j(\frac{n}{2}+\epsilon)}(1+|t|)^{n/2}\sum_{\ell\leq 1/m}(2^{\ell/m}r_B)^{-n/2}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}\\
&\leq C2^{-j(\frac{n}{2}+\epsilon)}(1+|t|)^{n/2}.
\end{aligned}$$

Combining the estimates for $\mathrm{I}(k,j,s)$, $\mathrm{II}(k,j,s)$, $\mathrm{III}(k,j,s)$ and $\mathrm{IV}(k,j,s)$, we obtain the estimate (4.6) and therefore,

$$\sum_{k=0}^{M-1}\left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}\left|e^{itL}E_k(y)\right|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^1} \leq C\sum_{k=0}^{M-1}\sum_{j\geq0}\sup_{s\in[r_B,\sqrt[m]{2}r_B]}\|E(k,j,s)\|_{L^1(X)}$$

$$\begin{aligned}&\le C\sum_{j\ge 0}2^{-j\varepsilon'}(1+t)^{n/2}\\&\le C(1+t)^{n/2}.\end{aligned}$$

Case 2. $k=M$. Similarly to the proof of estimating $e^{itL}E_k$ for $k=1,2,\cdots,M-1$ as in **Case 1**, we conclude that

$$\left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}\left|e^{itL}E_M(y)\right|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{\frac12}\right\|_{L^1}\le C(1+|t|)^{n/2}.$$

This finishes the proof of (4.2) and then Proposition 1.5. □

4.2. **Proof of boundedness on $L^p(X)$.** This subsection will show how to apply the $H^1_L-H^1_L$ boundedness for Schrdinger groups and the complex interpolation method we obtain in the appendix to get the L^p boundedness for Schrödinger groups.

Proof of Theorem 1.3. Inspired by [7], we consider the analytic family of operators

$$T_z:=e^{(1-z)^2}(1+|t|)^{-\frac{(1-z)n}{2}}(I+L)^{-\frac{(1-z)n}{2}}e^{itL},\quad 0\le \mathrm{Re}z\le 1.$$

Then T_z is a holomorphic function of z in the sense that

$$z\to\int_X T_zf(x)g(x)d\mu(x)$$

for $f,g\in L^2(X)$.

By the spectral theorem,

$$\|T_{1+iy}f\|_2=e^{-y^2}\|(I+L)^{\frac{iyn}{2}}e^{itL}\|_2\le C\|f\|_2.$$

Besides, it follows from the Theorem 5.5 that

$$\|(I+L)^{\frac{iyn}{2}}\|_{H^1_L\to H^1_L}\le C(1+|y|)^{\kappa_0+1}.$$

This, in combination with Proposition 1.5, yields

$$\begin{aligned}\|T_{iy}f\|_{L^1}&=e^{1-y^2}(1+|t|)^{-n/2}\|e^{itL}(I+L)^{-n/2}(I+L)^{\frac{iyn}{2}}f\|_{L^1}\\&\le Ce^{1-y^2}\|(I+L)^{\frac{iyn}{2}}f\|_{H^1_L}\\&\le C\|f\|_{H^1_L}.\end{aligned}$$

By complex interpolation between $\mathrm{Re}z=0$ and $\mathrm{Re}z=1$, we obtain that for $\theta\in(0,1)$ and $p\in(1,2)$,

$$\|T_\theta(f)\|_{L^p}\le C\|f\|_{[H^1_L,H^2_L]_\theta}\le C\|f\|_{H^p_L}.$$

where the parameter θ satisfies $1/p=1-\theta/2$. This, together with Lemma 2.9, shows that for any $p_0<p\le 2$,

$$\|(I+L)^{-\sigma_p n}e^{itL}f\|_p=\|e^{-(1-\theta)^2}(1+|t|)^{\frac{(1-\theta)n}{2}}T_\theta f\|_p\le C(1+|t|)^{\sigma_p n}\|f\|_p.$$

By duality, we can obtain the corresponding results for $2\le p<p_0'$. Then the L^p boundedness for Schrödinger groups is proven. □

5. Results on Hardy space $H^1_L(X)$: Proof of Lemmas 2.7, 2.8 and 2.9

In this section, with the help of the estimate (3.1), we will give the proof of Lemmas 2.7, 2.8, 2.9 and establish the spectral theorem on H^1_L.

5.1. **Tent space.** We recall some preliminaries on tent spaces of homogeneous type.

Let F be a measurable function defined on $X \times (0,\infty)$. Denote

$$\mathcal{A}(F)(x) = \left(\int_0^\infty \int_{d(x,y)<\tau} |F(y,\tau)|^2 \frac{d\mu(y)}{V(x,t)}\frac{d\tau}{\tau}\right)^{\frac{1}{2}}.$$

Following [9], for any $1 \le p < \infty$, the tent space $T^p_2(X)$ is defined as the space of measurable functions F on $X \times (0,\infty)$ such that $\mathcal{A}(F) \in L^p(X)$, equipped with the norm:

$$\|F\|_{T^p_2(X)} = \|\mathcal{A}(F)\|_{L^p(X)}.$$

Next, we recall the atomic decomposition theory for tent spaces, which was originally studied in [9].

Definition 5.1. *A measurable function $A(x,\tau)$ on $X \times (0,\infty)$ is said to be a T^2_1-atom if there exists a ball $B \subset X$ such that A is supported in $\hat{B}$ and satisfies*

$$\left(\int_0^\infty \int_X |A(x,\tau)|^2 \frac{d\mu(x)d\tau}{\tau}\right)^{\frac{1}{2}} \le \mu(B)^{-\frac{1}{2}}.$$

The following atomic decomposition theorem for tent spaces was obtained in [34].

Lemma 5.2. *For every $F \in T^1_2(X)$ there exists a constant $C > 0$, a sequence $\{\lambda_j\}_{j=0}^\infty \in \ell^1$ and a sequence of T^1_2-atoms $\{A_j\}_{j=0}^\infty$ such that*

$$F = \sum_{j=0}^\infty \lambda_j A_j \ \text{ in } T^1_2(X) \text{ a.e. in } X \times (0,\infty)$$

and

$$\sum_{j=0}^\infty |\lambda_j| \le C\|F\|_{T^1_2(X)}.$$

In addition, if $F \in T^1_2(X) \cap T^2_2(X)$, then the summation also converges in $T^2_2(X)$.

5.2. **Molecular decompositions for Hardy spaces.** This subsection is devoted to giving molecular decompositions for Hardy spaces H^1_L. To begin with, we consider the operator: $\pi_L : T^2_2(X) \to L^2(X)$, given by

$$\pi_L(F)(x) := \int_0^\infty \phi(\tau^m L)(F(\cdot,\tau))(x)\frac{d\tau}{\tau},$$

where the improper integral converges weakly in $L^2(X)$. By duality and the boundedness of square function, it is not difficult to see that π_L is bounded from $T^2_2(X)$ to $L^2(X)$.

Lemma 5.3. *For any $T^1_2(X)$-atom $A(y,\tau)$ associated to a ball B(or more precisely, to its tent $\hat{B}$), there is a uniform constant $C > 0$ such that $C^{-1}\pi_L(A)$ is a $(1,2,M,\epsilon)$-molecule associated to B for some $\epsilon > 0$.*

Proof. By the definition of $T_2^1(X)$-atom,

$$\left(\int_{X\times(0,\infty)} |A(y,\tau)|^2 \frac{d\mu(y)d\tau}{\tau}\right)^{\frac{1}{2}} \le \mu(B)^{-\frac{1}{2}}.$$

We write $a = \pi_L(A) = L^M b$, where $b(x) = \int_0^\infty L^{-M}\phi(\tau^m L)(A(\cdot,\tau))(x)\frac{d\tau}{\tau}$. Next for any $\ell \ge 0$, $k = 0, 1, \ldots, M$, we estimate $\|(r_B^m L)^k b\|_{L^2(U_\ell(B))}$ by duality. Consider $h \in L^2(U_\ell(B))$ such that $\|h\|_{L^2(U_\ell(B))} = 1$. Then since L is self-adjoint, we can apply Proposition 3.1 to conclude that when $\ell \ge 5$,

$$\begin{aligned}
\left|\int_X (r_B^m L)^k b(x)h(x)d\mu(x)\right| &\le r_B^{mk}\int_{\hat{B}} |A(y,\tau)||L^{k-M}\phi(\tau^m L)h(y)|\frac{d\mu(y)d\tau}{\tau}\\
&\le r_B^{mk}\|A\|_{T_2^2(X)}\left(\int_{\hat{B}} |(\tau^m L)^{k-M}\phi(\tau^m L)h(y)|^2\frac{d\mu(y)d\tau}{\tau^{2m(k-M)+1}}\right)^{\frac{1}{2}}\\
&\le r_B^{mk}\mu(B)^{-\frac{1}{2}}\left(\int_0^{r_B} \|\chi_B(\tau^m L)^{k-M}\phi(\tau^m L)\chi_{U_\ell(B)}h\|_2^2\frac{d\tau}{\tau^{2m(k-M)+1}}\right)^{\frac{1}{2}}\\
&\le Cr_B^{mk}\mu(B)^{-\frac{1}{2}}\left(\int_0^{r_B}\left(\frac{2^\ell r_B}{\tau}\right)^{-2\kappa_0}\frac{d\tau}{\tau^{2m(k-M)+1}}\right)^{\frac{1}{2}}\\
&\le C2^{-\ell(\kappa_0-\frac{n}{2})}r_B^{mM}\mu(2^\ell B)^{-\frac{1}{2}}.
\end{aligned}$$

Taking supremum over all $h \in L^2(U_\ell(B))$ satisfying $\|h\|_{L^2(U_\ell(B))} = 1$ and choosing $\frac{n}{2} < \epsilon < \kappa_0$, we obtain that for any $\ell \ge 0$, $k = 0, 1, \ldots, M$,

$$\|(r_B^m L)^k b\|_{L^2(U_\ell(B))} \le C2^{-\ell(\kappa_0-\frac{n}{2})}r_B^{mM}\mu(2^\ell B)^{-\frac{1}{2}} \le C2^{-\ell\epsilon}r_B^{mM}\mu(2^\ell B)^{-\frac{1}{2}}.$$

This implies that $C^{-1}\pi_L(A)$ is a $(1, 2, M, \epsilon)$-molecule. □

Proof of Lemma 2.7. Set $\mathcal{H}^2(X) = \overline{\{Lu \in L^2(X) : u \in L^2(X)\}}$. Recall that $H^1_{L,mol,M,\epsilon}(X)$ and $H^1_L(X)$ are the completions of $\mathbb{H}^1_{L,mol,M,\epsilon}(X)$ and $H^1_L(X) \cap \mathcal{H}^2(X)$, respectively. It suffices to show that $H^1_{L,mol,M,\epsilon}(X)$ and $H^1_L(X)$ have the same dense subset $\mathbb{H}^1_{L,mol,M,\epsilon}(X) = H^1_L(X) \cap \mathcal{H}^2(X)$ with equivalent norms.

Step I: $\mathbb{H}^1_{L,mol,M,\epsilon}(X) \subset (H^1_L(X) \cap \mathcal{H}^2(X))$.

By definition, $\mathbb{H}^1_{L,mol,M,\epsilon}(X) \subset \mathcal{H}^2(X)$. Therefore, by a standard density argument, it will be enough to show that for every $(1, 2, M, \epsilon)$-molecule $a(x)$ associated to a ball $B = B(x_B, r_B)$ of X, we have

$$\|S_{L,\phi}(a)\|_{L^1(X)} \le C.$$

Denote $F(y,\tau) = \phi(\tau^m L)a$. By a simple change of variable, it is enough to show that

$$\|F\|_{T_2^1(X)} \le C. \tag{5.1}$$

Now let $\eta_0 = \chi_{2B\times(0,2r_B)}$ and for all $j \ge 1$, define $\eta_j = \chi_{U_{j+1}(B)\times(0,r_B)}$, $\eta'_j = \chi_{U_{j+1}(B)\times(r_B,2^{j+1}r_B)}$ and $\eta''_j = \chi_{2^jB\times(2^jr_B,2^{j+1}r_B)}$. Then we decompose F as follows.

$$F = \eta_0 F + \sum_{j=1}^\infty \eta_j F + \sum_{j=1}^\infty \eta'_j F + \sum_{j=1}^\infty \eta''_j F.$$

Next, we will show that there exist constants $C, \sigma > 0$, such that

(a) For any $j \geq 0$, $\|\eta_j F\|_{T_2^2(X)} \leq C2^{-j\sigma}\mu(2^jB)^{-\frac{1}{2}}$;

(b) For any $j \geq 1$, $\|\eta'_j F\|_{T_2^2(X)} \leq C2^{-j\sigma}\mu(2^jB)^{-\frac{1}{2}}$;

(c) For any $j \geq 1$, $\|\eta''_j F\|_{T_2^2(X)} \leq C2^{-j\sigma}\mu(2^jB)^{-\frac{1}{2}}$.

Since each $\eta_j F$, $\eta'_j F$, $\eta''_j F$ are supported in $\widehat{2^{j+2}B}$, these three estimates will imply that $\frac{1}{C}2^{j\sigma}\eta_j F$, $\frac{1}{C}2^{j\sigma}\eta'_j F$ and $\frac{1}{C}2^{j\sigma}\eta''_j F$ are atoms in $T_2^1(X)$, respectively, and thus the estimate (5.1) will be done.

Now we show the estimates (a),(b),(c). To show (a), we first apply the L^2 boundedness of the square function to obtain that

$$\|\eta_0 F\|_{T_2^2(X)} \leq C\left(\int_0^\infty \int_X |\phi(\tau^m L)a(y)|^2 \frac{d\mu(y)d\tau}{\tau}\right)^{\frac{1}{2}} \leq C\|a\|_{L^2(X)} \leq C\mu(B)^{-\frac{1}{2}}.$$

For $j \geq 1$, we apply the formula (4.4) to obtain that

$$\|\chi_{U_{j+1}(B)}\phi(\tau^m L)a\|_2 \leq \sum_{\ell=0}^{\infty} \mathrm{I}(\ell,\tau) + \sum_{\ell=0}^{\infty} \mathrm{II}(\ell,\tau), \tag{5.2}$$

where

$$\mathrm{I}(\ell,\tau) = \sum_{k=0}^{M-1} r_B^{-m} \int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}\|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^M G_{k,r_B,M}(L)\phi(\tau^m L)\chi_{U_\ell(B)}(r_B^{-mk}L^{M-k}b)\|_2 ds$$

and

$$\mathrm{II}(\ell,\tau) = \sum_{\nu=1}^{(2M-1)M} \|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^M e^{-\nu r^m L}\phi(\tau^m L)\chi_{U_\ell(B)}(r_B^{-mM}b)\|_2.$$

We apply Proposition 3.1 to conclude that when $\ell \leq j-5$,

$$\begin{aligned}
\mathrm{I}(\ell,\tau) &\leq C\sum_{k=0}^{M-1} \sup_{s\in[r_B,\sqrt[m]{2}r_B]} \|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^M G_{k,r_B,M}(L)\phi(\tau^m L)\chi_{U_\ell(B)}(r_B^{-mk}L^{M-k}b)\|_2 \\
&\leq C\left(\frac{2^j r_B}{\tau}\right)^{-\kappa_0} \|r_B^{-mk}L^{M-k}b\|_{L^2(U_\ell(B))} \leq C2^{-\ell\epsilon}\left(\frac{2^j r_B}{\tau}\right)^{-\kappa_0} \mu(2^\ell B)^{-\frac{1}{2}} \\
&\leq C2^{-\ell\epsilon}\left(\frac{2^j r_B}{\tau}\right)^{-\kappa_0} 2^{(j-\ell)n/2}\mu(2^jB)^{-\frac{1}{2}},
\end{aligned}$$

and thus

$$\sum_{\ell=0}^{j-5}\left(\int_0^{r_B} \left|\mathrm{I}(\ell,\tau)\right|^2 \frac{d\tau}{\tau}\right)^{\frac{1}{2}} \leq C\sum_{\ell=0}^{j-5} 2^{-\ell(\epsilon+\frac{n}{2})}2^{-j(\kappa_0-\frac{n}{2})}\mu(2^jB)^{-\frac{1}{2}} \leq C2^{-j(\kappa_0-\frac{n}{2})}\mu(2^jB)^{-\frac{1}{2}}. \tag{5.3}$$

Besides, the self-adjoint property of the operator L allows us to apply Proposition 3.1 to conclude that when $j \leq \ell - 5$,

$$\begin{aligned}
\mathrm{I}(\ell,\tau) &\leq C\sum_{k=0}^{M-1} \sup_{s\in[r_B,\sqrt[m]{2}r_B]} \|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^M G_{k,r_B,M}(L)\phi(\tau^m L)\chi_{U_\ell(B)}(r_B^{-mk}L^{M-k}b)\|_2 \\
&\leq C\left(\frac{2^\ell r_B}{\tau}\right)^{-\kappa_0} \|r_B^{-mk}L^{M-k}b\|_{L^2(U_\ell(B))}
\end{aligned}$$

$$\leq C2^{-\ell\epsilon}\left(\frac{2^\ell r_B}{\tau}\right)^{-\kappa_0} 2^{(j-\ell)n/2}\mu(2^jB)^{-\frac12},$$

and therefore

$$\sum_{\ell=j+5}^{\infty}\left(\int_0^{r_B}\left|\mathrm{I}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac12} \leq C\sum_{\ell=j+5}^{\infty}2^{-\ell\epsilon}2^{(j-\ell)n/2}\mu(2^jB)^{-\frac12}\left(\int_0^{r_B}\left(\frac{2^\ell r_B}{\tau}\right)^{-2\kappa_0}\frac{d\tau}{\tau}\right)^{\frac12}$$

$$\leq C2^{-j(\kappa_0+\epsilon)}\mu(2^jB)^{-\frac12}. \tag{5.4}$$

Finally, by the L^2 boundedness of the square function, we have

$$\begin{aligned}\sum_{\ell=j-4}^{j+4}\left(\int_0^{r_B}\left|\mathrm{I}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac12} &\leq C\sum_{\ell=j-4}^{j+4}\left(\int_0^{\infty}\|\phi(\tau^mL)\chi_{U_\ell(B)}(r_B^{-mk}L^{M-k}b)\|_2^2\frac{d\tau}{\tau}\right)^{\frac12}\\ &\leq C\sum_{\ell=j-4}^{j+4}\|r_B^{-mk}L^{M-k}b\|_{L^2(U_\ell(B))}\\ &\leq C2^{-j\epsilon}\mu(2^jB)^{-\frac12}.\end{aligned} \tag{5.5}$$

The term $\mathrm{II}(\ell,\tau)$ can be handled in a similar way. This, in combination with the estimates (5.2), (5.3), (5.4) and (5.5), imply that for any $j\geq 1$,

$$\|\eta_jF\|_{T_2^2}\leq C\left(\int_0^{r_B}\|\chi_{U_{j+1}(B)}\phi(\tau^mL)a\|_2^2\frac{d\tau}{\tau}\right)^{\frac12}\leq C2^{-j\sigma}\mu(2^jB)^{-\frac12}$$

for some $\sigma>0$.

Next, we turn to show the estimate (b). We decompose $M=M_0+M_1$, where M_0, M_1 are two constants to be chosen large enough later. Then, we iterate the formula (4.3) M_0 times to conclude that

$$\|\chi_{U_{j+1}(B)}\phi(\tau^mL)a\|_2 \leq \sum_{\ell=0}^{\infty}\mathrm{III}(\ell,\tau)+\sum_{\ell=0}^{\infty}\mathrm{IV}(\ell,\tau), \tag{5.6}$$

where

$$\mathrm{III}(\ell,\tau)=\sum_{k=0}^{M_0-1}r_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B}s^{m-1}\|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^{M_0}G_{k,r_B,M_0}(L)L^{M_1}\phi(\tau^mL)\chi_{U_\ell(B)}(r_B^{-mk}L^{M_0-k}b)\|_2ds$$

and

$$\mathrm{IV}(\ell,\tau)=\sum_{\nu=1}^{(2M_0-1)M_0}\|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^{M_0}e^{-\nu r^mL}L^{M_1}\phi(\tau^mL)\chi_{U_\ell(B)}(r_B^{-mM_0}b)\|_2.$$

We apply Proposition 3.1 to conclude that when $\ell\leq j-5$,

$$\begin{aligned}&\sup_{s\in[r_B,\sqrt[m]{2}r_B]}\|\chi_{U_{j+1}(B)}(1-e^{-s^mL})^{M_0}G_{k,r_B,M_0}(L)(\tau^mL)^{M_1}\phi(\tau^mL)\chi_{U_\ell(B)}\|_{2\to2}\\ &\leq C\tau^{-mM_1}\left(\frac{2^jr_B}{\tau}\right)^{-\kappa_0}\min\{1,(\tau^{-1/m}r_B)^{mM_0-c_0}\}\leq C\tau^{-mM_1}\left(\frac{2^jr_B}{\tau}\right)^{-\kappa_0}\end{aligned}$$

for some $M_0>c_0/m$, which implies that

$$\mathrm{III}(\ell,\tau)\leq C\tau^{-mM_1}\left(\frac{2^jr_B}{\tau}\right)^{-\kappa_0}\|r_B^{-mk}L^{M_0-k}b\|_{L^2(U_\ell(B))}\leq C2^{-\ell\epsilon}\tau^{-mM_1}\left(\frac{2^jr_B}{\tau}\right)^{-\kappa_0}r_B^{mM_1}\mu(2^\ell B)^{-\frac12}$$

$$\leq C2^{-\ell\epsilon}\tau^{-mM_1}\left(\frac{2^j r_B}{\tau}\right)^{-\kappa_0} r_B^{mM_1}2^{(j-\ell)n/2}\mu(2^jB)^{-\frac{1}{2}}.$$

Hence,

$$\sum_{\ell=0}^{j-5}\left(\int_{r_B}^{2^{j+1}r_B}\left|\mathrm{III}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac{1}{2}} \leq C\sum_{\ell=0}^{j-5}2^{-\ell\epsilon}r_B^{mM_1}2^{(j-\ell)n/2}\mu(2^jB)^{-\frac{1}{2}}\left(\int_{r_B}^{2^{j+1}r_B}\left(\frac{2^jr_B}{\tau}\right)^{-2\kappa_0}\frac{d\tau}{\tau^{2mM_1+1}}\right)^{\frac{1}{2}}$$

$$\leq C\sum_{\ell=0}^{j-5}2^{-\ell\epsilon}2^{(j-\ell)n/2}\mu(2^jB)^{-\frac{1}{2}}2^{-j\kappa_0} \leq C2^{-j(\kappa_0-\frac{n}{2})}\mu(2^jB)^{-\frac{1}{2}}, \tag{5.7}$$

where in the next to the last inequality we choose $M_1 > \frac{\kappa_0}{m}$ such that the integral can be bounded by a uniform constant $C > 0$. Similarly,

$$\sum_{\ell=j+5}^{\infty}\left(\int_{r_B}^{2^{j+1}r_B}\left|\mathrm{III}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac{1}{2}} \leq C\sum_{\ell=j+5}^{\infty}2^{-\ell\epsilon}r_B^{mM_1}2^{(j-\ell)n/2}\mu(2^jB)^{-\frac{1}{2}}\left(\int_{r_B}^{2^{j+1}r_B}\left(\frac{2^\ell r_B}{\tau}\right)^{-2\kappa_0}\frac{d\tau}{\tau^{2mM_1+1}}\right)^{\frac{1}{2}}$$

$$\leq C\sum_{\ell=j+5}^{\infty}2^{-\ell(\kappa_0+\frac{n}{2}+\epsilon)}2^{\frac{jn}{2}}\mu(2^jB)^{-\frac{1}{2}} \leq C2^{-j(\kappa_0+\epsilon)}\mu(2^jB)^{-\frac{1}{2}}. \tag{5.8}$$

Also we have that

$$\begin{aligned}\sum_{\ell=j-4}^{j+4}\left(\int_{r_B}^{2^{j+1}r_B}\left|\mathrm{III}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac{1}{2}} &\leq C\sum_{\ell=j-4}^{j+4}r_B^{-mM_1}\left(\int_{r_B}^{2^{j+1}r_B}\|(\tau^mL)^{M_1}\phi(\tau^mL)\chi_{U_\ell(B)}(r_B^{-mk}L^{M_0-k}b)\|_2^2\frac{d\tau}{\tau}\right)^{\frac{1}{2}}\\ &\leq \sum_{\ell=j-4}^{j+4}r_B^{-mM_1}\|r_B^{-mk}L^{M_0-k}b\|_{L^2(U_\ell(B))}\\ &\leq C2^{-j\epsilon}\mu(2^jB)^{-\frac{1}{2}}.\end{aligned}$$

The term $\mathrm{IV}(\ell,\tau)$ can be handled in a similar way. This, in combination with the estimates (5.6), (5.7), (5.8) and (5.9), implies that for any $j \geq 1$,

$$\|\eta_j'F\|_{T_2^2} \leq C\left(\int_{r_B}^{2^{j+1}r_B}\|\chi_{U_{j+1}(B)}\phi(\tau^mL)a\|_2^2\frac{d\tau}{\tau}\right)^{\frac{1}{2}} \leq C2^{-j\sigma}\mu(2^jB)^{-\frac{1}{2}}$$

for some $\sigma > 0$.

Finally, it remains to show the estimate (c). Similar to the proof of the estimate (b), we decompose $M = M_0 + M_1$, where M_0, M_1 are two constants to be chosen large enough later. Then, we iterate the formula (4.3) M_0 times to conclude that

$$\|\chi_{2^jB}\phi(\tau^mL)a\|_2 \leq \sum_{\ell=0}^{\infty}\mathrm{V}(\ell,\tau)+\sum_{\ell=0}^{\infty}\mathrm{VI}(\ell,\tau), \tag{5.9}$$

where

$$\mathrm{V}(\ell,\tau) = \sum_{k=0}^{M_0-1}r_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B}s^{m-1}\|\chi_{2^jB}(1-e^{-s^mL})^{M_0}G_{k,r_B,M_0}(L)L^{M_1}\phi(\tau^mL)\chi_{U_\ell(B)}(r_B^{-mk}L^{M_0-k}b)\|_2ds$$

and

$$\mathrm{VI}(\ell,\tau)=\sum_{\nu=1}^{(2M_0-1)M_0}\|\chi_{2^jB}(1-e^{-s^mL})^{M_0}e^{-\nu r^mL}L^{M_1}\phi(\tau^mL)\chi_{U_\ell(B)}(r_B^{-mM_0}b)\|_2.$$

It follows from the L^2 boundedness of square function that when $\ell\leq j+4$,

$$\begin{aligned}\left(\int_{2^jr_B}^{2^{j+1}r_B}\left|\mathrm{V}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac12}&\leq C(2^jr_B)^{-mM_1}\|r_B^{-mk}L^{M_0-k}\|_{L^2(U_\ell(B))}\\&\leq C2^{-\ell\epsilon}2^{(j-\ell)n/2}2^{-jmM_1}\mu(2^jB)^{-\frac12}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\ell=0}^{j+4}\left(\int_{2^jr_B}^{2^{j+1}r_B}\left|\mathrm{V}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac12}&\leq C\sum_{\ell=0}^{j+4}2^{-\ell\epsilon}2^{(j-\ell)n/2}2^{-jmM_1}\mu(2^jB)^{-\frac12}\\&\leq C2^{-j(mM_1-\frac n2)}\mu(2^jB)^{-\frac12}.\end{aligned}\tag{5.10}$$

Besides, if we choose M_0 sufficient large, then the self-adjoint property of the operator L allows us to apply Proposition 3.1 to conclude that when $\ell\geq j+5$,

$$\begin{aligned}\left(\int_{2^jr_B}^{2^{j+1}r_B}\left|\mathrm{V}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac12}&\leq C(2^jr_B)^{-mM_1}\left(\int_{2^jr_B}^{2^{j+1}r_B}\left(\frac{2^\ell r_B}{\tau}\right)^{-2\kappa_0}\frac{d\tau}{\tau}\right)^{\frac12}\|r_B^{-mk}L^{M_0-k}b\|_{L^2(U_\ell(B))}\\&\leq C2^{-\ell\epsilon}2^{(j-\ell)(\kappa_0+\frac n2)}2^{-jmM_1}\mu(2^jB)^{-\frac12}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{\ell=j+5}^{\infty}\left(\int_{2^jr_B}^{2^{j+1}r_B}\left|\mathrm{V}(\ell,\tau)\right|^2\frac{d\tau}{\tau}\right)^{\frac12}&\leq C\sum_{\ell=j+5}^{\infty}2^{-\ell\epsilon}2^{(j-\ell)(\kappa_0+\frac n2)}2^{-jmM_1}\mu(2^jB)^{-\frac12}\\&\leq C2^{-j(mM_1+\epsilon)}\mu(2^jB)^{-\frac12}.\end{aligned}\tag{5.11}$$

The term $\mathrm{VI}(\ell,\tau)$ can be handled in a similar way. This, in combination with the estimates (5.9), (5.10) and (5.11), implies that if we choose $M_1>\frac{n}{2m}$, then for any $j\geq1$,

$$\|\eta_j''F\|_{T_2^2}\leq C\left(\int_{2^jr_B}^{2^{j+1}r_B}\|\chi_{2^jB}\phi(\tau^mL)a\|_2^2\frac{d\tau}{\tau}\right)^{\frac12}\leq C2^{-j\sigma}\mu(2^jB)^{-\frac12}$$

for some $\sigma>0$. This finishes the proof of the estimate (c) and then the **Step I**.

Step II: $(H_L^1(X)\cap\mathcal{H}^2(X))\subset\mathbb{H}^1_{L,mol,M,\epsilon}(X)$.

Let $f\in H_L^1(X)\cap\mathcal{H}^2(X)$, we will establish a molecular $(1,2,M,\epsilon)$-representation for f. To this end, we modify the argument in [13] and set $F(x,\tau)=\phi(\tau^mL)f(x)$. Then the definition of $H_L^1(X)$ and the L^2 boundedness of square function imply that $F\in T_2^1(X)\cap T_2^2(X)$. Therefore, it follows from the Lemma 5.2 that

$$F=\sum_j\lambda_jA_j,$$

where each A_j is a $T_2^1(X)$-atom, the sum converges in both $T_2^1(X)$ and $T_2^2(X)$, and

$$\sum_j |\lambda_j| \le C\|F\|_{T_2^1(X)} \le C\|f\|_{H_L^1(X)}.$$

Besides, by L^2-functional calculus, we have

$$f(x) = c\int_0^\infty \phi(\tau^m L)\phi(\tau^m L)f(x)\frac{d\tau}{\tau} = c\pi_L(F)(x) = c\sum_j \lambda_j \pi_L(A_j)(x), \tag{5.12}$$

where the last sum converges in $L^2(X)$ (see [23, Lemma 3.22]). Lemma 5.3 implies that up to a harmless constant $C > 0$, each $\pi_L(A_j)$ is a $(1, 2, M, \epsilon)$-molecule associated to B for some $\epsilon > 0$, which indicates that (5.12) gives a molecular $(1, 2, M, \epsilon)$-representation of f so that $f \in \mathbb{H}^1_{L,mol,M,\epsilon}(X)$. This finishes the proof of Lemma 2.7. □

5.3. **Interpolation.** The goal of this subsection is to establish the theory of complex interpolation for Hardy spaces.

Proof of Lemma 2.8. For any $f \in H_L^p$, $1 \le p < \infty$, we consider

$$Q_{\tau,L} f(x,\tau) := \phi(\tau^m L)f(x), \quad \tau > 0, \ x \in X.$$

Then by the definition of $H_L^p(X)$, $Q_{\tau,L}$ embeds the Hardy space $H_L^p(X)$ isometrically into the tent space $T_2^p(X)$ for $1 \le p < \infty$. Besides, from Lemma 5.3 we can easily see that the condition ($\mathrm{PEV}_{2,2}^{\kappa,m}$) implies that for any $1 \le p < \infty$, π_L is bounded from $T_2^p(X)$ to $H_L^p(X)$. By the L^2-functional calculus, for any $f \in L^2(X)$, there exists a constant $c > 0$ such that the following Calderón reproducing formula holds:

$$f(x) = c\pi_L(Q_{\tau,L} f)(x).$$

Then Lemma 2.8 can be shown by following a similar outline in [23]. □

Lemma 5.4. *Let $1 \le p_0 < 2$. Suppose that T is a sublinear operator bounded on $L^2(X)$, and let $\{A_r\}_{r>0}$ be a family of linear operators acting on $L^2(X)$. Assume that there exists a constant $N > \frac{n}{2}$ such that for $j \ge 6$,*

$$\|\chi_{U_j(B)} T(I - A_{r_B})\chi_B f\|_2 \le C2^{-jN}\mu(B)^{-\sigma_{p_0}}\|f\|_{p_0} \tag{5.13}$$

and for $j \ge 0$,

$$\|\chi_{U_j(B)} A_{r_B}\chi_B f\|_2 \le C2^{-jN}\mu(B)^{-\sigma_{p_0}}\|f\|_{p_0} \tag{5.14}$$

for all ball B with r_B the radius of B and all f supported in B. Then for any $p_0 < p \le 2$, T is bounded on L^p.

Proof. The proof is a slight modification of Theorem 2.1 in [1]. We omit the details and leave it to the readers. □

Next, we give the proof of Lemma 2.9.

Proof of Lemma 2.9. We first show that $L^p(X) \subset H^p_L(X)$, or equivalently, for any $p \in (p_0, 2]$, $S_{L,\phi}$ is bounded on $L^p(X)$. By Lemma 5.4, it is enough to verify that there exists a sufficient large constant M such that the operator $T = S_{L,\phi}$ and $A_{r_B} = I - (I - e^{-r_B^m L})^M$ satisfy the estimates (5.13) and (5.14). Indeed,

$$
\begin{aligned}
&\|\chi_{U_j(B)} S_{L,\phi}(I - e^{-r_B^m L})^M \chi_B f\|_2 \\
\leq\ & \left(\int_{U_j(B)} \int_0^{2^{m(j-2)}r_B^m} \int_{d(x,y)<\tau^{1/m}} |\phi(\tau L)(I - e^{-r_B^m L})^M \chi_B f(y)|^2 \frac{d\mu(y) d\tau d\mu(x)}{V(x, \tau^{1/m})\tau}\right)^{\frac{1}{2}} \\
+\ & \left(\int_{U_j(B)} \int_{2^{m(j-2)}r_B^m}^{\infty} \int_{d(x,y)<\tau^{1/m}} |\phi(\tau L)(I - e^{-r_B^m L})^M \chi_B f(y)|^2 \frac{d\mu(y) d\tau d\mu(x)}{V(x, \tau^{1/m})\tau}\right)^{\frac{1}{2}} \\
=:\ & \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

Note that if $x \in U_j(B)$, $\tau < 2^{m(j-2)} r_B^m$ and $d(x,y) < \tau^{1/m}$, then $y \in U_j'(B)$, where

$$U_j'(B) := U_{j-1}(B) \cup U_j(B) \cup U_{j+1}(B).$$

Hence, it follows from the doubling condition (1.5) and Proposition 3.1 that there exist constants $C, c > 0$ such that

$$
\begin{aligned}
\mathrm{I} &\leq C\left(\int_0^{2^{m(j-2)}r_B^m} \|\chi_{U_j'(B)} \phi(\tau L)(I - e^{-r_B^m L})^M \chi_B f\|_2^2 \frac{d\tau}{\tau}\right)^{\frac{1}{2}} \\
&\leq C 2^{-j\kappa_0} \mu(B)^{-\sigma_{p_0}} \left(\int_0^{2^{m(j-2)}r_B^m} (\tau^{-1/m} r_B)^{-2\kappa_0 + 2n\sigma_{p_0}} \min\{1, (\tau^{-1/m} r_B)^{2mM - c_0}\} \frac{d\tau}{\tau}\right)^{\frac{1}{2}} \|f\|_{p_0} \\
&\leq C 2^{-j\kappa_0} \mu(B)^{-\sigma_{p_0}} \|f\|_{p_0}.
\end{aligned}
$$

Next, we apply Lemma 2.2 and the doubling condition (1.5) to obtain that

$$
\begin{aligned}
\mathrm{II} &\leq \left(\int_{2^{m(j-2)}r_B^m}^{\infty} \|\phi(\tau L)(I - e^{-r_B^m L})^M \chi_B f(y)\|_2^2 \frac{d\tau}{\tau}\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{2^{m(j-2)}r_B^m}^{\infty} \min\{1, (\tau^{-1/m} r_B)^{2mM}\} \|V_{\tau^{1/m}}^{-\sigma_{p_0}} f\|_{p_0}^2 \frac{d\tau}{\tau}\right)^{\frac{1}{2}} \\
&\leq C\mu(B)^{-\sigma_{p_0}} \left(\int_{2^{m(j-2)}r_B^m}^{\infty} \min\{1, (\tau^{-1/m} r_B)^{2mM}\} \frac{d\tau}{\tau}\right)^{\frac{1}{2}} \|f\|_{p_0} \\
&\leq C 2^{-jmM} \mu(B)^{-\sigma_{p_0}} \|f\|_{p_0}.
\end{aligned}
$$

Hence, (5.13) is proved after we choose $M > \frac{n}{2m}$.

Besides, observe that $[I - (I - e^{-r_B^m L})^M]$ is a finite combination of the terms $e^{-kr_B^m L}$, $k = 1, \ldots, M$, and that there exists a constant $j_0 > 0$ (only depending on M) such that for any $j > j_0$, the semigroup $e^{-kr_B^m L}$ satisfies the following estimate

$$
\|\chi_{U_j(B)} e^{-kr_B^m L} \chi_B f\|_2 \leq \left(\sum_{\alpha \in \mathcal{I}_{\sqrt[m]{k} r_B},\ d(x_\alpha, x_B) \geq 2^{j-j_0} \sqrt[m]{k} r_B} \|\chi_{Q_\alpha(\sqrt[m]{k} r_B)} e^{-kr_B^m L} \chi_{B(x_B, \sqrt[m]{k} r_B)} f\|_2^2\right)^{\frac{1}{2}}
$$

$$\leq C\mu(B)^{-\sigma_{p_0}}\left(\sum_{\alpha\in\mathcal{I}_{\sqrt[m]{k}r_B},\ d(x_\alpha,x_B)\geq 2^{j-j_0}\sqrt[m]{k}r_B}\left(1+\frac{d(x_\alpha,x_B)}{\sqrt[m]{k}r_B}\right)^{-2n-2\kappa}\right)^{\frac{1}{2}}\|f\|_{p_0}$$
$$\leq C2^{-\frac{j(n+2\kappa)}{2}}\mu(B)^{-\sigma_{p_0}}\|f\|_{p_0}.$$

This, together with the Lemma 2.1, shows (5.14).

Next, by the same argument as in the proof of [36, Theorem 4.19], our proof can be reduced to showing the L^p boundedness of square function $G_{L,\phi}$, where $p\in(2,p_0')$ and $G_{L,\phi}$ is defined by

$$G_{L,\phi}f(x):=\left(\int_0^\infty|\phi(tL)f(x)|^2\frac{dt}{t}\right)^{1/2},$$

while this boundedness can be obtained by verifying the condition in [1, Theorem 2.2] as verifying the estimates (5.13) and (5.14) (see also [1, p.78]). This ends the proof of Lemma 2.9. □

5.4. **Spectral multipliers theorem on the Hardy space $H^1_L(X)$.** Under the assumption that the operator L satisfies the Gaussian upper bounds ($\mathrm{GE_m}$), the spectral multipliers theorem on the Hardy space $H^1_L(X)$ was shown in [17]. Now, with the help of Proposition 3.1 and Lemma 2.7, we can extend this result under a weaker assumption that L satisfies the inequality ($\mathrm{PEV}^{\kappa,m}_{2,2}$). Such a result is a helpful tool to obtain the boundedness on L^p for Schrödinger groups.

Theorem 5.5. *Suppose that L satisfies the estimate* ($\mathrm{PEV}^{\kappa,m}_{2,2}$) *for some $m>0$ and $\kappa>\kappa_0:=[\frac{n}{2}]+1$. Assume in addition that F is an even bounded Borel function such that $\sup_{R>0}\|\eta\delta_RF\|_{C^{\kappa_0+1}}<\infty$ and some nonzero cutoff function $\eta\in C_c^\infty(\mathbb{R}_+)$. Then the operator $F(L)$ is bounded on $H^1_L(X)$. More precisely,*

$$\|F(L)\|_{H^1_L(X)\to H^1_L(X)}\leq C\left(\sup_{R>0}\|\eta\delta_RF\|_{C^{\kappa_0+1}}+F(0)\right).$$

Proof. We will modify the proof of Proposition 1.5 and use the same notation as before (except that we will use $F(\lambda)$ to denote an even bounded Borel function instead of $(I+\lambda)^{-n/2}$) to show this theorem.

It suffices to show that

$$\left\|\left(\int_0^{+\infty}\int_{d(x,y)<\tau^{1/m}}|\phi(\tau L)F(L)a(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{\frac{1}{2}}\right\|_{L^1(X)}\leq C\left(\sup_{R>0}\|\eta\delta_RF\|_{C^{\kappa_0+1}}+F(0)\right).\tag{5.15}$$

We may assume in the sequel that $F(0)=0$. Otherwise, we may replace F by $F-F(0)$.

By the formula (4.4), the theorem can be reduced to showing that for $k=0,1,\cdots,M$,

$$\left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|E_k(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{\frac{1}{2}}\right\|_{L^1}\leq C\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}.$$

Case 1. $k=0,1,\cdots,M-1$. In this case, we see that

$$\left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|E_k(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^1}$$

$$
\begin{aligned}
&\leq C\sum_{j\geq 0}\sup_{s\in[r_B,\sqrt[m]{2}r_B]}\left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^1} \\
&=: C\sum_{j\geq 0}\sup_{s\in[r_B,\sqrt[m]{2}r_B]}\|E'(k,j,s)\|_{L^1(X)}, \qquad (5.16)
\end{aligned}
$$

where

$$
E'(k,j,s)=\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}.
$$

Let us estimate the term $\|E'(k,j,s)\|_{L^1(X)}$. To begin with, let $\psi\in C_c^\infty(\mathbb{R})$ such that supp $\psi\subseteq(1/8,2)$ and supp$\psi=1$ on $(1/4,1)$. Noting that $\|G_{k,r_B,M}\|_{L^\infty}\leq C$, we apply the estimate (4.1), the L^2-boundedness of the square function and the doubling condition (1.5) to see that

$$
\begin{aligned}
&\|E'(k,j,s)\|_{L^1(64\cdot 2^jB)} \\
&\leq \left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2}\right\|_{L^2(64\cdot 2^jB)}\mu(64\cdot 2^jB)^{\frac12} \\
&\leq C\left(\int_0^\infty\|\psi_\tau(L)F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)\|_2^2\frac{d\tau}{\tau}\right)^{\frac12}\mu(64\cdot 2^jB)^{\frac12} \\
&\leq C\left\|r_B^{-mk}L^{M-k}b\right\|_{L^2(U_j(B))}\mu(64\cdot 2^jB)^{\frac12}\sup_{R>0}\|\phi\delta_RF\|_{L^\infty} \\
&\leq C2^{-j\epsilon}\sup_{R>0}\|\phi\delta_RF\|_{L^\infty}. \qquad (5.17)
\end{aligned}
$$

Next we show that for some $\varepsilon'>0$,

$$
\|E'(k,j,s)\|_{L^1((64\cdot 2^jB)^c)}\leq C2^{-j\varepsilon'}\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}.
$$

To begin with, we have a decomposition according to the frequency,

$$
\begin{aligned}
E'(k,j,s)&=\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2} \\
&\leq\sum_{\ell\in\mathbb{Z}}\left(\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<\tau^{1/m}}|F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{1/2} \\
&=:\sum_{\ell\in\mathbb{Z}}E'(k,j,s,\ell). \qquad (5.18)
\end{aligned}
$$

Let $\nu_0\in\mathbb{Z}_+$ be a positive integer such that

$$
\begin{aligned}
8<2^{\nu_0+j+(\ell-1)/m}r_B\leq 16,\ \text{ if } 2^{(\ell-1)/m+j}r_B\leq\frac18; \\
\nu_0=7,\ \text{ if } 2^{(\ell-1)/m+j}r_B>\frac18. \qquad (5.19)
\end{aligned}
$$

Then it follows from (5.19) that if $2^{(\ell-1)/m+j}r_B\leq\frac18$,

$$
\begin{aligned}
\|E'(k,j,s)\|_{L^1((64\cdot 2^jB)^c)}&\leq\sum_{\ell\in\mathbb{Z}}\|E'(k,j,s,\ell)\|_{L^1(B(x_B,8\cdot 2^{-(\ell-1)/m}))}+\sum_{\ell\in\mathbb{Z}}\sum_{\nu\geq\nu_0}\|E'(k,j,s,\ell)\|_{L^1(U_{\nu+j}(B))} \\
&=: \mathrm{I}(k,j,s)+\mathrm{II}(k,j,s). \qquad (5.20)
\end{aligned}
$$

Note that there is no term $\mathrm{I}(k,j,s)$ if $2^{(\ell-1)/m+j}r_B>\frac{1}{8}$. Besides, when $2^{(\ell-1)/m+j}r_B\leq\frac{1}{8}$, similar to the procedure of dealing with the term $\mathrm{I}(k,j,s)$ defined by (4.9), we can easily show that

$$\mathrm{I}(k,j,s)\leq C2^{-j\epsilon}\sup_{R>0}\|\phi\delta_R F\|_{C^{\kappa_0+1}}. \tag{5.21}$$

By the estimates (5.20) and (5.21), it remains to estimate the second term $\mathrm{II}(k,j,s)$. To estimate this term, we first note that (5.19) implies that for $\tau\in[2^{-\ell},2^{-\ell+1}]$, we have $\tau^{1/m}\leq 2^{1/m}2^{-\ell/m}\leq 2^{\nu+j-2}r_B$. Hence, if $d(x,y)<\tau^{1/m}$ and $x\in U_{\nu+j}(B)$, then $y\in U''_{\nu+j}(B)$, where

$$U''_{\nu+j}(B):=U_{\nu+j+1}(B)\cup U_{\nu+j}(B)\cup U_{\nu+j-1}(B).$$

Then we have

$$\begin{aligned}
&\|E'(k,j,s,\ell)\|^2_{L^2(U_{\nu+j}(B))}\\
&\quad\leq C\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{U''_{\nu+j}(B)}|F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)(y)|^2\int_{d(x,y)<\tau^{1/m}}d\mu(x)\frac{d\mu(y)}{V(y,\tau^{1/m})}\frac{d\tau}{\tau}\\
&\quad\leq C\sum_{w=-1}^{1}\int_{2^{-\ell}}^{2^{-\ell+1}}\|\chi_{U_{\nu+j+w}(B)}F_{\tau,s}(L)\chi_{U_j(B)}(r_B^{-mk}L^{M-k}b)\|_2^2\frac{d\tau}{\tau}\\
&\quad\leq C\sum_{w=-1}^{1}\int_{2^{-\ell}}^{2^{-\ell+1}}\|\chi_{U_{\nu+j+w}(B)}F_{\tau,s}(L)\chi_{U_j(B)}\|^2_{2\to2}\|r_B^{-mk}L^{M-k}b\|^2_{L^2(U_j(B))}\frac{d\tau}{\tau}.
\end{aligned}\tag{5.22}$$

It follows from Proposition 3.1 that there exist constants $C,c_0>0$ such that for $w\in\{-1,0,1\}$ and $\nu\geq7$, $2^{-\ell}\leq\tau\leq2^{-l+1}$,

$$\|\chi_{U_{\nu+j+w}(B)}F_{\tau,s}(L)\chi_{U_j(B)}\|_{2\to2}\leq C2^{-\nu\kappa_0}(2^{\ell/m}2^jr_B)^{-\kappa_0}(2^{\ell/m}r_{j,\ell})^{-c_0}\|\delta_{\tau^{-1}}F_{\tau,s}\|_{C^{\kappa_0+1}}, \tag{5.23}$$

where $r_{j,\ell}=\min\{2^jr_B,2^{-\ell/m}\}$. Note that the conditions $\mathrm{supp}\phi\subset(1/4,1)$ and $r_B\leq s\leq\sqrt[m]{2}r_B$ implies if $2^{-\ell}\leq\tau\leq2^{-l+1}$, then

$$\|\delta_{\tau^{-1}}F_{\tau,s}\|_{C^{\kappa_0+1}}\leq C\min\{1,(2^{\ell/m}r_B)^{mM}\}\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}. \tag{5.24}$$

Combining the estimates (5.22), (5.23), (5.24), (4.14) with (4.1), we conclude that

$$\begin{aligned}
&\|E'(k,j,s,\ell)\|_{L^2(U_{\nu+j}(B))}\\
&\quad\leq C2^{-j\epsilon}2^{-\nu\kappa_0}\mu(2^jB)^{-\frac{1}{2}}(2^{\ell/m}2^jr_B)^{-\kappa_0}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}.
\end{aligned}$$

This, in combination with the doubling condition (1.5), yields

$$\begin{aligned}
\mathrm{II}(k,j,s)&\leq C\sum_{\ell\in\mathbb{Z}}\sum_{\nu\geq\nu_0}\|E'(k,j,s,\ell)\|_{L^2(U_{\nu+j}(B))}\mu(2^{\nu+j}B)^{\frac{1}{2}}\\
&\leq C2^{-j(\epsilon+\kappa_0)}\sum_{\ell\in\mathbb{Z}}\sum_{\nu\geq\nu_0}2^{-\nu(\kappa_0-\frac{n}{2})}(2^{\ell/m}r_B)^{-\kappa_0}\min\{1,(2^{\ell/m}r_B)^{mM-c_0}\}\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}\\
&\leq C2^{-j(\epsilon+\kappa_0)}\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}.
\end{aligned}\tag{5.25}$$

It follows from the estimates (5.17), (5.20), (5.21) and (5.25) that for $k=0,1,\cdots,M-1$,

$$\left\|\left(\int_0^\infty\int_{d(x,y)<\tau^{1/m}}|E_k(y)|^2\frac{d\mu(y)}{V(x,\tau^{1/m})}\frac{d\tau}{\tau}\right)^{\frac{1}{2}}\right\|_{L^1}\leq C\sup_{R>0}\|\phi\delta_RF\|_{C^{\kappa_0+1}}.$$

Case 2. $k = M$. Similarly to the proof of estimating E_k for $k = 1, 2, \cdots, M-1$ as in **Case 1**, we conclude that

$$\left\| \left(\int_0^\infty \int_{d(x,y)<\tau^{1/m}} |E_M(y)|^2 \frac{d\mu(y)}{V(x,\tau^{1/m})} \frac{d\tau}{\tau} \right)^{\frac{1}{2}} \right\|_{L^1} \le C \sup_{R>0} \|\phi\delta_R F\|_{C^{\kappa_0+1}}.$$

This finishes the proof of (5.15) and then Theorem 5.5. □

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