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The L^p-boundedness of wave operators for two dimensional Schrödinger operators with threshold singularities

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Abstract. We generalize the recent result of Erdoğan, Goldberg and Green on the L^p-boundedness of wave operators for two dimensional Schrödinger operators and prove that they are bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$ if and only if the Schrödinger operator possesses no p-wave threshold resonances, viz. Schrödinger equation $(-\Delta + V(x))u(x) = 0$ possesses no solutions which satisfy $u(x) = (a_1x_1 + a_2x_2)|x|^{-2} + o(|x|^{-1})$ as $|x| \to \infty$ for an $(a_1, a_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and, otherwise, they are bounded in $L^p(\mathbb{R}^2)$ for $1 < p \leq 2$ and unbounded for $2 < p < \infty$. We present also a new proof for the known part of the result.

1. Introduction and main result

Let $H_0 = -\Delta$, $D(H_0) = W^{2,2}(\mathbb{R}^2)$ be the free Schrödinger operator on $\mathbb{R}^d$, $d \geq 1$ and $V(x)$ a real measurable function on $\mathbb{R}^d$. Suppose that, for some $\gamma > 1/2$, $\langle x\rangle^\gamma |V(x)|^{1/2}(H_0+1)^{-\frac12}$, $\langle x\rangle = (1+|x|^2)^{\frac12}$, is a compact operator on $L^2(\mathbb{R}^d)$. Then, Schrödinger operator $H = -\Delta + V$ defined via the quadratic form is selfadjoint ([28]); the spectrum $\sigma(H)$ consists of the absolutely continuous part $[0,\infty)$ and the point spectrum which is discrete in $\mathbb{R}\setminus\{0\}$ ([2]); $L^2(\mathbb{R}^d)$ is the orthogonal sum of the absolutely continuous subspace $L^2_{ac}(\mathbb{R}^d)$ for H and the space of eigenfunctions of H. The scattering theory compares the large time behavior of scattering solutions $e^{-itH}\varphi$, $\varphi \in L^2_{ac}(\mathbb{R}^d)$ of the time dependent Schrödinger equation:

$$i\partial_t u(t) = Hu(t), \quad u(0) = \varphi \in L^2_{ac}(\mathbb{R}^d) \tag{1.1}$$

with that of free solutions $e^{-itH_0}\varphi_0$ and wave operators are defined by the strong limits:

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\,. \tag{1.2}$$

It is well known ([2, 24]) that $W_\pm$ exist, Image $W_\pm = L^2_{ac}(H)$ and, all scattering solutions $e^{-itH}P_{ac}(H)\varphi$ become asymptotically free in the remote past and far future:

$$\lim_{t\to\pm\infty} \|e^{-itH}P_{ac}(H)\varphi - e^{-itH_0}\varphi_\pm\| = 0, \quad \varphi_\pm = W_\pm^*\varphi,$$

where $P_{ac}(H)$ is the orthogonal projection onto $L^2_{ac}(H)$.

Wave operators $W_\pm$ satisfy the intertwining property, viz. for Borel functions $f(\lambda)$

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of $\lambda \in \mathbb{R}^1$

$$f(H)P_{ac}(H) = W_\pm f(H_0) W_\pm^* \tag{1.3}$$

and various properties of $f(H)P_{ac}(H)$ may be derived from those of the Fourier multiplier $f(H_0)$ if $W_\pm$ satisfy appropriate properties. If $W_\pm$ are bounded in $L^p(\mathbb{R}^d)$ for $p \in I \subset [1,\infty]$, then (1.3) produces a set of estimates that for $\{p,q\} \in I \times I^*$, $I^* = \{p/p-1, p \in I\}$,

$$\|f(H)P_{ac}(H)\|_{\mathbf{B}(L^q,L^p)} \le C\|f(H_0)\|_{\mathbf{B}(L^q,L^p)},$$
$$\|f(H_0)\|_{\mathbf{B}(L^p,L^q)} \le C^{-1}\|f(H)P_{ac}(H)\|_{\mathbf{B}(L^p,L^q)}$$

where $C = C_{p,q}$ is independent of f. Here $\mathbf{B}(X,Y)$ is the Banach space of bounded operators from X to Y and $\mathbf{B}(X) = \mathbf{B}(X,X)$. Such estimates are very useful and have many applications (e.g. [32]).

Thus, the problem of whether or not $W_\pm$ are bounded in $L^p(\mathbb{R}^d)$ has attracted interest of many authors and various results have been obtained under various assumptions. We briefly review here some results under simplified assumption that $|V(x)| \le C\langle x\rangle^{-\sigma}$ for a sufficiently large $\sigma > 2$. We need some notation: Multiplication operator M_m with a function $m(x)$ is often denoted simply by m. $\mathbb{C}^+ = \{\lambda \in \mathbb{C}\colon \Im\lambda > 0\}$ and $\overline{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{R}$; $G_0(\lambda) = (H_0 - \lambda^2)^{-1}$ for $\lambda \in \mathbb{C}^+$; $\mathcal{G}_\lambda(x)$ is the convolution kernel of $G_0(\lambda)$: $G_0(\lambda)u(x) = (\mathcal{G}_\lambda * u)(x)$; if $d = 2$, $\mathcal{G}_\lambda(x)$ is given by

$$\mathcal{G}_\lambda(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\xi}d\xi}{\xi^2 - \lambda^2} = \frac{i}{4}H_0^{(1)}(\lambda|x|)\,, \tag{1.4}$$

where $H_0^{(1)}(z)$ is the Hankel function of the first kind.

$$\langle u, v\rangle = \int_{\mathbb{R}^d} u(x)v(x)dx$$

without complex conjugation.

$$U(x) = \begin{cases} 1 & \text{if } V(x) \ge 0, \\ -1 & \text{if } V(x) < 0, \end{cases} \quad v(x) = |V(x)|^{1/2}, \quad w(x) = U(x)v(x)\,.$$

The limiting absorption priniciple (e.g. [2, 24]) implies that the holomorphic function $\mathbb{C}^+ \ni \lambda \to vG_0(\lambda)v \in \mathbf{B}_c(L^2(\mathbb{R}^d))$, the space of compact operators on $L^2(\mathbb{R}^d)$, has a continuous extension to $\overline{\mathbb{C}}^+$ if $d \ge 3$ and to $\lambda \in \overline{\mathbb{C}}^+ \setminus \{0\}$ if $d = 1$ and $d = 2$;

$$M(\lambda) \overset{\text{def}}{=} U + vG_0(\lambda)v, \quad \lambda \in \overline{\mathbb{C}}^+ \setminus \{0\} \tag{1.5}$$

is invertible unless λ^2 is an eigenvalue of H and the absence of positive eigenvalues ([20, 23]) impies that $M(\lambda)^{-1} \in \mathbf{B}(L^2(\mathbb{R}^d))$ exists for any $\lambda \in \mathbb{R} \setminus \{0\}$.

For $d \ge 3$, we say H is regular at zero if $M(0)^{-1} \in \mathbf{B}(L^2)$ exists and is singular at zero otherwise. It is known that H is singular if and only if for some $1/2 < \gamma < \sigma - 1/2$

$$\mathcal{N} \overset{\text{def}}{=} \{u \in \langle x\rangle^\gamma L^2(\mathbb{R}^d)\colon (-\Delta + V(x))u(x) = 0\} \ne \{0\}; \tag{1.6}$$

$\mathcal{N}$ is independent of γ and $u \in \mathcal{N}$ is called threshold resonance; $u(x) = O(|x|^{2-d})$ as

$|x| \to \infty$ and $u(x) = O(|x|^{1-d-j})$ if it satisfies $\langle x^\alpha V, u\rangle = 0$ for $|\alpha| \le j$, $j = 0, 1, \dots$; $u \in \mathcal{N}$ is an eigenfunction of H with eigenvalue 0 if $d \ge 5$. When $d = 1$ or 2, we say H is regular at zero if $\mathcal{N}_\infty = \{0\}$ where

$$\mathcal{N}_\infty = \{u \in L^\infty(\mathbb{R}^d) \colon (-\Delta + V(x))u(x) = 0\} \tag{1.7}$$

and H is singular at zero if otherwise. If $d = 2$, it is known (see Lemma 5.1) that $u \in \mathcal{N}_\infty$ satisfies for constants c, b_1, b_2 and $\varepsilon > 0$ that

$$u(x) = c + \frac{b_1x_1 + b_2x_2}{|x|^2} + O(|x|^{-1-\varepsilon}), \quad (|x| \to \infty) \tag{1.8}$$

and $u \in \mathcal{N}_\infty \setminus \{0\}$ is called s-wave resonance if $c \ne 0$, p-wave resonance if $c = 0$ but $(b_1, b_2) \ne (0, 0)$ and it is a zero energy eigenfunction of H if $c = b_1 = b_2 = 0$.

We list some known results. If $d = 1$, $W_\pm$ are bounded in $L^p(\mathbb{R}^1)$ for $1 < p < \infty$ and are in general unbounded in $L^p(\mathbb{R}^1)$ for $p = 1, \infty$ ([36, 13, 8]).

If H is regular at zero, we have almost complete results: If $d = 2$, $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^2))$ for $1 < p < \infty$ ([39, 18] but no results for $p = 1$ and $p = \infty$); if $d \ge 3$, $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^d))$ for $1 \le p \le \infty$ ([37, 38, 5]).

If H is singular at zero, a rather complete result is known for $d \ne 2, 4$. If $d \ge 5$, $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^d))$ for $1 \le p < d/2$, for $1 \le p < d$ if and only if $\langle V, u\rangle = 0$ for all $u \in \mathcal{N}$ and, for $1 \le p < \infty$ if and only if $\langle x^\alpha V, u\rangle = 0$ for all $|\alpha| \le 1$ ([15, 41, 12, 40]). If $d = 3$, $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^3))$ for $1 < p < 3$; for $p = 1$ if and only if all $u \in \mathcal{N}$ satisfy $\langle V, u\rangle = 0$; for $1 \le p < \infty$ if and only if $\langle x^\alpha V, u\rangle = 0$ for all $|\alpha| \le 1$; for $p = \infty$ if $\langle x^\alpha V, u\rangle = 0$ for all $|\alpha| \le 2$. ([42]).

However, only a partial result is known when $d = 2$ or $d = 4$: For $d = 2$, $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^2))$ for $1 < p < \infty$ if $\mathcal{N}_\infty$ consists only of zero energy eigenfunctions or only of s-wave resonances ([10]). If $d = 4$ and if all $u \in \mathcal{N}$ satisfy $\langle V, u\rangle = 0$, then $W_\pm \in \mathbf{B}(L^p(\mathbb{R}^4))$ for $1 \le p \le 4$ and, for $1 \le p < \infty$ if $\langle x^\alpha V, u\rangle = 0$ for $|\alpha| \le 1$.([14, 19]).

The purpose of this paper is to prove the following theorem for the case $d = 2$ which fills the missing part in the results of [39, 18] and [10].

Theorem 1.1. *Suppose $\langle x\rangle^2 V \in L^{\frac{4}{3}}(\mathbb{R}^2)$ and $\langle x\rangle^\gamma |V(x)| \in L^1(\mathbb{R}^2)$ for a constant $\gamma > 8$. Then, $W_\pm$ are bounded in $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ if and only if H has no p-wave resonances. $W_\pm$ are otherwise bounded in $L^p(\mathbb{R}^2)$ for $1 < p \le 2$ and unbounded for $2 < p < \infty$.*

We shall also give a new proof for the known parts ([39, 18, 10]) under slightly weaker assumptions. The problem for $p = 1$ and $p = \infty$, however, is left open. A similar result has recently been obtained for Schrödinger operators with point interactions on $\mathbb{R}^2$ ([6], [7] and [43]) and the main idea of the proof is borrowed from [6] and [43].

We briefly explain here the basic strategy for the proof of Theorem 1.1, introducing some more notation and displaying the plan of the paper. We prove it only for W_+. The result for W_- then follows via the complex conjugation $\mathcal{C}u(x) = \overline{u(x)}$: $W_- = \mathcal{C}W_+\mathcal{C}^{-1}$.

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\xi}u(x)dx \quad \text{and} \quad \check{u}(\xi) = (\mathcal{F}^{-1}u)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\xi}u(x)dx$$

are the Fourier transform and its inverse respectively; $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space of rapidly decreasing functions;

$$\mathcal{D}_* = \{u \in \mathcal{S}(\mathbb{R}^2) \mid \hat{u} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})\}; \tag{1.9}$$

$\mathcal{D}_*$ is dense in $L^p(\mathbb{R}^2)$, $1 \le p < \infty$; $\|u\|_p = \|u\|_{L^p(\mathbb{R}^2)}$;

$$\tau_y u(x) = u(x-y), \quad y \in \mathbb{R}^2.$$

For a Borel function $f(\lambda)$ of $\lambda > 0$, $f(|D|)$ is the Fourier multiplier

$$f(|D|)u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\xi} f(|\xi|)\hat{u}(\xi)d\xi.$$

We take and fix $\chi \in C_0^\infty(\mathbb{R})$ such that

$$\chi(\lambda) = 1 \text{ for } |\lambda| \le 1/2 \text{ and } \chi(\lambda) = 0 \text{ for } |\lambda| \ge 1 \tag{1.10}$$

and define for $a > 0$

$$\chi_{\le a}(\lambda) = \chi(\lambda/a), \quad \chi_{>a}(\lambda) = 1 - \chi_{\le a}(\lambda). \tag{1.11}$$

We decompose W_+ into the high and the low energy parts:

$$W_+ = W_+\chi_{>2a}(|D|) + W_+\chi_{\le 2a}(|D|) \tag{1.12}$$

and study them separately. Here $a > 0$ is arbitrary.

The proof is based on the stationary formula for W_+ (e.g. [24]):

$$W_+u(x) = u(x) - \Omega_+u(x), \tag{1.13}$$

$$\Omega_+u(x) = \int_0^\infty (G_0(-\lambda)vM(\lambda)^{-1}v\Pi(\lambda)u)(x)\lambda d\lambda\,, \tag{1.14}$$

where $\Pi(\lambda)u(x) = (i\pi)^{-1}(G_0(\lambda) - G_0(-\lambda))u(x)$ and for $\lambda > 0$

$$\Pi(\lambda)u(x) = \frac{1}{2\pi}\int_{\mathbb{S}^1} e^{i\lambda\omega x}\hat{u}(\lambda\omega)d\omega = \frac{1}{2\pi}\int_{\mathbb{S}^1}(\mathcal{F}\tau_{-x}u)(\lambda\omega)d\omega\,. \tag{1.15}$$

We define $\Pi(\lambda)u(x)$ also for $\lambda = 0$ by (1.15). Then, for $f \in C([0,\infty))$,

$$f(\lambda)\Pi(\lambda)u = \Pi(\lambda)f(|D|)u, \quad \lambda \ge 0\,. \tag{1.16}$$

We have $W_+\chi_{>2a}(|D|)u = \chi_{>2a}(|D|)u + \Omega_{\text{high},2a}u$ and $W_+\chi_{\le 2a}(|D|)u = \chi_{\le 2a}(|D|)u + \Omega_{\text{low},2a}u$ where

$$\Omega_{\text{high},2a}u = \int_0^\infty G_0(-\lambda)vM(\lambda)^{-1}v\Pi(\lambda)u\chi_{>2a}(\lambda)\lambda d\lambda\,, \tag{1.17}$$

$$\Omega_{\text{low},2a}u = \int_0^\infty G_0(-\lambda)vM(\lambda)^{-1}v\Pi(\lambda)u\chi_{\le 2a}(\lambda)\lambda d\lambda\,. \tag{1.18}$$

and we have only to study $\Omega_{\text{high},2a}$ and $\Omega_{\text{low},2a}$.

In section 2, we state and prove some estimates related to the Hankel function and recall some properties of the integral operator K introduced in [6]:

$$Ku(x) = \frac{1}{2\pi}\int_0^{+\infty} \mathcal{G}_{-\lambda}(x)\lambda \left(\int_{\mathbb{S}^1} (\mathcal{F}u)(\lambda\omega)d\omega\right) d\lambda\,; \tag{1.19}$$

K is bounded in $L^p(\mathbb{R}^2)$ for all $1<p<\infty$ and is closely connected to Ω_+:

$$(\tau_y K \tau_{-z} u)(x) = \int_0^\infty \mathcal{G}_{-\lambda}(x-y)\Pi(\lambda)u(z)\lambda d\lambda\,. \tag{1.20}$$

We shall give a simpler proof of the $L^p(\mathbb{R}^2)$-boundedness of K in section 2.

In section 3 we prove the following Proposition 1.4 which will be repeatedly used for estimating the operators produced by (1.14), by replacing $vM(\lambda)v$ by operator valued functions related to it. $\mathcal{H}_2$ is the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^2)$. $\mathcal{L}_1$ is the Banach space of integral operators T with $T(x,y)\in L^1(\mathbb{R}^2\times\mathbb{R}^2)$ with the norm $\|T\|_{\mathcal{L}_1} = \|T\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$.

Definition 1.2. We use the following terminology. We say:

(1) X is a *good operator* if $X\in \mathbf{B}(L^p(\mathbb{R}^2))$ for all $1<p<\infty$.

(2) $f\in C^2((0,\infty))$ is a *good multiplier* if it satisfies $|f^{(j)}(\lambda)|\le C_j\lambda^{-j}$ for $j=0,1,2$. $f(|D|)$ is then a good operator by virtue of Mikhlin's theorem (cf. [34]). f is a *good multiplier for small* $\lambda>0$ if $\chi_{\le a}(\lambda)f(\lambda)$ is a good multiplier for a $a>0$. $\mathcal{M}(\mathbb{R}^2)$ is the space of good multipliers. We denote $\|f\|_{\mathcal{M},p} = \|f(|D|)\|_{\mathbf{B}(L^p)}$, $1<p<\infty$.

(3) Let $k=0,1,\dots$ and $h(\lambda)>0$. $T(\lambda)\in \mathcal{O}^{(k)}_{\mathcal{L}_1}(h)$ as $\lambda\to 0$ (or $\lambda\to\infty$) if $T(\lambda,x,y)$ is a function of $\lambda>0$ of C^k-class for a.e. (x,y) and simultaneously as an $\mathcal{L}_1$-valued function and $\|\partial_\lambda^j T(\lambda)\|_{\mathcal{L}_1}\le C|h(\lambda)|\lambda^{-j}$ for $0\le j\le k$ as $\lambda\to 0$ (or $\lambda\to\infty$).

(4) $T(\lambda)\in\mathcal{O}_2(h)$ if $T(\lambda,x,y)$ satisfies the properties of (3) with $k=2$ and $L^2(\mathbb{R}^2\times\mathbb{R}^2)$ and $\mathcal{H}_2$ replacing $L^1(\mathbb{R}^2\times\mathbb{R}^2)$ and $\mathcal{L}_1$ respectively. If $T(\lambda)\in\mathcal{O}_2(h)$ and $v,w\in L^2(\mathbb{R}^2)$, then $vT(\lambda)v\in\mathcal{O}^{(2)}_{\mathcal{L}_1}(h)$.

We often denote by $\mathcal{O}^{(k)}_{\mathcal{L}_1}(h)$ an operator in the class $\mathcal{O}^{(k)}_{\mathcal{L}_1}(h)$ and likewise for $\mathcal{O}_2(h)$.

Definition 1.3. The operators defined by (1.14) with T or $T(\lambda)$ replacing $vM(\lambda)^{-1}v$ are denoted respectively by $W(T)$ and $\mathcal{W}(T(\lambda))$:

$$W(T)u(x) = \int_0^\infty (G_0(-\lambda)T\Pi(\lambda)u)(x)\lambda d\lambda,\ u\in\mathcal{D}_*\,, \tag{1.21}$$

$$\mathcal{W}(T(\lambda))u(x) = \int_0^\infty (G_0(-\lambda)T(\lambda)\Pi(\lambda)u)(x)\lambda d\lambda,\ u\in\mathcal{D}_*\,. \tag{1.22}$$

We say T or $T(\lambda)$ is a *good producer* if $W(T)$ (resp.$\mathcal{W}(T(\lambda))$) extends to a good operator and it is a *good producer for small* $\lambda>0$ *or large* λ if $\chi_{\le 2a}(\lambda)T$ or $\chi_{>2a}(\lambda)T$ (resp. $\chi_{\le 2a}(\lambda)T(\lambda)$ or $\chi_{>2a}(\lambda)T(\lambda)$) is a good producer for some $a>0$.

The one dimensional operator $f\mapsto v(x)\int_{\mathbb{R}^2} w(y)f(y)dy$ *without complex conjugate* will be indiscriminately denoted by $v\otimes w$ or $|v\rangle\langle w|$.

PROPOSITION 1.4. *Let $f \in \mathcal{M}(\mathbb{R}^2)$, $F \in L^1(\mathbb{R}^2)$ and $T \in \mathcal{L}_1$. Suppose $a>0$, $\varepsilon>0$ and $k \in \mathbb{N}$. Then, we have the following statements for all $1<p<\infty$:*

(1) $\|W(M_F)u\|_p \le C\|F\|_1\|u\|_p$.

(2) $\|\mathcal{W}(f(\lambda)M_F)u\|_p \le C\|f\|_{\mathcal{M},p}\|F\|_1\|u\|_p$.

(3) $\|W(T)u\|_p \le C\|T\|_{\mathcal{L}_1}\|u\|_p$.

(4) $\|\mathcal{W}(f(\lambda)T)u\|_p \le C\|f\|_{\mathcal{M},p}\|T\|_{\mathcal{L}_1}\|u\|_p$.

(5) $\chi_{\le 2a}(\lambda)T(\lambda)$ *is a good producer if* $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^{1+\varepsilon})$ *as* $\lambda \to 0$.

(6) $\chi_{>2a}(\lambda)T(\lambda)$ *is a good producer if* $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^{-\varepsilon})$ *as* $\lambda \to \infty$.

(7) *Let* $I^{\psi,\varphi}_{k,a}(\lambda) = f(\lambda)\chi_{\le 2a}(\lambda)(\log\lambda)^k(\psi\otimes\varphi)$ *for* $\varphi,\psi \in L^1(\mathbb{R}^2)$. *Assume that either* $\langle x\rangle\varphi \in L^1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2}\varphi dx = 0$ *or* $\langle x\rangle\psi \in L^1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2}\psi dx = 0$. *Then,* $I^{\psi,\varphi}_{k,a}(\lambda)$ *is a good producer and we respectively have*

$$\|\mathcal{W}(I^{\psi,\varphi}_{k,a}(\lambda))u\|_p \le C_p\|f\|_{\mathcal{M},p}\|\langle x\rangle\varphi\|_1\|\psi\|_1\|u\|_p, \tag{1.23}$$

$$\|\mathcal{W}(I^{\psi,\varphi}_{k,a}(\lambda))u\|_p \le C_p\|f\|_{\mathcal{M},p}\|\varphi\|_1\|\langle x\rangle\psi\|_1\|u\|_p . \tag{1.24}$$

In section 4 we prove that $W_+\chi_{>2a}(|D|)$ is a good operator if $a>0$ under the condition that $\langle x\rangle^2 V \in L^{\frac43}(\mathbb{R}^2)$. If we expand $M(\lambda)^{-1}$ in (1.17) as

$$M(\lambda)^{-1} = \sum_{j=0}^{4}(-1)^j U(vG_0(\lambda)w)^j - U(vG_0(\lambda)w)^5(1+vG_0(\lambda)w)^{-1}, \tag{1.25}$$

then $\Omega_{\text{high},2a}$ becomes the sum $\sum_{j=0}^5(-1)^j\Omega_{h,j}$ and we prove $\Omega_{h,j}$, $0\le j\le 5$ are good operators separately. $\Omega_{h,0} = Z_V$ is a good operator by virtue of Proposition 1.4 (1). For proving the same for $\Omega_{h,1}$ we represent it as

$$\Omega_{h,1}u = \int_{\mathbb{R}^2} W(M_{V^{(2)}_y})(\mathcal{H}(|y||D|)\tau_y\chi_{>2a}(|D|)u)dy\,, \tag{1.26}$$

where we used the short hand notation

$$\mathcal{H}(\lambda) = (i/4)H^{(1)}_0(\lambda) \text{ and } V^{(2)}_y(x) = V(x)V(x-y).$$

Since $\mathcal{H}(\lambda) = e^{i\lambda}\omega(\lambda)$ with symbol $\omega(\lambda)$ of order $-1/2$ (see (2.6)), the theory of the spatially homogeous Fourier integral operators ([27, 35]) implies

$$\|\mathcal{H}(|y||D|)\chi_{>2a}(|D|)\|_{\mathbf{B}(L^p(\mathbb{R}^2))} \le C_p(1+|\log|y||)$$

(see Lemma 4.5). Then, Proposition 1.4 (1) implies

$$\|\Omega_{h,1}\|_{\mathbf{B}(L^p)} \le C_p\int_{\mathbb{R}^2}|V(x)V(x-y)|(1+|\log|y||)dxdy < \infty$$

and $\Omega_{h,1}$ is a good operator. For obtaning the expression (1.26) it is important to observe that $VG_0(\lambda)V$ is the superposition of the product of mulitiplication, multiplier and translation:

$$VG_0(\lambda)Vu(x) = \int_{\mathbb{R}^2} M_{V_y^{(2)}} \mathcal{H}(|y|\lambda)(\tau_y u)(x)dy\,, \tag{1.27}$$

and then (1.26) follows by substituting (1.27) for $vM(\lambda)^{-1}v$ in (1.17) and by applying (1.16) which implies $\mathcal{H}(|y|\lambda)\Pi(\lambda) = \Pi(\lambda)\mathcal{H}(|y||D|)$. We show in Proposition 4.6 that $\Omega_{h,j}$, $j = 2, 3, 4$ are represented likewise (see (4.13)) and that they are good operators. We prove that $\Omega_{h,5}$ is a good operator by showing that $\|\partial_\lambda^j vG_0(\lambda)w\|_{\mathcal{H}_2} \le C\lambda^{-1/2}$ as $\lambda \to \infty$ for $j = 0, 1, 2$ (see Lemma 4.7) and, if sandwitched by v, the final term on the right of (1.25) is of class $\mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^{-\frac12})$.

We study $\Omega_{\mathrm{low},2a}$ in section 5, which will be divided into seven subsections. We need study $M(\lambda)^{-1}$ near $\lambda = 0$ in detail. Define three integral operators N_0, G_1 and G_2 by

$$N_0u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|u(y)dy\,, \tag{1.28}$$

$$G_1u(x) = \frac{1}{4}\int_{\mathbb{R}^2} |x-y|^2 u(y)dy\,, \tag{1.29}$$

$$G_2u(x) = \frac{1}{8\pi}\int_{\mathbb{R}^2} |x-y|^2 \log\left(\frac{e}{|x-y|}\right) u(y)dy\,. \tag{1.30}$$

The expansion (2.1) of the Hankel function $\mathcal{H}(\lambda)$ and the definition (2.3) of $g(\lambda)$ then imply that, as $\lambda \to 0$,

$$M(\lambda) = U + g(\lambda)v\otimes v + vN_0v + \lambda^2 g(\lambda)vG_1v + \lambda^2 vG_2v + \mathcal{O}_2(g\lambda^4). \tag{1.31}$$

We define projections P, Q and the operator T_0 by

$$P = (v/\|v\|_2)\otimes(v/\|v\|_2), \quad Q = 1-P, \quad T_0 = U + vN_0v.$$

Following definition is due to [17] (see also [31, 11, 10]):

Definition 1.5. We say H is regular at zero if $QT_0Q|_{QL^2(\mathbb{R}^2)}$ is invertible. Otherwise, H is singular at zero. If H is singular at zero, let S_1 be the projection in $QL^2(\mathbb{R}^2)$ onto $\mathrm{Ker}_{\,\mathrm{QL^2(\mathbb{R}^2)}}\mathrm{QT_0Q}$.

(1) We say H has singularities of the first kind at zero, if $T_1 = S_1QT_0PT_0QS_1|_{S_1L^2}$ is non-singular.

(2) If T_1 is singular, let S_2 be the projection in $S_1L^2(\mathbb{R}^2)$ onto $\mathrm{Ker}\,\mathrm{T_1}$. We say H has singularities of the second kind at zero, if $T_2 = S_2(vG_1v)S_2|_{S_2L^2(\mathbb{R}^2)}$ is non-singular.

(3) We say H has singularities of the third kind at zero if T_2 is singular. Let S_3 be the projection in $S_2L^2(\mathbb{R}^2)$ onto $\mathrm{Ker}\,\mathrm{T_2}$.

In subsection 5.1, we recall from [17] the relation between the kind of singularities of H at zero and the existence/absence of specific kinds of resonances. We shall give a brief proof for readers' convenience and add some remarks. In subsection 5.2 we recall without proof the Feshbach formula, Jensen-Nenciu's lemma ([17]) and two estimates from [11, 10] respectively on

$$M_0(\lambda) = M(\lambda) - g(\lambda)v\otimes v - T_0 \ \text{ and } \ M_1(\lambda) = M_0(\lambda) - v\lambda^2(gG_1+G_2)v.$$

In subsection 5.3 we prove that $\Omega_{\text{low},2a}$ is a good operator if H is regular at zero by showing that

$$M(\lambda)^{-1} = (g(\lambda)+c)^{-1}L + \mathcal{B} + \mathcal{O}_2(g\lambda^2) \quad (\lambda\to 0) \tag{1.32}$$

and applying Proposition 1.4, where c is a constant, L is an operator of finite rank and $\mathcal{B}$ is the sum of the multiplication by a bounded function and a Hilbert-Schmidt operator. This is a considerably simpler proof than the one in [39].

In subsection 5.4 we present a few identities which are necessary for studying $M(\lambda)^{-1}$ near $\lambda = 0$. It will be important to observe that operators which appear as the coefficients of the singularities of $M(\lambda)^{-1}$ at $\lambda = 0$ act in the finite dimensional subspace $S_1L^2(\mathbb{R}^2)$ and that all $\zeta \in S_1L^2(\mathbb{R}^2)$ satisfy

$$\int_{\mathbb{R}^2} v(x)\zeta(x) = 0, \tag{1.33}$$

which will cancel some singularities of $M(\lambda)^{-1}$ at $\lambda = 0$.

In section 5.5 we give a new proof of Erdoğan, Goldberg and Green's result ([10]) that $\Omega_{\text{low},2a}$ is a good operator if H has singularities of the first kind at zero under the condition $\langle x\rangle^{4+\varepsilon}V \in L^1(\mathbb{R}^2)$ for an $\varepsilon > 0$. In this case $\operatorname{rank} S_1 = 1$ and $S_1 = \zeta\otimes\zeta$ for a normalized $\zeta \in S_1L^2(\mathbb{R}^2)$. We then show that $vM(\lambda)^{-1}v \equiv -c_1\log\lambda(v\zeta\otimes\zeta v)$ modulo a good producer for small $\lambda > 0$ and hence, is itself a good producer for small $\lambda > 0$ by virtue of Proposition 1.4.

In subsection 5.6 and 5.7 we asume $\langle x\rangle^{\gamma}V \in L^1(\mathbb{R}^2)$ for $\gamma > 8$. We study the case that H has singularities of the second kind in subsection 5.6. Then $1 \le \operatorname{rank} T_2 = \operatorname{rank} S_2 \le 2$ and we assume $\operatorname{rank} S_2 = 2$ as the case $\operatorname{rank} S_2 = 1$ is easier. Let $\{\zeta_1, \zeta_2\}$ be the orthonormal basis of $S_2L^2(\mathbb{R}^2)$ such that $T_2\zeta_j = -\kappa_j^2\zeta_j$, $\kappa_j > 0$, $j = 1, 2$ and $\tilde{\mathcal{R}}_1(\lambda) = S_2v(G_1 + g(\lambda)^{-1}G_2)vS_2$. Then, the representation matrix $C(\xi) = (c_{jk}(\lambda))$ for $\tilde{\mathcal{R}}_1(\lambda)$ on $S_2L^2(\mathbb{R}^2)$ with respect to the basis $\{\zeta_1, \zeta_2\}$ satisfies

$$c_{jk}(\lambda) = -\kappa_j^2\delta_{jk} + \mathcal{O}_2(g(\lambda)^{-1}), \quad j,k = 1,2.$$

Write $D(\lambda) = (d_{jk}(\lambda))$ for $C(\lambda)^{-1}$. $d_{jk}(\lambda)$ are good multiplers for small $\lambda > 0$. Via the threshold analysis, we first prove that modulo a good operator $\Omega_{\text{low},2a}u(x)$ is equal to

$$-\sum_{j,k=1}^{2}\int_0^\infty g(\lambda)^{-1}\lambda^{-2}d_{jk}(\lambda)(G_0(-\lambda)v\zeta_j)(x)\langle\zeta_k v, \Pi(\lambda)u\rangle\lambda\chi_{\le 2a}(\lambda)d\lambda, \tag{1.34}$$

whose integrand has the strong singularity $g(\lambda)^{-1}\lambda^{-2}$ at $\lambda = 0$. Recalling (1.33) which implies $\langle\zeta_k v, 1\rangle = 0$ for $k = 1, 2$, we replace $\Pi(\lambda)u(z)$ by

$$\Pi(\lambda)u(z) - \Pi(\lambda)u(0) = \frac{1}{2\pi}\int_{\mathbb{S}^1}(e^{i\lambda z\omega x} - 1)\hat{u}(\lambda\omega)d\omega\,,$$

which we decompose into the sum of *good part* $\tilde{g}(\lambda, z)$ and *bad part* $\tilde{b}(\lambda, z)$ as follows by

Taylor expanding $e^{i\lambda z\omega x}$ upto the second order:

$$\tilde{g}(\lambda,z) = \frac{-\lambda^2}{2\pi}\int_{\mathbb{S}^1}\left(\int_0^1 (1-\theta)(z\omega)^2 e^{i\lambda z\omega\theta}d\theta\right)\hat{u}(\lambda\omega)d\omega \tag{1.35}$$
$$= \sum_{j,k=1}^{2} z_j z_k \lambda^2 \int_0^1 (1-\theta)\left(\frac{-1}{2\pi}\int_{\mathbb{S}^1}\mathcal{F}(\tau_{-\theta z}R_jR_k u)(\lambda\omega)d\omega\right)d\theta.$$

$$\tilde{b}(\lambda,z) = \frac{i\lambda}{2\pi}\int_{\mathbb{S}^1}(z\omega)\hat{u}(\lambda\omega)d\omega = \frac{i\lambda}{2\pi}\sum_{l=1}^{2} z_l \int_{\mathbb{S}^1}\mathcal{F}(R_l u)(\lambda\omega)d\omega. \tag{1.36}$$

Let $\tilde{\Omega}_{(g)}$ and $\tilde{\Omega}_{(b)}$ be respectively defined by (1.34) by replacing $\Pi(\lambda)u(z)$ by $\tilde{g}(\lambda,z)$ and $\tilde{b}(\lambda,z)$. Then, $\tilde{\Omega}_{(g)}$ is a good operator because the singularity λ^{-2} is cancelled by $\tilde{g}(\lambda,z)$, $\mu_{jk}(\lambda){=}\tilde{g}(\lambda)^{-1}d_{jk}(\lambda)\chi_{\leq 2a}(\lambda) \in \mathcal{M}(\mathbb{R}^2)$ and $\|\langle y\rangle^2 v\zeta\|_1 \leq C\|\langle y\rangle^{4+\varepsilon}V\|_1$ for p-wave resonances. The bad part $\tilde{b}(\lambda,z)$ has only the factor λ and we show $\tilde{\Omega}_{(b)}$ is bounded in $L^p(\mathbb{R}^2)$ only for $1<p\leq 2$ and is unbounded for $2<p<\infty$. We avoid outlining the proof of the boundedness part for not making the introduction too long.

For proving that $\tilde{\Omega}_{(b)}$ is unbounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$ it suffices to prove the same for $\chi_{>4a}(|D|)\tilde{\Omega}_{(b)}$ since $\chi_{>4a}(|D|)$ is a good operator.

$$\chi_{>4a}(|D|)\tilde{\Omega}_{(b)}u(x) = -\sum_{j,k=1}^{2}\int_0^\infty \lambda^{-2}\mu_{jk}(\lambda)(\chi_{>4a}(|D|)\mathcal{G}_{-\lambda}*v\zeta_j)(x)\langle\zeta_k v,\tilde{b}(\lambda,z)u\rangle\lambda d\lambda$$

and, with $\mu(\xi) = \chi_{>4a}(|\xi|)|\xi|^{-2}$, $\chi_{>4a}(|D|)\mathcal{G}_{-\lambda}(x) = (2\pi)^{-1}\hat{\mu}(x) + \lambda^2\mu(|D|)\mathcal{G}_{-\lambda}(x)$. Then, $\lambda^2\mu(|D|)\mathcal{G}_{-\lambda}(x)$ produces a good operator cancelling the singularity λ^{-2} and we show that the operator produced by $(2\pi)^{-1}\hat{\mu}(x)$ is unbounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$. Thanks to the fact that $(2\pi)^{-1}\hat{\mu}(x)$ is λ-independent, the operator in question becomes

$$-\sum_{j,k=1}^{2} a_j(x)\int_0^\infty \lambda^{-2}\mu_{jk}(\lambda)\langle\zeta_k v,\tilde{b}(z,\lambda)\rangle\lambda d\lambda = -\sum_{j=1}^{2}a_j(x)\ell_j(u) \tag{1.37}$$

where, for $j=1,2$, $a_j(x){=}(2\pi)^{-1}(\hat{\mu}*v\zeta_j)(x) \in L^p(\mathbb{R}^2)$ for $1<p<\infty$ and Parseval's identity implies that the linear functional $\ell_j(u)$ is equal to

$$\ell_j(u) = \frac{i}{2\pi}\sum\langle z_l v,\zeta_k\rangle\left(\int_0^\infty\int_{\mathbb{S}^1}\mu_{jk}(\lambda)\mathcal{F}(R_l u)(\lambda\omega)d\omega d\lambda\right) \tag{1.38}$$
$$= \frac{i}{2\pi}\sum\langle z_l v,\zeta_k\rangle\int_{\mathbb{R}^2}u(x)\mathcal{F}(\mu_{jk}(|\xi|)\xi_l|\xi|^{-2})(x)dx,$$

where the sum is taken over $k,l=1,2$. We show $a_1,a_2 \in L^p(\mathbb{R}^2)$ are linearly independent if $a>0$ is sufficiently small. It follows by the Hahn-Banach theorem that, if $-\sum_{j=1}^{2}a_j(x)\ell_j(u)$ were bounded operator in $L^p(\mathbb{R}^2)$ for such $a>0$, ℓ_1 and ℓ_2 must be bounded on $L^p(\mathbb{R}^2)$ hence, by virtue of the Riesz theorem, it must be that for $q=p/(p-1)$

$$\sum_{k,l=1}^{2}\langle z_l v|\zeta_k\rangle\mathcal{F}(\mu_{jk}(|\xi|)\xi_l|\xi|^{-2}) \in L^q(\mathbb{R}^2), \quad j=1,2.$$

Then, since $1 < q < 2$ for $2 < p < \infty$, Hausdorff-Young's inequality implies

$$\sum_{k=1}^{2} d_{jk}(|\xi|) \sum_{l=1}^{2} \langle z_l v|\zeta_k\rangle \chi_{\leq 2a}(\xi)\xi_l g(|\xi|)^{-1}|\xi|^{-2} \in L^p(\mathbb{R}^2)$$

and, since $C(|\xi|) = D(|\xi|)^{-1}$ is bounded for $|\xi| \leq 2a$,

$$\frac{\chi_{\leq 2a}(|\xi|)}{g(|\xi|)|\xi|^2} \begin{pmatrix} \langle z_1 v|\zeta_1\rangle\xi_1 + \langle z_1 v|\zeta_2\rangle\xi_2 \\ \langle z_2 v|\zeta_1\rangle\xi_1 + \langle z_2 v|\zeta_2\rangle\xi_2 \end{pmatrix} \in L^p(\mathbb{R}^2, \mathbb{C}^2).$$

But, this can happen only when $\langle z_l v|\zeta_k\rangle = 0$ for $1 \leq j, k \leq 2$, which is impossible if T_2 is non-singular, see (5.4) and $-\sum_{j=1}^2 a_j(x)\ell_j(u)$ is unbounded in $L^p(\mathbb{R}^2)$ for $2 < p < \infty$.

In the final subsection 5.7 we assume that H has singularities of the third kind at zero. Then, $T_3 = S_3 G_2 S_3$ is necessarily non-singular and the Feshbach formula implies $(S_2\tilde{\mathcal{R}}_1 S_2)^{-1} = g(\lambda)S_3 T_3^{-1} S_3 + L_4(\lambda)$, $L_4(\lambda) \in \mathcal{M}(\mathbb{R}^2)$ being an operator in $S_2 L^2(\mathbb{R}^2)$; $\zeta \in S_3 L^2(\mathbb{R}^2)$ satisfies the extra moment conditions

$$\int_{\mathbb{R}^2} x_1\zeta(x)v(x)dx = \int_{\mathbb{R}^2} x_2\zeta(x)v(x)dx = 0. \tag{1.39}$$

Modulo a good operator $\Omega_{\text{low},2a}$ is still given by (1.34) and, if $(d_{jk}(\lambda))$ is replaced by the matrix for $g(\lambda)S_3 T_3^{-1} S_3$ then, it produces a good operator. This is because we have $\langle \zeta v, \Pi(\lambda)u\rangle = \langle \zeta v, \tilde{g}(\lambda, \cdot)\rangle$ for $\zeta \in S_3 L^2(\mathbb{R}^2)$ without the bad part by virtue of (1.39) and the factor λ^2 of $\tilde{g}(\lambda, x)$ cancels the singularity $\lambda^{-2}g(\lambda)^{-1}$ leaving $g(\lambda)^{-1}$ which then cancels $g(\lambda)$ in the front of $g(\lambda)S_3 T_3^{-1} S_3$. If $S_3 = S_2$, in which case p-wave resonances are absent, then $L_4(\lambda) = 0$ and $\Omega_{\text{low},2a}$ becomes a good operator. If $S_2 \neq S_3$, then $L_4(\lambda) \neq 0$ but $L_4(\lambda)$ has the structure similar to that of $\tilde{\mathcal{R}}_1(\lambda)^{-1}$ of subsection 5.6, and we prove by slightly modifying the argument in subsection 5.6 that it produces an operator which is bounded in L^p for $1 < p \leq 2$ and unbounded for $2 < p < \infty$. In the rest of the paper we shall give the details of the proof.

2. Preparatory estimates

We often write λ for $z \in \overline{\mathbb{C}}^+$ when we want to emphasize that z can also be real. $a \leq_{|\cdot|} b$ means that $|a| \leq |b|$. Recall that $G_0(\lambda)u(x) = (\mathcal{G}_\lambda * u)(x)$, $\mathcal{G}_\lambda(x) = \mathcal{H}(\lambda|x|)$ and $\mathcal{H}(\lambda) = (i/4)H_0^{(1)}(\lambda)$. The Hankel function $\mathcal{H}(\lambda)$ has the well known series expansion ([1], p.358, (9.1.12) and (9.1.13)) and the integral representation ([1],p. 360, (9.1.23)):

$$\mathcal{H}(\lambda) = g(\lambda)\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\Big(\frac{\lambda^2}{4}\Big)^k + \sum_{k=1}^{\infty} \frac{1}{2\pi}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)\frac{(-1)^{k-1}(\frac{1}{4}\lambda^2)^k}{(k!)^2} \tag{2.1}$$

$$= \frac{e^{i\lambda}}{2^{\frac{3}{2}}\pi}\int_0^\infty e^{-t}t^{-\frac{1}{2}}\left(\frac{t}{2} - i\lambda\right)^{-\frac{1}{2}} dt \tag{2.2}$$

where $z^{\frac{1}{2}}$ is positive for positive z and, with Euler's constant γ and with principal branch of $\log z$

$$g(\lambda) = -\frac{1}{2\pi}\log\Big(\frac{\lambda}{2}\Big) + \frac{i}{4} - \frac{\gamma}{2\pi}\,. \tag{2.3}$$

The following lemma is obvious from (2.1) and (2.2).

LEMMA 2.1. (1) *$\mathcal{H}(\lambda)$ and $\mathcal{G}_\lambda(x)$ satisfy for any $\delta > 0$*

$$\mathcal{H}(\lambda) = g(\lambda) + \lambda^2\Big(-\frac{g(\lambda)}{4} + \frac{1}{8\pi}\Big) + O(g(\lambda)\lambda^4), \lambda \to 0, \tag{2.4}$$

$$\mathcal{G}_\lambda(x) = g(\lambda) + N_0(x) + O((\lambda|x|)^{2-\delta}), \quad \lambda|x| \to 0. \tag{2.5}$$

(2) *For $\lambda \geq 1$, $\mathcal{H}(\lambda) = e^{i\lambda}\omega(\lambda)$ with symbol $\omega(\lambda)$ of order $-1/2$:*

$$|\omega^{(j)}(\lambda)| \leq C_j \lambda^{-\frac{1}{2}-j}, \quad \lambda \geq 1, \ j = 0, 1, \dots. \tag{2.6}$$

(3) *There exists a constant $C > 0$ such that for*

$$|\mathcal{G}_\lambda(x)| \leq C \begin{cases} \langle \log \lambda|x|\rangle, & |\lambda|x|| \leq 1, \\ \langle \lambda|x|\rangle^{-1/2}, & |\lambda|x|| \geq 1\,. \end{cases} \tag{2.7}$$

LEMMA 2.2. *Let $0 < \alpha < \beta < \infty$. Then, for a constant $C_{\alpha,\beta}$,*

$$\int_\alpha^\beta |\mathcal{G}_\lambda(x)| d\lambda \leq C_{\alpha,\beta}(|x|^{-\frac{1}{2}} + |x|^{-1}), \quad x \in \mathbb{R}^2\,. \tag{2.8}$$

Proof. Applying (2.7), we estimate the integral by

$$C\int_{\alpha<\lambda<|x|^{-1}} (|\log \lambda|x|| + 1) d\lambda + C\int_{|x|^{-1}<\lambda<\beta} (\lambda|x|)^{-1/2} d\lambda. \tag{2.9}$$

For $|x| < \beta^{-1}$ the second integral vanishes and via change of variable

$$\int_\alpha^\beta |\mathcal{G}_\lambda(x)| d\lambda \leq \frac{C}{|x|}\int_{\alpha|x|}^1 (-\log s + 1) ds \leq \frac{C}{|x|}\int_0^1 (-\log s + 1) ds = \frac{C}{|x|}.$$

When $|x| > \alpha^{-1}$ the first integral vanishes and

$$\int_\alpha^\beta |\mathcal{G}_\lambda(x)| d\lambda \leq \frac{2C}{|x|^{\frac{1}{2}}}(\beta^{\frac{1}{2}} - |x|^{-\frac{1}{2}}) \leq \frac{2C\beta^{\frac{1}{2}}}{|x|^{\frac{1}{2}}}\,.$$

Since the left side is continuous for $\beta^{-1} \leq |x| \leq \alpha^{-1}$, (2.8) follows. □

LEMMA 2.3. *Let $u \in \mathcal{D}_*$. Then, $\Pi(\lambda)u(x) \in C^\infty(\mathbb{R}_\lambda \times \mathbb{R}^2_x)$ and there exist constants $0 < \alpha < \beta < \infty$ such that*

$$\mathrm{supp}_\lambda \Pi(\lambda)u(x) \subset (\alpha, \beta). \tag{2.10}$$

$$|\Pi(\lambda)u(x)| \leq \min(\|u\|_1,\ C_u\langle x\rangle^{-1/2}). \tag{2.11}$$

Proof. The definition of $\mathcal{D}_*$ trivially implies (2.10). The estimate (2.11) is well known, see e.g. [34], p.348. □

The following lemma is proved in [6]. K is defined by (1.19).

LEMMA 2.4. *K satisfies the identity (1.20) for $x, y, z \in \mathbb{R}^2$. Moreover:*

(1) *$Ku(x)$ is a rotationary invariant and*

$$Ku(x) = \lim_{\varepsilon \downarrow 0} \frac{-1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{u(y)dy}{x^2 - y^2 - i\varepsilon}. \tag{2.12}$$

(2) *K is bounded in $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$:*

$$\|Ku\|_p \leq C_p \|u\|_p \,. \tag{2.13}$$

Proof. We give a simpler proof of (2.12). Let $u, v \in \mathcal{D}_*$. Then for intervals $I \Subset (0, \infty)$

$$\langle \mathcal{G}_{-\lambda}, v \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\hat{v}(\xi)}{\xi^2 - \lambda^2 + i\varepsilon} d\xi$$

uniformly with respect to $\lambda \in I$ and the dominated convergence theorem implies

$$\begin{aligned} \langle Ku, v \rangle &= \frac{1}{2\pi} \int_0^{+\infty} \lambda \left(\int_{\mathbb{S}^1} \hat{u}(\lambda\omega) d\omega \right) \langle \mathcal{G}_{-\lambda}, v \rangle d\lambda \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \int_0^{+\infty} \lambda \left(\int_{\mathbb{S}^1} \hat{u}(\lambda\omega) d\omega \right) \left(\int_{\mathbb{R}^2} \frac{\hat{v}(\xi)}{\xi^2 - \lambda^2 + i\varepsilon} d\xi \right) d\lambda \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^4} \frac{\hat{u}(\eta)\hat{v}(\xi)}{\xi^2 - \eta^2 + i\varepsilon} d\xi d\eta, \end{aligned} \tag{2.14}$$

where we used that $d\eta = \lambda d\lambda d\omega$ in the polar coordinates $\eta = \lambda\omega$, $\lambda > 0$, $\omega \in \mathcal{S}^1$ in the last step. By virtue of Fubini's theorem (2.14) is equal to

$$\lim_{\varepsilon \downarrow 0} \frac{-i}{2} \int_0^\infty e^{-t\varepsilon} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-it\eta^2/2} \hat{u}(\eta) d\eta \right) \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{it\xi^2/2} \hat{v}(\xi) d\xi \right) dt \,. \tag{2.15}$$

The inner integrals are equal to $e^{-itH_0}u(0)$ and $e^{itH_0}v(0)$ respectively and for $w \in \mathcal{D}_*$

$$e^{-itH_0} w(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-it\eta^2/2} \hat{w}(\zeta) d\zeta = \frac{\mp i}{2|t|\pi} \int_{\mathbb{R}^2} e^{iz^2/2t} w(z) dz, \quad \pm t > 0,$$

is a smooth function of $t \in \mathbb{R}$ and is rapidly decreasing as $t \to \infty$. Thus

$$\begin{aligned} (2.15) &= \lim_{\varepsilon \downarrow 0} \frac{-i}{2(2\pi)^2} \int_0^\infty e^{-\varepsilon/2t} \left(\iint_{\mathbb{R}^4} e^{i(x^2-y^2)/2t} u(x) v(y) dx dy \right) \frac{dt}{t^2} \\ &= \lim_{\varepsilon \downarrow 0} \frac{-i}{(2\pi)^2} \int_0^\infty \left(\iint_{\mathbb{R}^4} e^{is(x^2-y^2+i\varepsilon)} u(x) v(y) dx dy \right) ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^4} \frac{u(x)v(y)}{x^2 - y^2 + i\varepsilon} dx dy \,. \end{aligned}$$

This proves (2.12). Denote $Ku(|x|) = Ku(x)$. Then,

$$Ku(\sqrt{r}) = \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \int_0^\infty \frac{1}{r - \rho - i\varepsilon} \left(\frac{1}{2\pi} \int_{\mathbb{S}^1} u(\sqrt{\rho}\omega) d\omega \right) d\rho.$$

This is essentially the Hilbert transform and

$$\begin{aligned}\|Ku\|_p^p &= \pi\int_0^\infty |Ku(\sqrt{r})|^p dr \le C\int_0^\infty \left|\frac{1}{2\pi}\int_{\mathbb{S}^1} u(\sqrt{\rho}\omega)d\omega\right|^p d\rho \\ &\le \frac{C}{2\pi}\int_0^\infty\int_{\mathbb{S}^1}|u(\sqrt{\rho}\omega)|^p d\omega d\rho = \frac{C}{\pi}\|u\|_p^p,\end{aligned}$$

where Hölder's inequality is used in the third step. □

3. Integral estimates

In this section we prove Proposition 1.4. We often identify integral operators with their integral kernels and say e.g. the integral operator $T(x,y)$. We always suppose $u \in \mathcal{D}_*$ which will not be mentioned anymore. In this section we assume $V \in L^1(\mathbb{R}^2)$ if otherwise stated explicitly

Lemma 3.1. *For $a, b \in \mathbb{R}^2$, define*

$$\Omega_{a,b}u(x) = \int_0^\infty \mathcal{G}_{-\lambda}(x-a)(\Pi(\lambda)u)(b)\lambda d\lambda. \tag{3.1}$$

Then, for a constant C_p independent of a, b and u,

$$\|\Omega_{a,b}u\|_p \le C_p\|u\|_p. \tag{3.2}$$

Proof. By virtue of Lemmas 2.2 and 2.3, the integral is absolutely convergent unless $x = a$ and (1.20) implies $\Omega_{a,b}u(x) = (\tau_a K\tau_{-b}u)(x)$. Since translations are isometries of L^p, (2.13) implies (3.2). □

Lemma 3.2. *Let $F \in L^1(\mathbb{R}^2)$ and $f \in \mathcal{M}(\mathbb{R}^2)$. Then:*

$$\|\mathcal{W}(f(\lambda)M_F)u\|_p \le C\|F\|_1\|f\|_{\mathcal{M},p}\|u\|_p, \quad 1 < p < \infty. \tag{3.3}$$

In particular M_F is good producer.

Proof. We need prove the lemma when $f(\lambda) = 1$ since $f(\lambda)\Pi(\lambda)u = \Pi(\lambda)f(|D|)u$ by virtue of (1.16) and $\|f(|D|)u\|_p \le \|f\|_{\mathcal{M},p}\|u\|_p$. We have

$$W(M_F)u(x) = \int_0^\infty \left(\int_{\mathbb{R}^2}\mathcal{G}_{-\lambda}(x-y)F(y)\Pi(\lambda)u(y)dy\right)\lambda d\lambda. \tag{3.4}$$

If we integrate with respect to λ first, (3.1) implies

$$W(M_F)u(x) = \int_{\mathbb{R}^2} F(y)(\Omega_{y,y}u)(x)dy, \quad \text{a.e. } x \in \mathbb{R}^2 \tag{3.5}$$

and Minkowski's inequality and (3.2) yield the desired estimate:

$$\|W(M_F)u\|_p \le \int_{\mathbb{R}^2}|F(y)|\|\Omega_{y,y}u(x)\|_p dy \le C_p\|F\|_1\|u\|_p.$$

To change the order of integrations in (3.4) it suffices to show that the integral is absolutely convergent to a.e. $x \in \mathbb{R}^2$. The following argument which is standard be repeatedly used in the rest of the paper. Let $B_R = \{x\colon |x| \le R\}$. Since $u \in \mathcal{D}_*$, $\Pi(\lambda)u(z) = 0$ for $\lambda \notin (\alpha, \beta)$ and $|\Pi(\lambda)u(z)| \le C$ by virtue of Lemma 2.3. It follows by virtue of (2.8) that

$$\begin{aligned} \int_{B_R} \left(\int_0^\infty |\mathcal{G}_{-\lambda}(x-y)F(y)(\Pi(\lambda)u)(y)|\lambda d\lambda \right) dx \\ \le C|F(y)| \int_{B_R} (|x-y|^{-1/2} + |x-y|^{-1}) dx \le C|F(y)|\,. \end{aligned} \tag{3.6}$$

and the following 5-dimensional integral is (absolutely) convergent:

$$\int_{\mathbb{R}^2 \times B_R \times [0,\infty)} |\mathcal{G}_{-\lambda}(x-y)F(y)(\Pi(\lambda)u)(y)\lambda| d\lambda dx dy < \infty.$$

Then, Fubini's theorem implies that for a.e. $x \in \mathbb{R}^2$ (3.4) is integrable on $[0,\infty) \times \mathbb{R}^2$. This concludes the proof. □

LEMMA 3.3. *Let $T \in \mathcal{L}_1$ and $f \in \mathcal{M}(\mathbb{R}^2)$. Then for a constant $C_p > 0$*

$$\|\mathcal{W}(f(\lambda)T)u\|_p \le C_p \|T\|_1 \|f\|_{\mathcal{M},p} \|u\|_p\,, \quad 1 < p < \infty. \tag{3.7}$$

In particular $W(T)$ is a good operator

Proof. As previously it suffices to show (3.7) for $W(T)u$:

$$W(T)u(x) = \int_0^\infty \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} \lambda \mathcal{G}_{-\lambda}(x-y) T(y,z) \Pi(\lambda) u(z) dz dy \right) d\lambda \tag{3.8}$$

If the integral on the right side is absolutely integrable for a.e. $x \in \mathbb{R}^2$, we may integrate with respect to λ first, and then (1.20) implies

$$\begin{aligned} W(T)u(x) &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} T(y,z) \left(\int_0^\infty \mathcal{G}_{-\lambda}(x-y) \Pi(\lambda) u(z) \lambda d\lambda \right) dy dz \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} T(y,z) (\tau_y K \tau_{-z} u)(x) dy dz\,. \end{aligned}$$

Then, Minkowski's inequality and Lemma 2.4 imply

$$\|W(T)u\|_p \le \int_{\mathbb{R}^2 \times \mathbb{R}^2} |T(y,z)| \|K\|_{\mathbf{B}(L^p)} \|u\|_p dy dz \le C\|u\|_p.$$

To see that (3.8) is absolutely integrable for a.e. $x \in \mathbb{R}^2$, we repeat the argument of the proof of the previous lemma. Using the notation of there and applying (2.10), (2.11) and (2.8), we estimate as in (3.6)

$$\begin{aligned} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |T(y,z)| \left(\int_{B_R \times [0,\infty)} \lambda |\mathcal{G}_{-\lambda}(x-y)| |\Pi(\lambda) u(z)| dx d\lambda \right) dy dz \\ \le C \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle z \rangle^{-\frac{1}{2}} |T(y,z)| dy dz \le C \|T\|_1 < \infty. \end{aligned} \tag{3.9}$$

This completes the proof. $\square$

Lemma 3.4. *Let $\sigma > 1$ and $a > 0$. Suppose $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^\sigma)$ as $\lambda \to 0$. Then, for a constant $C_p > 0$*

$$\|\mathcal{W}(\chi_{\le a}(\lambda)T(\lambda))u\|_p \le C_p\|u\|_p, \quad 1 < p < \infty. \tag{3.10}$$

Proof. Let $T'(\lambda, x, y) = \partial_\lambda T(\lambda, x, y)$ and $T''(\lambda, x, y) = \partial^2_\lambda T(\lambda, x, y)$. Since $T(0, x, y) = T'(0, x, y) = 0$ for a.e. $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, integration by parts shows that

$$T(\lambda, x, y) = \int_0^\infty (\lambda - \mu)_+ T''(\mu, x, y) d\mu, \tag{3.11}$$

as a Riemann integral in $L^1(\mathbb{R}^4)$ and simulataneously pointwise for a.e.$(x, y) \in \mathbb{R}^4$. Hence, inserting $\chi_{\le 2a}(\mu)$ which is equal to 1 when $(\lambda - \mu)_+\chi_{\le a}(\lambda) \neq 0$, we see that $\mathcal{W}(\chi_{\le a}(\lambda)T(\lambda))u(x)$ is equal to

$$\int_0^\infty \lambda\chi_{\le a}(\lambda) \left[\int_{\mathbb{R}^2} \mathcal{G}_{-\lambda}(x - y) \left\{\int_0^\infty (\lambda - \mu)_+ \times \left(\int_{\mathbb{R}^2} T''(\mu, y, z)\Pi(\lambda)u(z)dz\right) \chi_{\le 2a}(\mu)d\mu\right\} dy\right] d\lambda. \tag{3.12}$$

Since $\chi_{\le a}(\lambda)\lambda(\lambda-\mu)_+ \le a^2\chi_{\le a}(\lambda)$, $\Pi(\lambda)u(z) = 0$ for $\lambda \notin (\alpha, \beta)$ and $|\Pi(\lambda)u(z)| \le C$, the argument similar to the one which is used in the proof of previous lemmas implies that (3.12) is absolutely integrable for a.e. $x \in \mathbb{R}^2$. Then, changing the order of integration with respect to $d\lambda$ and $d\mu$ and inserting

$$(\lambda - \mu)_+ = \lambda - \mu + (\mu - \lambda)_+, \tag{3.13}$$

we obtain $\mathcal{W}(\chi_{\le a}(\lambda)T(\lambda))u = \sum_{j=1}^3 \mathcal{W}_{(j)}u$ with

$$\mathcal{W}_{(1)}u = \int_0^\infty \left(\int_0^\infty G_0(-\lambda)T''(\mu)\Pi(\lambda)|D|\chi_{\le a}(|D|)u\lambda d\lambda\right) \chi_{\le 2a}(\mu)d\mu,$$
$$\mathcal{W}_{(2)}u = -\int_0^\infty \left(\int_0^\infty G_0(-\lambda)T''(\mu)\Pi(\lambda)\chi_{\le a}(|D|)u\lambda d\lambda\right) \mu\chi_{\le 2a}(\mu)d\mu,$$
$$\mathcal{W}_{(3)}u = \int_0^\infty \left(\int_0^\infty G_0(-\lambda)T''(\mu)\Pi(\lambda)(1 - |D|/\mu)_+\chi_{\le a}(|D|)u\lambda d\lambda\right) \mu\chi_{\le 2a}(\mu)d\mu.$$

We apply Lemma 3.3. Then, since $\lambda\chi_{\le a}(\lambda)$ and $\chi_{\le a}(\lambda) \in \mathcal{M}(\mathbb{R}^2)$, Minkowski's inequality and $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^\sigma)$ for $\sigma > 1$ as $\lambda \to 0$ jointly imply that for $j = 1, 2$

$$\|\mathcal{W}_{(j)}u\|_p \le C\|u\|_p \int_0^\infty \|T''(\mu)\|_{\mathcal{L}_1}\mu^{j-1}\chi_{\le 2a}(\mu)d\mu \le C\|u\|_p. \tag{3.14}$$

We have $\sup_{\mu>0} \|(1-|D|/\mu)_+\|_{\mathbf{B}(L^p)} \le C$ for any $1 \le p \le \infty$ since the Fourier transform of $(1-|\xi|/\mu)_+$ is integrable with μ-indendent $L^1(\mathbb{R}^2)$-norm (see p. 426 of [34]). It follows as previously that

$$\|\mathcal{W}_{(3)}u\|_p \le C\|u\|_p \int_0^\infty \|T''(\mu)\|_{\mathcal{L}_1}\mu\chi_{\le 2a}(\mu)d\mu \le C\|u\|_p. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain the lemma. □

Lemma 3.5. *Let $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^{-\sigma})$ as $\lambda \to \infty$ for a $\sigma > 0$ and let $a > 0$. Then, for a constant $C_p > 0$ independent of u,*

$$\|\mathcal{W}(\chi_{>a}(\lambda)T(\lambda))u\|_p \le C_p\|u\|_p, \quad 1 < p < \infty. \tag{3.16}$$

Proof. By integration by parts we have as previously

$$T(\lambda, x, y) = \int_\lambda^\infty (\mu-\lambda)_+ T''(\mu, x, y)d\mu, \quad \text{a.e. } (x, y). \tag{3.17}$$

If we use (3.17) in place of (3.11), then $\mathcal{W}(\chi_{>a}(\lambda)T(\lambda))u(x)$ may be expressed as in (3.12) with $(\mu-\lambda)_+$, $\chi_{>a}(\lambda)$ and $\chi_{>a/2}(\mu)$ in place of $(\lambda-\mu)_+$, $\chi_{\le a}(\lambda)$ and $\chi_{\le 2a}(\mu)$ respectively. Then, we repeat the argument in the proof of Lemma 3.4. The argument actually is simpler here because we do not have to use the splitting as in (3.13). We should safely be able to omit the repetitious details. □

For proving statement (7) of Proposition 1.4, we need the following lemma. The lemma must be well known, however, we present a proof for readers' convenience.

Lemma 3.6. *Let $\kappa \in C_0^\infty(\mathbb{R}^2)$. Then for $k = 0, 1, \dots$.*

$$\mathcal{F}(g(|\xi|)^{k+1}\kappa(\xi))(x) \le_{|\,\cdot\,|} C_k \langle \log|x|\rangle^k \langle x\rangle^{-2} \tag{3.18}$$

Proof. It suffices to show (3.18) with $\log|\xi|$ in place of $g(|\xi|)$ for sufficiently large $|x| > 100$. We prove the lemma by induction. Since $(-\Delta)\log|\xi| = 2\pi\delta(\xi)$, $\mathcal{F}(\log|\xi|)(x) = |x|^{-2}$ in $\mathbb{R}^2 \setminus \{0\}$. Hence, if we define the regularization $S(x) \in \mathcal{S}'(\mathbb{R}^2)$ of $|\xi|^{-2}$ by

$$\langle S, u\rangle = \int_{\mathbb{R}^2} \frac{u(x) - \chi(x)u(0)}{|x|^2}dx,$$

then $\mathcal{F}(\log|\xi|)(x) - S(x)$ is a finite sum of $C_\alpha D_x^\alpha \delta(x)$ and

$$\mathcal{F}(\log|\xi|\kappa(\xi))(x) = (2\pi)(S * \hat\kappa)(x) + \sum C_\alpha D_x^\alpha \hat\kappa(x).$$

$D_x^\alpha\hat\kappa(x)$ is clearly rapidly decreasing and

$$(S*\hat\kappa)(x) = \left(\int_{|y|<1} + \int_{|y|\ge 1}\right) \frac{\hat\kappa(x-y) - \chi(y)\hat\kappa(x)}{|y|^2}dy = I_1(x) + I_2(x).$$

For $|y| \le 1$, we have $\hat\kappa(x-y) - \chi(y)\hat\kappa(x) = (1-\chi(y))\hat\kappa(x) + y\nabla\hat\kappa(x-\theta y)$ for a $0 < \theta < 1$ and

$$|I_1(x)| \le C|\hat\kappa(x)| + \int_{|y|<1} \frac{|\nabla\hat\kappa(x-\theta y)|}{|y|}dy$$

is rapidly decreasing.

$$I_2(x) = \frac{1}{|x|^2}\int_{\mathbb{R}^2} \hat\kappa(y)dy - \frac{1}{|x|^2}\int_{|x-y|\le 1}\hat\kappa(y)dy + \int_{|x-y|\ge 1}\hat\kappa(y)\left(\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\right)dy$$

and the second and the third integrals are bounded by $C|x|^{-3}$ for large $|x|$. Thus, (3.18) is satisfied when $k=0$.

Suppose that (3.18) has been proved for $0,\dots,k-1$. Let $\kappa_1 \in C_0^\infty(\mathbb{R}^2)$ be such that $\kappa_1(\xi)=1$ when $\kappa(\xi)\neq 0$. Then $\mathcal{F}(g(|\xi|)^{k+1}\kappa(\xi))(x) = \{\mathcal{F}(g(|\xi|)^k\kappa(\xi)) * \mathcal{F}(g(|\xi|)\kappa_1(\xi))\}(x)$ and the induction hypothesis implies that

$$\begin{aligned}
&|\mathcal{F}(g(|\xi|)^{k+1}\kappa(\xi))(x)| \le C\int_{\mathbb{R}^2}\langle \log|x-y|\rangle^{k-1}\langle x-y\rangle^{-2}\langle y\rangle^{-2}dy \\
&\le C\left(\int_{D_1}+\int_{D_2}+\int_{D_3}\right)\langle \log|x-y|\rangle^{k-1}\langle x-y\rangle^{-2}\langle y\rangle^{-2}dy = I_1+I_2+I_3,
\end{aligned}$$

where $D_1=\{y\colon |x-y|\le |x|/2\}$, $D_2=\{y\colon 2|x|\le |y|\}$ and $D_3=\{y\colon |x-y|>|x|/2, |y|<2|x|\}$. We have $|x|/2\le |y|\le 3|x|/2$ for $y\in D_1$ and

$$I_1 \le C\langle \log(|x|/2)\rangle^{k-1}\langle x\rangle^{-2}\int_{|x-y|\le |x|/2}\langle x-y\rangle^{-2}dy \le C\langle \log|x|\rangle^k\langle x\rangle^{-2}.$$

For $y\in D_2$, we have $|y|/2\le |x-y|\le 3|y|/2$ and, for large $|x|$, we have

$$\begin{aligned}
I_2 &\le C\int_{|y|\ge 2|x|}\frac{\langle \log|y|\rangle^{k-1}}{\langle y\rangle^4}dy \le C\int_{2|x|}^\infty \frac{(\log r)^{k-1}}{r^3}dr \\
&= C\int_2^\infty \frac{(\log|x|+\log r)^{k-1}}{|x|^2 r^3}dr \le C\frac{\langle \log|x|\rangle^{k-1}}{|x|^2}.
\end{aligned}$$

For $y\in D_3$, we estimate $|x|/2<|x-y|\le 3|x|$ and

$$I_3 \le \frac{C\langle \log|x|\rangle^{k-1}}{|x|^2}\int_{|y|<2|x|}\frac{dy}{\langle y\rangle^2}dy \le \frac{C\langle \log|x|\rangle^k}{|x|^2}$$

Combining these estimates yields (3.18) and the proof is completed. □

The idea of the proof of the following two lemmas is borrowed from [43]. We first show that the statement (7) is satisfied more generally if φ on the right satisfies the vanishing moment condition $\langle 1,\varphi\rangle = 0$. Define for $u\in\mathcal{D}_*$

$$\Omega(\psi,\varphi,\rho)u(x) = \int_0^\infty G_0(-\lambda)|\psi\rangle\langle\varphi,\Pi(\lambda)u\rangle\rho(\lambda)\lambda d\lambda. \tag{3.19}$$

Note that $\mathcal{W}(I_{k,a}^{\psi,\varphi}) = \Omega(\psi,\varphi,f(\lambda)g(\lambda)^k\chi_{\le 2a}(\lambda))$ and $\lambda f(\lambda)g(\lambda)^k\chi_{\le 2a}(\lambda)\in\mathcal{M}(\mathbb{R}^2)$.

Lemma 3.7. *Suppose $\psi(x), \langle x\rangle\varphi(x)\in L^1(\mathbb{R}^2)$, $\int_{\mathbb{R}^2}\varphi(x)dx=0$ and $\rho(\lambda)$ be such that $\tilde\rho(\lambda)=\lambda\rho(\lambda)\in\mathcal{M}(\mathbb{R}^2)$. Then:*

$$\|\Omega(\psi,\varphi,\rho)u\|_p \le C_p\|\langle x\rangle\varphi\|_1\|\psi\|_1\|u\|_p, \quad 1<p<\infty. \tag{3.20}$$

In particular, (1.23) *is satisfied for* $\mathcal{W}(I_{k,a}^{\psi,\varphi})$.

Proof. Since $\int_{\mathbb{R}^2}\varphi(x)dx = 0$, $\langle\varphi(y),\Pi(\lambda)u(y)\rangle = \langle\varphi(y),\Pi(\lambda)u(y)-\Pi(\lambda)u(0)\rangle$ and via Taylor's formula

$$\begin{aligned}\Pi(\lambda)u(y)-\Pi(\lambda)u(0) &= \frac{1}{2\pi}\int_0^1\left(\int_{\mathbb{S}^1} i\lambda y\omega e^{i\lambda y\omega\theta}(\mathcal{F}u)(\lambda\omega)d\omega\right)d\theta \\ &= i\sum_{j=1}^2\frac{\lambda}{2\pi}\int_0^1\left(\int_{\mathbb{S}^1} y_j\mathcal{F}(\tau_{-\theta y}R_j u)(\lambda\omega)d\omega\right)d\theta,\end{aligned}$$

where $R_j = D_j/|D|$, $j=1,2$ are Riesz transforms. It follows by virtue of (1.16) that

$$\rho(\lambda)\langle\varphi,\Pi(\lambda)u\rangle = \frac{i}{2\pi}\sum_{j=1}^2\int_{\mathbb{R}^2} y_j\varphi(y)\left(\int_0^1\int_{\mathbb{S}^1}(\mathcal{F}R_j\tau_{-\theta y}\tilde\rho(|D|)u)(\lambda\omega)d\omega d\theta\right)dy\,. \tag{3.21}$$

Multiply (3.21) by $G_0(-\lambda)\psi(x) = \int_{\mathbb{R}^2}\mathcal{G}_{-\lambda}(x-z)\psi(z)dz$ from the left and integrate with respect to $\lambda d\lambda$ first, which is regitimated as in the proof of Lemma 3.2. Then

$$\Omega(\psi,\varphi,\rho)u(x) = \sum_{j=1}^2\int_0^1\left(\iint_{\mathbb{R}^4} y_j\varphi(y)\psi(z)(\tau_z K\tau_{-\theta y}\mathcal{R}_j\tilde\rho(|D|)u)(x)dydz\right)d\theta$$

and Minkowki's inequality, Riesz theorem, Mikhlin's multiplier theorem and Lemma 2.4 jointly imply

$$\|\Omega(\psi,\varphi,\rho)u\|_p \le C\sum_{j=1}^2\int_0^1\left(\iint_{\mathbb{R}^4}|y_j\varphi(y)\psi(z)|\|K\|_{\mathbf{B}(L^p)}\|u\|_p dydz\right)d\theta.$$

Estimate (3.20) follows. □

The proof of Proposition 1.4 (7) for the case when ψ satisfies $\langle\psi,1\rangle = 0$ is much more involved than that of Lemma 3.7 for the case $\langle\varphi,1\rangle = 0$.

LEMMA 3.8. *Suppose* $\langle x\rangle\psi(x),\varphi(x)\in L^1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2}\psi(x)dx = 0$. *Then,* (1.24) *is satisfied for* $k=0,1,\dots$*:*

$$\|\mathcal{W}(I_{k,a}^{\psi,\varphi})u\|_p \le C_{k,p}\|\langle x\rangle\psi\|_1\|\varphi\|_1\|u\|_p, \quad 1<p<\infty. \tag{3.22}$$

Proof. We may and do assume $f(\lambda)=1$ as previously. We have $\mathcal{W}(I_{k,a}^{\psi,\varphi})u = \chi_{>4a}(|D|)\mathcal{W}(I_{k,a}^{\psi,\varphi})u + \chi_{\le 4a}(|D|)\mathcal{W}(I_{k,a}^{\psi,\varphi})u$ and we deal with $\chi_{>4a}(|D|)\mathcal{W}(I_{k,a}^{\psi,\varphi})u$ first. Define $\mu(\xi) = \chi_{>4a}(\xi)|\xi|^{-2}$. It is obvious that $\hat\mu\in L^p(\mathbb{R}^2)$for $1\le p<\infty$. For $\lambda<2a$, functions of both sides of

$$\chi_{>4a}(|\xi|)(\xi^2-\lambda^2)^{-1} = \mu(\xi)+\lambda^2\mu(\xi)(\xi^2-\lambda^2)^{-1}$$

are smooth and the Fourier transform yields

$$\chi_{>4a}(|D|)\mathcal{G}_{-\lambda}(x) = \frac{1}{2\pi}\hat\mu(x)+\lambda^2\mu(|D|)\mathcal{G}_{-\lambda}(x).$$

It follows that

$$\chi_{>4a}(|D|)G_0(-\lambda)\psi(x) = \frac{1}{2\pi}(\hat{\mu} * \psi)(x) + \lambda^2\mu(|D|)G_0(-\lambda)\psi(x) \tag{3.23}$$

and $\chi_{>4a}(|D|)\mathcal{W}(I_{k,a}^{\psi,\varphi})u(x) = T_1u(x) + T_2u(x)$ with

$$T_1u(x) = \frac{1}{2\pi}(\hat{\mu} * \psi)(x) \cdot \int_0^\infty \langle\varphi, \Pi(\lambda)u\rangle g(\lambda)^k \chi_{\le 2a}(\lambda)\lambda d\lambda\,, \tag{3.24}$$

$$T_2u(x) = \mu(|D|)\int_0^\infty G_0(-\lambda)|\varphi\rangle\langle\psi|\Pi(\lambda)u\rangle\rho_k(\lambda)\lambda d\lambda\,, \tag{3.25}$$

where $\rho_k(\lambda) = \lambda^2 g(\lambda)^k\chi_{\le 2a}(\lambda) \in \mathcal{M}(\mathbb{R}^2)$. The second factor on the right of (3.24) which we denote by $\ell(u)$ is a inear functional and

$$\ell(u) = \frac{1}{2\pi}\int_{\mathbb{R}^2}(\mathcal{F}^{-1}\varphi)(\xi)\hat{u}(\xi)g(|\xi|)^k\chi_{\le 2a}(|\xi|)d\xi = \langle u, \varphi * \mathcal{F}(g^k \cdot \chi_{\le 2a})\rangle\,.$$

Since $\mathcal{F}(g^k \cdot \chi_{\le 2a}) \in L^q(\mathbb{R}^2)$ for any $1 < q < \infty$ by virtue of Lemma 3.6, Young's and Hölder's inequalities imply

$$\|T_1u\|_p \le (2\pi)^{-1}\|\hat{\mu}\|_p\|\psi\|_1\|\varphi\|_1\|\mathcal{F}(g^k \cdot \chi_{\le 2a})\|_q\|u\|_p, \quad q = p/p-1.$$

Since $T_2u = \mu(|D|)\mathcal{W}(\rho_k(\lambda)\psi \otimes \varphi)u$, Lemma 3.3 implies $\|T_2u\|_p \le C\|\varphi\|_1\|\psi\|_1\|u\|_p$. Thus, $\chi_{>4a}(|D|)\mathcal{W}(I_{k,a}^{\psi,\varphi})$ is bounded by the right side of (3.22).

We next estimate $\chi_{\le 4a}(|D|)\mathcal{W}(I_{k,a}^{\psi,\varphi})u(x)$ which is equal to

$$\int_0^\infty (\chi_{\le 4a}(|D|)G_0(-\lambda)(\psi \otimes \varphi)\Pi(\lambda)u)(x)g(\lambda)^k\chi_{\le 2a}(\lambda)\lambda d\lambda\,. \tag{3.26}$$

By the definition of $G_0(-\lambda)$ for $\lambda \in \mathbb{R}$, $\chi_{\le 4a}(|D|)G_0(-\lambda)\psi(x)$ is given by

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)\hat{\psi}(\xi)}{\xi^2 - (-\lambda + i0)^2}d\xi = \lim_{\sigma\to+0}\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)\hat{\psi}(\xi)}{\xi^2 - (-\lambda + i\sigma)^2}d\xi\,. \tag{3.27}$$

Since $\langle\psi, 1\rangle = 0$,

$$\hat{\psi}(\xi) = \frac{1}{2\pi}\sum_{m=1}^2\int_0^1\left(\int_{\mathbb{R}^2}(-iz\xi)e^{-i\theta z\xi}\psi(z)dz\right)d\theta \tag{3.28}$$

and, for λ in the support of (3.26) which is compact in $(0, \infty)$, (3.27) becomes $\sum_{m=1}^2$ of

$$\begin{aligned}&\frac{-i}{(2\pi)^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_0^1\frac{\xi_m\chi_{\le 4a}(|\xi|)e^{i(x-\theta z)\xi}z_m\psi(z)}{\xi^2 - (-\lambda + i0)^2}d\theta dz\right)d\xi \\ &\qquad = \frac{-i}{2\pi}\int_0^1\int_{\mathbb{R}^2}z_m\psi(z)\tau_{\theta z}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{ix\xi}\frac{\xi_m\chi_{\le 4a}(|\xi|)}{\xi^2 - (-\lambda + i0)^2}d\xi\right)d\theta dz.\end{aligned} \tag{3.29}$$

Here the change of order of integral is trivially justified if $i0$ is replaced by $i\sigma$; the change of the order of the limit $\sigma \to 0$ and the integral with respect to $dzd\theta$ can also be justified by observing that $z_m\varphi(z) \in L^1(\mathbb{R}^2)$ and that for λ in compact intervals of $(0,\infty)$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\xi}\frac{\xi_m\chi_{\le 4a}(|\xi|)}{\xi^2-(-\lambda+i\sigma)}d\xi = G_0(-\lambda+i\sigma)\mathcal{F}(\xi_m\chi_{\le 4a})(x)$$

is bounded in $\mathbb{R}^2$ uniformly for $0<\sigma<1$ by virtue of (2.7) and it converges uniformly as $\sigma\to +0$. Elementary estimate involving Cauchy's principal value implies that

$$\lim_{\sigma\to 0+}\int_{\mathbb{R}^2}\frac{f(\xi)(|\xi|-\lambda)}{|\xi|^2-(-\lambda+i\sigma)^2}d\xi = \int_{\mathbb{R}^2}\frac{f(\xi)}{|\xi|+\lambda}d\xi, \quad f\in\mathcal{S}(\mathbb{R}^2),$$

and we may set in (3.29) as

$$\frac{\xi_m}{\xi^2-(-\lambda+i0)^2} = \frac{\lambda\omega_m}{\xi^2-(-\lambda+i0)^2}+\frac{\omega_m}{|\xi|+\lambda}, \quad \omega_m=\frac{\xi_m}{|\xi|}.$$

Then the inner integral in the right of (3.29) becomes

$$2\pi R_m\lambda\chi_{\le 4a}(|D|)\mathcal{G}_{-\lambda}(x)+R_m\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{ix\xi}\frac{\chi_{\le 4a}(|\xi|)}{|\xi|+\lambda}d\xi\right). \tag{3.30}$$

The contribution of the first term of (3.30) to (3.26) is given by $\sum_{m=1}^2\int_0^1 d\theta$ of

$$\begin{aligned} W^{(1)}_{m,\theta}u(x) &\stackrel{\text{def}}{=} -i\int_{\mathbb{R}^2} z_m\psi(z)\varphi(y)\tau_{\theta z}R_m\chi_{\le 4a}(|D|) \\ &\quad\times\left\{\int_0^\infty \mathcal{G}_{-\lambda}(x)\left(\int_{\mathbb{S}^1}(\mathcal{F}\tau_{-y}u)(\lambda\omega)d\omega\right)\lambda g(\lambda)^k\chi_{\le 2a}(\lambda)\lambda d\lambda\right\}dzdy \\ &= -2\pi i\int_{\mathbb{R}^2} z_m\psi(z)\varphi(y)(R_m\chi_{\le 4a}(|D|)K(\tau_{-y}\kappa(|D|)u)(x-\theta z)dzdy \end{aligned} \tag{3.31}$$

where $\kappa(\lambda)=\lambda g(\lambda)^k\chi_{\le a}(\lambda)$ is a good multiplier. Then, Minkowski's inequality, (2.13) and multiplier theory imply

$$\Big\|\sum_{m=1}^2\int_0^1 W^{(1)}_{m,\theta}d\theta\Big\|_p \le C\|\langle x\rangle\psi\|_1\|\varphi\|_1\|u\|_p, \quad 1<p<\infty \tag{3.32}$$

We define

$$F(x)=\int_0^\infty\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{ix\xi}\frac{\chi_{\le 4a}(|\xi|)}{|\xi|+\lambda}d\xi\right)\langle\varphi,\Pi(\lambda)u\rangle g(\lambda)^k\chi_{\le 2a}(\lambda)\lambda d\lambda$$

so that the contribution of the second term of (3.30) to (3.26) becomes

$$W^{(2)}u(x)=\frac{-i}{2\pi}\sum_{m=1}^2\int_0^1\int_{\mathbb{R}^2}z_m\psi(z)(R_mF)(x-\theta z)dzd\theta. \tag{3.33}$$

Then Minkowski's inequality implies

$$\|W^{(2)}u\|_p\le C\|\langle z\rangle\psi\|_1\|F\|_p, \quad 1<p<\infty \tag{3.34}$$

and for finishing the proof it suffices to show $\|F\|_p \le C\|\varphi\|_1\|u\|_p$. We have

$$F(x) = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} e^{ix\xi}\frac{\chi_{\le 4a}(|\xi|)}{|\xi|+|\eta|}d\xi\right)\check{\varphi}(\eta)\hat{u}(\eta)g(|\eta|)^k\chi_{\le 2a}(|\eta|)d\eta$$

$$= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^4} e^{ix\xi-iy\eta}\frac{\chi_{\le 4a}(|\xi|)g(|\eta|)^k\chi_{\le 2a}(|\eta|)\check{\varphi}(\eta)}{|\xi|+|\eta|}d\xi d\eta\right)u(y)dy. \tag{3.35}$$

Denote by $Q(x,y)$ the integral inside the parenthesis in (3.35) and define

$$\tilde{L}(x,y) = \int_{\mathbb{R}^4} e^{ix\xi-iy\eta}\frac{\chi_{\le 4a}(|\xi|)\chi_{\le 4a}(|\eta|)}{|\xi|+|\eta|}d\xi d\eta, \tag{3.36}$$

$$L(x,y) = \int_{\mathbb{R}^2}\tilde{L}(x,y-y')\mathcal{F}(g^k\chi_{\le 2a})(y')dy' \tag{3.37}$$

so that

$$Q(x,y) = \int_{\mathbb{R}^2} L(x,y-z)\varphi(z)dz \tag{3.38}$$

We will show $\|\|Q(x,y)\|_{L^p(\mathbb{R}^2_x)}\|_{L^q(\mathbb{R}^2_y)} \le C\|\varphi\|_1$ for $1<p<\infty$ and $q=p/(p-1)$, which will then yield the desired estimate via Minkowski's and Hölder's inequalities:

$$\|F\|_p \le \int_{\mathbb{R}^2}\|Q(x,y)\|_{L^p(\mathbb{R}^2_x)}|u(y)|dy \le \|\|Q(x,y)\|_{L^p(\mathbb{R}^2_x)}\|_{L^q(\mathbb{R}^2_y)}\|u\|_p = C\|u\|_p.$$

Since $\varphi\in L^1(\mathbb{R}^2)$, it suffices to show $\|L(x,y)\|_{L^p(\mathbb{R}^2_x)}\in L^q(\mathbb{R}^2_y)$. It is shown in Lemma B in Appendix of [43] that $|\tilde{L}(x,y)| \le C\langle x\rangle^{-1}\langle y\rangle^{-1}(\langle x\rangle+\langle y\rangle)^{-1}$ and elemetray estimates yield

$$\|\tilde{L}(x,y)\|_{L^p_x} \le C\begin{cases}\langle y\rangle^{-2}, & p>2,\\ \langle y\rangle^{-2}\langle\log y\rangle^{\frac12}, & p=2,\\ \langle y\rangle^{\frac{2}{p}-3}, & 1<p<2\,.\end{cases}$$

It follows by Minkowski's inequality that $\|L(x,y)\|_{L^p(\mathbb{R}^2_x)}$ is bounded respectively by

$$\int_{\mathbb{R}^2}\frac{\langle\log|y'|\rangle^k dy'}{\langle y-y'\rangle^2\langle y'\rangle^2},\ \int_{\mathbb{R}^2}\frac{\langle\log|y-y'|\rangle\langle\log|y'|\rangle^k}{\langle y-y'\rangle^2\langle y'\rangle^2}dy' \text{ and } \int_{\mathbb{R}^2}\frac{\langle\log|y'|\rangle^k dy'}{\langle y-y'\rangle^{3-\frac{2}{p}}\langle y'\rangle^2}$$

for $p>2$, $p=2$ and $1<p<2$ and Young's inequality implies that $\|L(x,y)\|_{L^p(\mathbb{R}^2_x)}\in L^q(\mathbb{R}^2_y)$ as desired. □

4. High energy estimate

In this section, we prove that $\Omega_{\text{high},2a}$ defined by (1.17):

$$\Omega_{\text{high},2a}u = \int_0^\infty G_0(-\lambda)vM(\lambda)^{-1}v\Pi(\lambda)u\chi_{>2a}(\lambda)\lambda d\lambda\,. \tag{4.1}$$

is a good operator for any $a>0$ under the condition $\langle x\rangle^2 V\in L^{\frac43}(\mathbb{R}^2)$ which will be assumed throughout this section. This has been known for some years under a slightly strongler assumption that for an $\varepsilon > 0$, $\langle x\rangle^{\frac{7}{2}+\varepsilon}|V(x)| \le C$ (cf. [39], Lemma 3.2). We give a new proof which replaces the argument via integration parts in [39] by the one via the Fourier multipliers and the singular integral operator K of (1.19).

THEOREM 4.1. *Assume $\langle x\rangle^2 V \in L^{\frac{4}{3}}(\mathbb{R}^2)$. For any $a > 0$, $\Omega_{\text{high},2a}$ is a good operator.*

We shall often omit the index $2a$ in $\Omega_{\text{high},2a}$ in what follows when no confusion is feared. Expanding $M(\lambda)^{-1} = U(1 + vG_0(\lambda)w)^{-1}$ as in (1.25) shows that $\Omega_{\text{high}}u$ is the sum of six operators:

$$\sum_{j=0}^{4} \Omega_{\text{high}}^{(j)} u + \Omega_{\text{high}(5)} u = \sum_{j=0}^{4} \int_0^\infty G_0(\lambda)w(-vG_0(\lambda)w)^j v\Pi(\lambda)u\lambda\chi_{>2a}(\lambda)d\lambda$$
$$+ \int_0^\infty G_0(\lambda)w(vG_0(\lambda)w)^5(1 + vG_0(\lambda)w)^{-1}v\Pi(\lambda)u\lambda\chi_{>2a}(\lambda)d\lambda. \tag{4.2}$$

and we show that $\Omega_{\text{high}}^{(j)}$, $0 \le j \le 4$ and $\Omega_{\text{high}(5)}$ are good operators separately.

4.1. Estimate of $\mathbf{\Omega_{high}^{(0)}}$

We remark that Hl̈der's inequality implies $\langle x\rangle^{-2}L^{\frac{4}{3}} \subset L^1(\mathbb{R}^2)$.

PROPOSITION 4.2. *Suppose $V \in L^1(\mathbb{R}^2)$. Then $\Omega_{\text{high},0}$ is a good operator.*

Proof. Since $\Omega_{\text{high}}^{(0)} = \mathcal{W}(\chi_{>2a}(\lambda)M_V)$, Proposition 1.4 (1) implies

$$\|\Omega_{\text{high}}^{(0)} u\|_p \le C_p\|V\|_1\|u\|_p, \quad 1 < p < \infty. \tag{4.3}$$

The proposition follows. □

4.2. Estimate of $\mathbf{\Omega_{high}^{(1)}}$

Define the Banch space $\mathcal{X}_{\varepsilon,r} = \langle x\rangle^{-\varepsilon}L^1(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$ with norm $\|u\|_{\mathcal{X}_{\varepsilon,r}} = \|\langle x\rangle^\varepsilon u\|_1 + \|u\|_r$. If $\langle x\rangle^2 V \in L^{\frac{4}{3}}(\mathbb{R}^2)$ then $V \in \mathcal{X}_{\varepsilon,r}$ and $\|V\|_{\mathcal{X}_{\varepsilon,r}} \le C\|\langle x\rangle^2 V\|_{\frac{4}{3}}$ for any $0 \le \varepsilon < 3/2$ and $1 \le r \le 4/3$ by virtue of Hölder's inequality.

PROPOSITION 4.3. *Suppose that $V \in \mathcal{X}_{\varepsilon,r}$ for some $0 < \varepsilon < 3/2$ and $1 < r \le 3/4$. Then, for any $1 < p < \infty$*

$$\|\Omega_{\text{high}}^{(1)} u\|_p \le C_p\|u\|_p \iint_{\mathbb{R}^4} |V(x)||V(y)|(1 + |\log|x-y||)dxdy \tag{4.4}$$

and the right side is bounded by $C_p\|V\|_{\mathcal{X}_{\varepsilon,r}}^2\|u\|_p$.

Define $V_y^{(2)}(x) = V(x)V(x-y)$. For a.e. $x \in \mathbb{R}^2$, $V_y^{(2)} \in L^1(\mathbb{R}^2)$.

LEMMA 4.4. *We have the following expression for a.e. $x \in \mathbb{R}^2$:*

$$\Omega_{\text{high}}^{(1)} u(x) = -\int_{\mathbb{R}^2} (W(M_{V_y^{(2)}})\mathcal{H}(|y||D|)\chi_{>2a}(|D|)\tau_y u)(x)dy. \tag{4.5}$$

Proof. For $\lambda \neq 0$, $\mathcal{G}_{-\lambda}(y)V(x-y) \in L^1(\mathbb{R}^2_y)$ and for $u \in \mathcal{D}_*$

$$
\begin{aligned}
VG_0(\lambda)Vu(x) &= \int_{\mathbb{R}^2} V(x)\mathcal{H}(\lambda|y|)V(x-y)u(x-y)dy \\
&= \int_{\mathbb{R}^2} V_y^{(2)}(x)\mathcal{H}(|y|\lambda)(\tau_y u)(x)dy, \quad \text{a.e. } x \in \mathbb{R}^2.
\end{aligned} \tag{4.6}
$$

It follows that

$$
\begin{aligned}
-\Omega^{(1)}_{\rm high}(x) &= \int_0^\infty (G_0(-\lambda)VG_0(\lambda)V\Pi(\lambda)u)(x)\lambda\chi_{>2a}(\lambda)d\lambda \\
&= \int_0^\infty \left(\int_{\mathbb{R}^4} \mathcal{G}_{-\lambda}(x-y)V_z^{(2)}(y)\mathcal{H}(\lambda|z|)(\Pi(\lambda)\tau_z u)(y)dzdy\right)\lambda\chi_{>2a}(\lambda)d\lambda.
\end{aligned}
$$

We show that the last integral converges absolutely for a.e. $x \in \mathbb{R}^2$ by repeating the argument used for (3.12): We let $B_R = \{x \in \mathbb{R}^2 \colon |x| \le R\}$. Lemma 2.3 implies $\Pi(\lambda)(\tau_z u)(y)$ vanishes for $\lambda \notin (\alpha,\beta)$ and is bounded; (2.7) that $|\mathcal{H}(|z|\lambda)| \le (|\log|z||+C)$ for $\lambda \in (\alpha,\beta)$. Then, (2.8) implies that

$$
\begin{aligned}
&\int_0^\infty \int_{B_R\times\mathbb{R}^4} |\mathcal{G}_{-\lambda}(x-y)V_z^{(2)}(y)\mathcal{H}(|z|\lambda)\Pi(\lambda)(\tau_z u)(y)|\lambda\chi_{>2a}(\lambda)dxdzdyd\lambda \\
&\le C\int_\alpha^\beta \int_{B_R\times\mathbb{R}^4} |\mathcal{G}_{-\lambda}(x-y)||V_z^{(2)}(y)|(|\log|z||+C)dxdzdyd\lambda \\
&\le C\int_{\mathbb{R}^4} |V(y)V(z)|(1+|\log|y-z||)dzdy < \infty.
\end{aligned}
$$

Thus, we can integrate with respect to $d\lambda$ first and, applying (1.16) for $f(\lambda) = \chi_{>2a}(\lambda)\mathcal{H}(|z|\lambda)$, we obtain

$$
\Omega^{(1)}_{\rm high}u(x) = -\int_{\mathbb{R}^2}\left(\int_0^\infty (G_0(-\lambda)V_y^{(2)}\Pi(\lambda)\mathcal{H}(|y||D|)\chi_{>2a}(|D|)\tau_y u)(x)\lambda d\lambda\right)dy
$$

which is nothing but (4.5). This proves the lemma. □

We define for an $a > 0$

$$
\mathcal{H}(\lambda) = \mathcal{H}_{\rm low}(\lambda) + \mathcal{H}_{\rm high}(\lambda) \stackrel{\rm def}{=} \chi_{\le a}(\lambda)\mathcal{H}(\lambda) + \chi_{>a}(\lambda)\mathcal{H}(\lambda).
$$

Lemma 4.5. *Let $a > 0$ and $y \in \mathbb{R}^2$. Then, $\mathcal{H}_{\rm high}(|y||D|)$, $\mathcal{H}_{\rm low}(|y||D|)\chi_{>2a}(|D|)$ and $\mathcal{H}_{\rm low}(|y||D|)\chi_{>2a}(|D|)$ are good operators and*

$$
\|\mathcal{H}_{\rm high}(|y||D|)\|_{\mathbf{B}(L^p)} \le C_p\,, \tag{4.7}
$$

$$
\|\mathcal{H}_{\rm low}(|y||D|)\chi_{>2a}(|D|)\|_{\mathbf{B}(L^p)} \le C_{a,p}(1+|\log|y||)\,, \tag{4.8}
$$

$$
\|\mathcal{H}(|y||D|)\chi_{>2a}(|D|)\|_{\mathbf{B}(L^p)} \le C_{a,p}(1+|\log|y||)\,. \tag{4.9}
$$

Proof. (1) Since $\mathcal{H}_{\rm high}(\lambda)$ satisfies (2.6), the theory of spatially homogenous Fourier integral operators (p.138 of [27], see also [33, 35]) implies that $\mathcal{H}_{\rm high}(|D|)$ is a good operator. Then, the scaling argument implies the same for $\mathcal{H}_{\rm high}(|y||D|)$ and the estimate (4.7) is satisfied. (2) By virtue of (2.4) $(\mathcal{H}(\lambda)-g(\lambda))\chi_{\le a}(\lambda) \in \mathcal{M}(\mathbb{R}^2)$ and we have for any

$1 < p < \infty$ that $\|\mathcal{H}_{\rm low}(|y||D|) - g(|y||D|)\chi_{\le a}(|y||D|)\|_{\mathbf{B}(L^p)} \le C_p$. Thus, it suffices to prove (4.8) for $g(|y||D|)\chi_{\le a}(|y||D|)\chi_{>2a}(|D|)$ which is equal to

$$(g(|y|) - (2\pi)^{-1}\log|D|)\chi_{\le a}(|y||D|)\chi_{>2a}(|D|).$$

Since $\|g(|y|)\chi_{\le a}(|y||D|)\chi_{>2a}(|D|)\|_{\mathbf{B}(L^p)} \le C(1+|\log|y||)$ is evident, we consider only $-(1/2\pi)(\log|D|)\chi_{\le a}(|y||D|)\chi_{>2a}(|D|)$. Define $F(\lambda) = (1/2\pi)(\log\lambda)\chi_{\le a}(|y|\lambda)\chi_{>2a}(\lambda)$. Then, we have

$$|F^{(j)}(\lambda)| \le C\lambda^{-j}(1+|\log|y||)\,, \quad j = 0,1,2. \tag{4.10}$$

Indeed $F(\lambda) \ne 0$ only if $|y| < 1$ and $a < \lambda < a/|y|$ and

$$|F(\lambda)| \le (2\pi)^{-1}\max(|\log a|, |\log a/|y||) \le (2\pi)^{-1}(|\log|y|| + |\log a|),$$

which implies (4.10) for $j = 0$. The proof for $j = 1, 2$ is similar. Then, Mikhlin's theorem implies that $\|F(|D|)\|_{\mathbf{B}(L^p)} \le C(1+|\log|y||)$ and (4.8) follows. (4.7) and (4.8) imply (4.9). □

Proof of Proposition 4.3. We apply Minkowski's inequality to (4.5). Then, Proposition 1.4 (1) and (4.9) imply that

$$\|\Omega^{(1)}_{\rm high}u\|_p \le C_p \int_{\mathbb{R}^2} \|V^{(2)}_y\|_1(1+|\log|y|)\|u\|_p dy$$

which is equivalent to (4.4). We have $\chi_{>2}(|x-y|)|\log|x-y|| \le C_\varepsilon\langle x\rangle^\varepsilon\langle y\rangle^\varepsilon$ for any $\varepsilon > 0$ and $\chi_{\le 2}(|x|)|\log|x|| \in L^p(\mathbb{R}^2)$ for any $1 \le p < \infty$ and, Young's and Hölder's inequalities imply that the right side of (4.4) is bounded by $C_p\|V\|^2_{\mathcal{X}_{\varepsilon,r}}$. □

4.3. Estimate of $\Omega^{(n)}_{\rm high}$

The following proposition implies that $\Omega^{(n)}_{\rm high}$, $n = 2,3,4$ are good operators. Note that $vG_0(-\lambda)w \in \mathcal{H}_2$ for $\lambda > 0$. We define for $n = 0,1,\dots$

$$\Omega^{(n)}_{\rm high}u = \int_0^\infty G_0(\lambda)w(-vG_0(\lambda)w)^j v\Pi(\lambda)u\lambda\chi_{>2a}(\lambda)d\lambda, \tag{4.11}$$

$$C^{(n)} = \int_{\mathbb{R}^{2(n+1)}} \left(\prod_{i=1}^{n+1}|V(y_i)|\right)\left(\prod_{i=1}^{n}(1+|\log|y_i - y_{i+1}||)\right)dy_1\cdots dy_{n+1}.$$

Proposition 4.6. *Suppose that $V \in \mathcal{X}_{\varepsilon,r}$ for some $0 \le \varepsilon < 3/2$ and $1 \le r \le 4/3$. Then, for any $1 < p < \infty$, there exists a constant C_p such that*

$$\|\Omega^{(n)}_{\rm high}u\|_p \le C_pC^{(n)}\|u\|_p, \quad C^{(n)} \le (C\|V\|_{\mathcal{X}_{\varepsilon,r}})^{n+1}, \quad n = 1,2,\dots. \tag{4.12}$$

Proof. We let $n \ge 2$. Repeating the argument which is used to derive (4.6), we obtain that for $u \in \mathcal{D}_*$

$$(vU(-vG_0(\lambda)w)^n vu)(x) = (-1)^n(VG_0(\lambda)V\cdots VG_0(\lambda)Vu)(x)$$

$$= (-1)^n \iint_{\mathbb{R}^{2n}} V^{(n+1)}_{y_1,\dots,y_n}(x) \left(\prod_{j=1}^n \mathcal{H}(\lambda|y_j|)\right) \tau_{y_1+\cdots+y_n} u(x)\, dy_1 \dots dy_n$$

where $V^{(n+1)}_{y_1,\dots,y_n}(x) = V(x)V(x-y_1)\cdots V(x-y_1-\cdots-y_n)$. This yields

$$\Omega^{(n)}_{\text{high}} u(x) = (-1)^n \iint_{\mathbb{R}^{2n}} \int_0^\infty \mathcal{G}_{-\lambda}(x-y) V^{(n+1)}_{y_1,\dots,y_n}(y) \times \left(\prod_{j=1}^n \mathcal{H}(|y_j|\lambda)\right) \Pi(\lambda)\tau_{y_1+\cdots+y_n} u(y)\lambda\chi_{>2a}(\lambda) dy_1\cdots dy_n dy d\lambda\,. \tag{4.13}$$

The argument similar to the one which is used in the proof of Lemma 4.4 and which we avoid to repeat here implies that (4.13) is absolutely integrable for a.e. $x \in \mathbb{R}^2$ and, integrating (4.13) by $d\lambda$ first after applying (1.16), we obtain that

$$\Omega^{(n)}_{\text{high}} u = \iint_{\mathbb{R}^{2n}} W(M_{V^{(n+1)}_{y_1,\dots,y_n}})\chi_{>2a}(|D|) \prod_{j=1}^n \mathcal{H}(|y_j||D|)\tau_{y_1+\cdots+y_n} u dy_1 \dots dy_n.$$

Minkowski's inequality, Proposition 1.4 (1) and Lemma 4.5 then imply

$$\|\Omega^{(n)}_{\text{high}} u\|_p \le C_p \int_{\mathbb{R}^{2n}} \|V^{(n+1)}_{y_1,\dots,y_n}\|_{L^1(\mathbb{R}^2_x)} \prod_{j=1}^n (1+|\log|y_j||)\|u\|_p dy_1\dots dy_n.$$

Repeating the argument in the last part of the proof of Proposition 4.3, we obtain $C^{(n)} \le (C\|V\|_{\mathcal{X}_{\varepsilon,r}})^{n+1}$. □

4.4. Estimate of $\Omega_{\text{high}(5)}$.

Estimate (4.12) implies that, if $\|V\|_{\mathcal{X}_{\varepsilon,r}}$ is, hence if $\|\langle x\rangle^2 V\|_{\frac43}$ is sufficiently small, the expansion $\Omega_{\text{high}} = \sum_{j=0}^\infty \Omega^{(n)}_{\text{high}}$ converges in $\mathbf{B}(L^p)$ for any $1<p<\infty$ and Ω_{high} becomes a good operator. However, we do not want the smallness assumption which makes H automatically regular at zero. We shall instead exploit the decay property of $\|vG_0(\lambda)w\|_{\mathcal{H}_2}$:

Lemma 4.7. (1) *Let $v, w \in L^{\frac83}(\mathbb{R}^2)$. Then, $vG_0(\lambda)w \in \mathcal{H}_2$ for any $\lambda>0$ and for $\varepsilon>0$ there exists a $C_\varepsilon>0$ such that*

$$\|vG_0(\lambda)w\|_{\mathcal{H}_2} \le C_\varepsilon \lambda^{-1/2}\|v\|_{8/3}\|w\|_{8/3}\,, \quad \lambda\ge\varepsilon. \tag{4.14}$$

(2) *Let $j=1,2$. Suppose that $\langle x\rangle^j v, \langle x\rangle^j w \in L^{\frac83}(\mathbb{R}^2)$. Then, $vG_0(\lambda)w$ is an $\mathcal{H}_2$-valued C^j function of $\lambda\in(0,\infty)$. For any $\varepsilon>0$, there exists a constant C_ε such that*

$$\|(d^j/d\lambda^j)vG_0(\lambda)w\|_{\mathcal{H}_2} \le C_\varepsilon \lambda^{-1/2}\|\langle x\rangle^j v\|_{8/3}\|\langle x\rangle^j w\|_{8/3}\,, \quad \lambda\ge\varepsilon. \tag{4.15}$$

Proof. (1) By virtue of (2.6) and (2.4), $\|vG_0(\lambda)w\|^2_{\mathcal{H}_2} \le C(F_1+F_2)$ where

$$F_1 = \int_{|x-y|>\varepsilon/\lambda} \frac{|v(x)|^2|w(y)|^2}{\lambda|x-y|} dxdy,$$

$$F_2 = \int_{|x-y|<\varepsilon/\lambda} (|\log|\lambda|x-y|| + C)^2 |v(x)|^2 |w(y)|^2 dxdy.$$

The generalized Young's inequality (e.g. [25]) implies

$$\begin{aligned} F_1 &\le C\lambda^{-1} \|v\|_{8/3}^2 \|w\|_{8/3}^2, \\ F_2 &\le C\|v\|_{8/3}^2 \|w\|_{8/3}^2 \left(\int_{|z|<\varepsilon/\lambda} (|\log|(\lambda z)|| + C)^2 dz \right)^2 \\ &\le C_\varepsilon \lambda^{-1} \|v\|_{8/3}^2 \|w\|_{8/3}^2. \end{aligned}$$

(4.14) follows.

(2) It is well known (see [1], pp. 360-364) that $\mathcal{G}_\lambda^{(1)}(x) = (d/d\lambda)\mathcal{G}_\lambda(x)$ and $\mathcal{G}_\lambda^{(2)}(x) = (d^2/d\lambda^2)\mathcal{G}_\lambda(x)$ are respectively given by $i/4$ times

$$|x|\Big(\frac{d}{dz}H_0^{(1)}\Big)(\lambda|x|) = -|x|H_{-1}^{(1)}(\lambda|x|) \le_{|\,\cdot\,|} C \begin{cases} \lambda^{-\frac12}|x|^{\frac12}, & |x|\lambda \ge 1\,. \\ \lambda^{-1}, & |x|\lambda < 1\,. \end{cases} \tag{4.16}$$

$$\begin{aligned} |x|^2\Big(\frac{d^2}{dz^2}H_0^{(1)}\Big)(\lambda|x|) &= |x|^2 \left(H_{-2}^{(1)}(\lambda|x|) + \frac{1}{\lambda|x|} H_{-1}^{(1)}(\lambda|x|) \right) \\ &\le_{|\,\cdot\,|} C \begin{cases} \lambda^{-\frac12}|x|^{\frac32}, & |x|\lambda \ge 1. \\ \lambda^{-2}, & |x|\lambda < 1. \end{cases} \end{aligned} \tag{4.17}$$

Define $G_0^{(j)}(\lambda)u(x) = (\mathcal{G}_\lambda^{(j)} * u)(x)$ for $j = 1, 2$ and $\mathcal{G}_\lambda^{(0)}(x) = \mathcal{G}_\lambda(x)$. If $\langle x\rangle^j v(x), \langle x\rangle^j w(x) \in L^{\frac83}(\mathbb{R}^2)$, the (4.16) and (4.17) imply as in the proof of (1) that $vG_0^{(j)}(\lambda)w \in \mathcal{H}_2$ and it satisfies (4.15). We have

$$v(x)\mathcal{G}_\lambda^{(j-1)}(x-y)w(y) - v(x)\mathcal{G}_\mu^{(j-1)}(x-y) = \int_\mu^\lambda v(x)\mathcal{G}_\rho^{(j)}(x-y)w(y)d\rho \quad j = 1, 2.$$

Then, (4.16) and (4.17) and the dominated convergence theorem imply that $\lambda \mapsto vG_0^{(j)}(\lambda)w \in \mathcal{H}_2$ is continuous for $0 < \lambda < \infty$. It follows that $vG_0(\lambda)w$ is $\mathcal{H}_2$-valued C^j and $(d/d\lambda)^j vG_0(\lambda)w = vG_0^{(j)}(\lambda)w$. This implies (2) of the lemma. □

PROPOSITION 4.8. *Assume $\langle x\rangle^2 V \in L^{\frac43}(\mathbb{R}^2)$. Then, $\Omega_{\rm high}^{(5)}$ is a good operator.*

Proof. Define $T(\lambda) = -w(vG_0(\lambda)w)^5(1 + vG_0(\lambda)w)^{-1}v$. It suffices by virtue of Proposition 1.4 (5) to show that $T(\lambda) \in \mathcal{O}_{\mathcal{L}_1}^{(2)}(\lambda^{-\varepsilon})$ for some $\varepsilon > 0$ as $\lambda \to \infty$. We first remark that, if $v, w \in L^{8/3}(\mathbb{R}^2)$ is such that $V = vw$ then

$$v(1 + wG_0(\lambda)v)^{-1}w = w(1 + vG_0(\lambda)w)^{-1}v \tag{4.18}$$

and it does not depend on the choice of v, w such that $V = vw$. Indeed, (4.18) holds for large $|\lambda|$ in the cone $\Gamma = \{\lambda \in \overline{\mathbb{C}}^+ : 0 \le \arg\zeta < \pi/4\}$ because $\|vG_0(\lambda)w\|_{\mathcal{H}_2} \to 0$ as $\lambda \to \infty$ in Γ by the proof of Lemma 4.7 (1) and both sides become $V + VG_0(\lambda)V + \cdots$ by the Neumann expansion. Then, the analyticity in $\mathbb{C}^+ \setminus \mathcal{E}$ and the continuity in $\overline{\mathbb{C}}^+ \setminus (\mathcal{E} \cup \{0\})$ imply that the same holds for all $\lambda > 0$, where $\mathcal{E} = \{i\kappa \colon \kappa > 0, \kappa^2 \in \sigma_p(H)\}$.

Let first $v(x) = |V(x)|^{\frac12}$ and $w(x) = U(x)v(x)$ so that $\langle x\rangle v, \langle x\rangle w \in L^{\frac83}(\mathbb{R}^2)$. Then, Lemma 4.7 implies that $T(\lambda) \in \mathcal{O}^{(1)}_{\mathcal{L}_1}(\lambda^{-\frac32})$ as $\lambda \to \infty$ and that

$$T'(\lambda) = -\sum_{j=0}^{4} w(vG_0(\lambda)w)^j vG_0'(\lambda)w(vG_0(\lambda)w)^{4-j}v \\ - w(vG_0(\lambda)w)^5(1+vG_0(\lambda)w)^{-1}vG_0'(\lambda)w(1+vG_0(\lambda)w)^{-1}v. \tag{4.19}$$

To see $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^{-\frac12})$, we further differentiate (4.19). The result is the sum of terms with contain two first derivatives $G_0'(\lambda)$ and the ones with a single $G_0''(\lambda)$. Letting $V = vw$ with $v(x) = |V(x)|^{\frac12}$ and $w(x) = U(x)v(x)$, we see the former terms are $\mathcal{L}_1$-valued continuous functions of $\lambda > 0$ with $\mathcal{L}_1$-noms bounded by $C\lambda^{-\frac52}$ as $\lambda \to \infty$. To see the same holds for the latter terms, we decompose $V = vw$ in such a way that $v, \langle x\rangle^2 w \in L^{\frac83}(\mathbb{R}^2)$ and sandwitch $G''(\lambda)$ by w like $w(vG_0(\lambda)w)^{j-1}vG_0(\lambda)vwG_0''(\lambda)w(vG_0(\lambda)w)^{4-j}v$. Thus, $T(\lambda) \in \mathcal{O}^{(2)}_{\mathcal{L}_1}(\lambda^{-\frac12})$ and the proof is completed. □

5. Low energy estimate

We now study the low energy part $\Omega_{\mathrm{low},2a}$ defined by (1.18):

$$\Omega_{\mathrm{low},2a}u = \int_0^\infty G_0(-\lambda)vM(\lambda)^{-1}v\Pi(\lambda)u\lambda\chi_{\le 2a}(\lambda)d\lambda,\ u \in \mathcal{D}_*. \tag{5.1}$$

In this section. *we shall often omit the phrase "for small $\lambda > 0$"* as we shall exclusively work for small $\lambda > 0$,

5.1. Resonances

The following lemma which has mostly been proved by [17] under a slightly different assumptions gives the relation between resonances and the singularities of H. We assume $V(x) \not\equiv 0$ define $v = |V|^{\frac12}$ and $w = Uv$ as in the introduction. Recall the notation in Definition 1.5.

LEMMA 5.1. *Suppose $\langle x\rangle^{1+\varepsilon}V \in L^1(\mathbb{R}^2)$ for an $\varepsilon > 0$. Then, $\mathcal{N}_\infty \neq \{0\}$ if and only if H is singular at zero. In this case $u \in \mathcal{N}_\infty$ satisfies $\langle wu, v\rangle = 0$ and $S_1L^2(\mathbb{R}^2) = \{wu\colon u \in \mathcal{N}_\infty\}$; $\mathcal{N}_\infty \ni u \mapsto \zeta = wu \in S_1L^2(\mathbb{R}^2)$ is an isomorphism and the inverse map is given by $u = N_0v\zeta - \|v\|^{-2}\langle PT_0S_1\zeta, v\rangle$; $u \in \mathcal{N}_\infty$ satisfies as $|x| \to \infty$ that*

$$u(x) = c + \sum_{j=1}^{2} \frac{x_j}{|x|^2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2} y_jV(y)u(y)dy\right) + O(|x|^{-1-\varepsilon}), \tag{5.2}$$

$$c = \|v\|_2^{-2}\langle PT_0S_1wu, v\rangle. \tag{5.3}$$

(1) *H has singularities of the first kind at zero if and only if $\mathcal{N}_\infty$ consists only of s-wave resonances. In this case* $\operatorname{rank} T_1 = \dim \mathcal{N}_\infty = 1$.

(2) *H has singularities of the second kind at zero then $\mathcal{N}_\infty$ consists of s-wave and p-wave resonances but no zero energy eigenfunctions. In this case* $1 \le \operatorname{rank} S_2 = \operatorname{rank} T_2 \le 2$. *$\mathcal{N}_\infty$ consists only of p-wave resonances if and only if $T_1 = 0$. If*

$T_1 \neq 0$, then $u \in \mathcal{N}_\infty$ is a p-wave resonance if $wu \in S_2L^2(\mathbb{R}^2)$ and an s-wave resonance otherwise.

(3) *H has singularities of the third kind at zero if and only if zero energy eigenfunctions exist. In this case $u \in \mathcal{N}_\infty$ is eigenfunction if $wu \in S_3L^2(\mathbb{R}^2)$, p-wave resonance if $wu \in S_2L^2(\mathbb{R}^2) \setminus S_3L^2(\mathbb{R}^2)$ and s-wave resonance if $wu \in S_1L^2(\mathbb{R}^2) \setminus S_2L^2(\mathbb{R}^2)$.*

Proof. We give a proof for readers' convenience. Suppose $u \in \mathcal{N}_\infty \setminus \{0\}$. Then, $(-\Delta)(u + N_0Vu) = 0$ and $u + N_0Vu$ is a harmonic polynomial. But, $u \in L^\infty$ implies $u + N_0Vu = O(\log|x|)$ as $|x| \to \infty$ and $u + N_0Vu = c$ for a constant. Hence, $N_0Vu(x) \in L^\infty$ and it must be that $\int Vudx = 0$ or $P(wu) = 0$. Thus, $cv = (v + vN_0V)u = T_0Q(wu)$. It follows $QT_0Q(wu) = 0$, viz $wu \in S_1L^2(\mathbb{R}^2)$ and $wu \neq 0$ because $wu = 0$ would imply $u = c$ and $w = 0$ and hence $V = 0$, which is a contradiction. Hence H is singular at zero. Moreover, $u + N_0Vu = c$ and $\int_{\mathbb{R}^2} V(x)u(x)dx = 0$ imply (5.2); from $T_0Q(wu) = cv$ we obtain (5.3).

Assume conversely that $QT_0Q\zeta = 0$ for a $\zeta \in QL^2(\mathbb{R}^2) \setminus \{0\}$. Then, $(U + vN_0v)\zeta = cv$ for a constant c and $u \stackrel{\text{def}}{=} N_0v\zeta - c$ satisfies both $U\zeta + vu = 0$ and $-\Delta u = v\zeta$. It follows $(-\Delta + V)u = 0$ and $\langle v, \zeta\rangle = 0$ implies $u \in L^\infty(\mathbb{R}^2)$. If $u = 0$, then $U\zeta = -vu = 0$ and $\zeta = 0$. It follows that $u \in \mathcal{N}_\infty \setminus \{0\}$, $\zeta = -wu$, viz. $\zeta \in \{wu \colon u \in \mathcal{N}_\infty\}$ and that $\mathcal{N}_\infty \ni u \to wu \in S_1L^2$ is an isomorphism. The first part of the lemma follows.

(1) (5.3) implies $c \neq 0$ for all $u \in \mathcal{N}_\infty$ if and only if $PT_0S_1Q \neq 0$ on $S_1L^2(\mathbb{R}^2)$, or T_1 is non-singular on $S_1L^2(\mathbb{R}^2)$.

(2) If $\zeta \in S_2L^2(\mathbb{R}^2) \setminus \{0\}$, then $\langle v, \zeta\rangle = 0$ and, since $T_2 = S_2vG_1vS_2$ is non-singular

$$\langle vG_1v\zeta, \zeta\rangle = -\sum_{j=1}^{2} \left| \int_{\mathbb{R}^2} x_jv(x)\zeta(x)dx \right|^2 > 0. \tag{5.4}$$

Moreover, $PT_0S_1\zeta = 0$ implies $c = 0$. Thus, the corresponding $u = N_0v\zeta - c\mathcal{N}_\infty$ is a p-wave resonance. (5.4) implies $1 \leq \operatorname{rank} S_2 = \operatorname{rank} T_2 \leq 2$.

(3) In this case, $u = -w\zeta$, $\zeta \in S_3L^2(\mathbb{R}^2)$ is an eigenfunction. The rest of the statement is obvious from the proof of (1) and (2). □

5.2. Preliminaries

For studying the behavior of $M(\lambda)^{-1}$ as $\lambda \to 0$, we repeatedly use the Feshbach formula and the lemma due to Jensen and Nencie ([17]) which we recall here. Let A be the operator matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

on the direct sum of Banach spaces $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$.

Lemma 5.2. *Suppose a_{11}, a_{22} are closed and a_{12}, a_{21} are bounded operators. Suppose that a_{22}^{-1} exists. Then A^{-1} exists if and only if $d = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$ exists. In this case we have*

$$A^{-1} = \begin{pmatrix} d & -da_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}d & a_{22}^{-1}a_{21}da_{12}a_{22}^{-1} + a_{22}^{-1} \end{pmatrix}. \tag{5.5}$$

LEMMA 5.3 ([17]). *Let A be a closed operator in a Hilbert space $\mathcal{X}$ and S a projection. Suppose $A+S$ has a bounded inverse. Then, A has a bounded inverse if and only if*

$$B = S - S(A+S)^{-1}S$$

has a bounded inverse in $S\mathcal{X}$. In this case,

$$A^{-1} = (A+S)^{-1} + (A+S)^{-1}SB^{-1}S(A+S)^{-1}. \tag{5.6}$$

Recall that $g(\lambda|x|) = N_0(x) + g(\lambda)$. The expansion (2.1) implies

$$\partial_\lambda^j(\mathcal{G}_\lambda(x) - g(\lambda|x|)) \le_{|\,\cdot\,|} C\partial_\lambda^j(\langle g(\lambda)\rangle\lambda^2)|x|^2\langle\log|x|\rangle, \tag{5.7}$$

$$\partial_\lambda^j\big(\mathcal{G}_\lambda(x) - g(\lambda|x|)(1 - \tfrac{1}{4}(\lambda|x|)^2) + \tfrac{1}{8\pi}(\lambda|x|)^2\big) \le_{|\,\cdot\,|} C\partial_\lambda^j(\langle g(\lambda)\rangle\lambda^4)|x|^4\langle\log|x|\rangle\,. \tag{5.8}$$

The following lemma has been proved in [11] (see also [10]) under slightly different assumptions. We set

$$g_1(\lambda) \stackrel{\text{def}}{=} g(\lambda)\|V\|_1. \tag{5.9}$$

LEMMA 5.4. (1) *Suppose $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$, $\gamma > 4$. Then, as $\lambda \to 0$,*

$$M(\lambda) = g_1(\lambda)P + T_0 + M_0(\lambda), \quad M_0(\lambda) = \mathcal{O}_2(g(\lambda)\lambda^2). \tag{5.10}$$

(2) *Suppose $\langle x\rangle^\sigma V \in L^1(\mathbb{R}^2)$ for some $\sigma > 8$. Then, as $\lambda \to 0$,*

$$M_0(\lambda) = -g(\lambda)\lambda^2 vG_1v - \lambda^2 vG_2v + \mathcal{O}_2(\lambda^4\langle\log\lambda\rangle)\,. \tag{5.11}$$

Proof. We may assume $0<\lambda<1$. (5.7) implies that for $j = 0,1,2$

$$\left(\int_{|x-y|\lambda<1} |(d/d\lambda)^j M_0(\lambda,x,y)|^2 dxdy\right)^{\frac12} \le C|g(\lambda)\lambda^{2-j}|\|\langle x\rangle^\gamma V\|_1.$$

For $\lambda|x-y| \ge 1$, (4.16) and (4.17) imply

$$\frac{d^j}{d\lambda^j}(M_0(\lambda) + v(x)g(\lambda|x-y|)v(y)) = O(\lambda^{-1/2}|x-y|^{j-1/2})v(x)v(y)$$

for $j = 0,1,2$. Since $\langle x\rangle^{-1}\langle y\rangle^{-1} \le C\lambda$, we have

$$\int_{|x-y|>\lambda^{-1}} |g(\lambda)|^2|v(x)v(y)|^2 dxdy \le C\lambda^4|g(\lambda)|^2\|\langle x\rangle^4 V\|_1^2,$$

$$\int_{|x-y|>\lambda^{-1}} |\log|x-y||^2|v(x)v(y)|^2 dxdy \le C\lambda^4|g(\lambda)|^2\|\langle x\rangle^\gamma V\|_1^2,$$

$$\int_{|x-y|>\lambda^{-1}} \lambda^{-1}|x-y|^{2j-1}|v(x)v(y)|^2 dxdy \le C\lambda^{\gamma-2j}\|\langle x\rangle^\gamma V\|_1^2.$$

These estimates yield the first statement of the lemma. Proof of (2) is similar and we omit the repetitious details. □

We often use the following trivial but important identity:

$$(1+X)^{-1} = 1 - X(1+X)^{-1} = 1 - X + X(1+X)^{-1}X. \tag{5.12}$$

5.3. The case H is regular at zero

In this section we assume QT_0Q is non-singular on $QL^2(\mathbb{R}^2)$ and prove the following theorem, which together with Theorem 4.1 implies that $W_\pm$ are bounded in $L^p(\mathbb{R}^2)$ for all $1<p<\infty$ if $V \in \langle x\rangle^{-2}L^{\frac{4}{3}}(\mathbb{R}^2) \cap \langle x\rangle^{-\gamma}L^1(\mathbb{R}^2)$ for a $\gamma>4$ and if H is regular at zero. This slightly improves the result of [39, 18] which assumes $|V(x)| \le C\langle x\rangle^{-6-\varepsilon}$, $\varepsilon>0$.

THEOREM 5.5. *Suppose that $\langle x\rangle^{\gamma}V \in L^1(\mathbb{R}^2)$ for a $\gamma>4$ and that H is regular at zero. Then, $\Omega_{\mathrm{low},2a}$ is a good operator for any $a>0$.*

Theorem 5.5 immediately follows from the next lemma and Proposition 1.4. We define $v_* = v/\|v\|_2$,

$$h(\lambda) = (g_1(\lambda)+c_1)^{-1}, \quad c_1 = \langle v_*|T_0 - T_0Q(QT_0Q)^{-1}QT_0|v_*\rangle. \tag{5.13}$$

Recall that $\mathcal{B} = \{M_m + T \colon m \in L^\infty(\mathbb{R}^2), T \in \mathcal{H}_2\}$.

LEMMA 5.6. *We have $Q(QT_0Q)^{-1}Q \in \mathcal{B}$ and there exists an operator L of rank at most two such that*

$$M(\lambda)^{-1} = h(\lambda)L + Q(QT_0Q)^{-1}Q + \mathcal{O}_2(g\lambda^2) \quad \lambda \to 0. \tag{5.14}$$

Proof. It has been proved ([31]) that $Q(QT_0Q)^{-1}Q \in \mathcal{B}$. In view of (5.10), we first show $g_1(\lambda)P + T_0$ is invertible. In the decomposition $L^2(\mathbb{R}^2) = PL^2(\mathbb{R}^2) \oplus QL^2(\mathbb{R}^2)$,

$$g_1(\lambda)P + T_0 = \begin{pmatrix} g_1(\lambda) + PT_0P & PT_0Q \\ QT_0P & QT_0Q \end{pmatrix}. \tag{5.15}$$

Here $a_{22} \overset{\text{def}}{=} QT_0Q$ is invertible by the assumption; for small $\lambda>0$,

$$g_1(\lambda)P + PT_0P - PT_0Qa_{22}^{-1}QT_0P = (g_1(\lambda)+c_1)P$$

is invertible in $PL^2(\mathbb{R}^2)$ and $(g_1(\lambda)P + PT_0P - PT_0Qa_{22}^{-1}QT_0P)^{-1} = h(\lambda)P$. Then, Lemma 5.2 implies that $(g_1(\lambda)P+T_0)^{-1}$ exists and it is equal to

$$\begin{aligned} & h(\lambda)\begin{pmatrix} P & -PT_0Qa_{22}^{-1} \\ -a_{22}^{-1}QT_0P & a_{22}^{-1}QT_0PT_0Qa_{22}^{-1} \end{pmatrix} + Qa_{22}^{-1}Q \\ & \overset{\text{def}}{=} h(\lambda)L + Q(QT_0Q)^{-1}Q. \end{aligned} \tag{5.16}$$

It is obvious that $\operatorname{rank} L \le 2$. It follows from (5.10) that

$$M(\lambda) = (1 + M_0(\lambda)(g_1(\lambda)P+T_0)^{-1})(g_1(\lambda)P+T_0)$$

and $M_0(\lambda)(g_1(\lambda)P+T_0)^{-1} = \mathcal{O}_2(g(\lambda)\lambda^2)$. Hence, for small $\lambda>0$, the series converges in $\mathcal{H}_2$ and

$$\begin{aligned} M(\lambda)^{-1} &= (g_1(\lambda)P+T_0)^{-1}(1+M_0(\lambda)(g_1(\lambda)P+T_0)^{-1})^{-1} \\ &= (g_1(\lambda)P+T_0)^{-1}+\sum_{j=1}^{\infty}(g_1(\lambda)P+T_0)^{-1}(M_0(\lambda)(g_1(\lambda)P+T_0)^{-1})^j, \end{aligned}$$

which implies (5.14) by virtue of (5.16). □

5.4. The case H is singular at zero. Threshold analysis 1

In the rest of the paper we assume that H is singular at zero: QT_0Q is singular in $QL^2(\mathbb{R}^2)$ and S_1 is the orthogonal projection in $QL^2(\mathbb{R}^2)$ onto $\mathrm{Ker}_{\mathrm{QL^2}}\mathrm{QT_0Q}$. In this subsection we assume that $\langle x\rangle^{\gamma}V \in L^1(\mathbb{R}^2)$ for $\gamma > 4$, however, for the results which we need in the following subsections we assume $\langle x\rangle^{\gamma}V \in L^1(\mathbb{R}^2)$ for a $\gamma > 8$.

LEMMA 5.7 ([17, 10]). *$\mathrm{Spec}(QT_0Q|_{QL^2(\mathbb{R}^2)})$ is discrete outside $\{-1,1\}$.*
(1) *The projection S_1 is of finite rank. Let $n = \mathrm{rank}\, S_1$.*
(2) *$D_0 = ((QT_0Q + S_1)|_{QL^2(\mathbb{R}^2)})^{-1}$ is of class $\mathcal{B}$.*

Proof. (1) is proved in [17]. Schlag([31]) proves $D_0 \in \mathcal{B}$. □

DEFINITION 5.8. We take and fix an orthonormal basis $\{\zeta_1,\dots,\zeta_n\}$ of $S_1L^2(\mathbb{R}^2)$. We regard the projection $S_1 = \zeta_1\otimes\zeta_1+\cdots+\zeta_n\otimes\zeta_n$ also as an orthogonal projection in $L^2(\mathbb{R}^2)$, viz. we identify QS_1Q and S_1.

We study $M(\lambda)^{-1}$ as $\lambda \to 0$ by applying Lemma 5.3 to the pair $(M(\lambda), S_1)$. We first study

$$M(\lambda)+S_1 = g_1(\lambda)P+T_0+S_1+M_0(\lambda). \tag{5.17}$$

Define a scalar function $h_1 \in \mathcal{M}(\mathbb{R}^2)$ and the operator matrix L_1 in the decomposition $L^2(\mathbb{R}^2) = PL^2(\mathbb{R}^2)\oplus QL^2(\mathbb{R}^2)$ by

$$h_1(\lambda) = (g_1(\lambda)+c_2)^{-1}, \quad c_2{=}\langle v_*|T_0 - T_0QD_0QT_0|v_*\rangle\,, \tag{5.18}$$

$$L_1 = \begin{pmatrix} P & -PT_0QD_0 \\ -D_0QT_0P & D_0QT_0PT_0QD_0 \end{pmatrix}. \tag{5.19}$$

L_1 is λ-independent and $\mathrm{rank}\, L_1 \le 2$.

LEMMA 5.9. *$g_1(\lambda)P+T_0+S_1$ is invertible in $L^2(\mathbb{R}^2)$ and*

$$(g_1(\lambda)P+T_0+S_1)^{-1} = h_1(\lambda)L_1 + QD_0Q. \tag{5.20}$$

We denote $N(\lambda) = (g_1(\lambda)P+T_0+S_1)^{-1}$. $vN(\lambda)v$ is a good producer.

Proof. We use Lemma 5.2. In the decomposition $L^2(\mathbb{R}^2) = PL^2\oplus QL^2$,

$$g_1(\lambda)P+T_0+S_1 = \begin{pmatrix} g_1(\lambda)P+PT_0P & PT_0Q \\ QT_0P & QT_0Q+S_1 \end{pmatrix}. \tag{5.21}$$

By virtue of Lemma 5.7, $D_0 = (QT_0Q + S_1)^{-1}$ exists and $D_0 \in \mathcal{B}$. It is obvious that

$$g_1(\lambda)P + PT_0P - PT_0QD_0QT_0P = (g_1(\lambda) + c_2)P$$

has the inverse $h_1(\lambda)P$. It follows that (5.21) is invertible for small $\lambda > 0$ and (5.5) implies (5.20). □

We define

$$\mathcal{R}_1(\lambda) \stackrel{\text{def}}{=} v(G_1 + g(\lambda)^{-1}G_2)v. \tag{5.22}$$

LEMMA 5.10. *$M(\lambda) + S_1$ is invertible for small $\lambda > 0$ and*

$$(M(\lambda) + S_1)^{-1} = N(\lambda) + \mathcal{O}_2(g(\lambda)\lambda^2), \quad \lambda \to 0. \tag{5.23}$$

We denote $\mathcal{A}_0(\lambda) = (M(\lambda) + S_1)^{-1}$. $v\mathcal{A}_0(\lambda)v$ is a good producer. If $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$ for a $\gamma > 8$, then (5.23) *is improved and as $\lambda \to 0$*

$$\mathcal{A}_0(\lambda) = N(\lambda) + \lambda^2 g(\lambda)N(\lambda)\mathcal{R}_1(\lambda)N(\lambda) + \mathcal{O}_2(\lambda^4 g(\lambda)^2). \tag{5.24}$$

Proof. We have $M(\lambda) + S_1 = (1 + M_0(\lambda)N(\lambda))(g_1(\lambda)P + T_0 + S_1)$ by virtue of Lemma 5.9. (5.10) and (5.20) then imply that it is invertible for small $\lambda > 0$ and

$$\begin{aligned} \mathcal{A}_0(\lambda) = (M(\lambda) + S_1)^{-1} = N(\lambda) - N(\lambda)M_0(\lambda)N(\lambda) \\ + N(\lambda)M_0(\lambda)N(\lambda)(1 + M_0(\lambda)N(\lambda))^{-1}M_0(\lambda)N(\lambda). \end{aligned} \tag{5.25}$$

This implies (5.23) and the second line of (5.25) is of $\mathcal{O}_2(\lambda^4 g(\lambda)^2)$. In particular, $v\mathcal{A}_0(\lambda)v$ is a good producer by virtiue of Proposition 1.4. If $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$ for a $\gamma > 8$, then (5.11) implies $M_0(\lambda) = -g(\lambda)\lambda^2\mathcal{R}_1(\lambda) + \mathcal{O}_2(g(\lambda)\lambda^4)$ and (5.24) follows from (5.25). □

We define $B_1(\lambda)$ on $S_1L^2(\mathbb{R}^2)$ by

$$B_1(\lambda) = S_1 - S_1\mathcal{A}_0(\lambda)S_1, \quad T_1 = S_1QT_0PT_0QS_1. \tag{5.26}$$

LEMMA 5.11. *On $S_1L^2(\mathbb{R}^2)$, we have as $\lambda \to 0$*

$$B_1(\lambda) = -h_1(\lambda)(T_1 - \lambda^2 X(\lambda)), \quad X(\lambda) \in \mathcal{O}_2(g(\lambda)^2). \tag{5.27}$$

If $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$ for a $\gamma > 8$, then in the right of (5.27)

$$\begin{aligned} X(\lambda) = -S_1h_1(\lambda)^{-1}g(\lambda)\big(\mathcal{R}_1(\lambda) + h_1(\lambda)(L_1\mathcal{R}_1(\lambda) + \mathcal{R}_1(\lambda)L_1) \\ + h_1(\lambda)^2 L_1\mathcal{R}_1(\lambda)L_1 + \mathcal{O}_2(g(\lambda)\lambda^2)\big)S_1. \end{aligned} \tag{5.28}$$

Proof. We have $S_1N(\lambda)S_1 = S_1 + T_1$ since $S_1QD_0QS_1 = S_1$ and

$$S_1L_1S_1 = \begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} P & -PT_0QD_0 \\ -D_0QT_0P & D_0QT_0PT_0QD_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix} = T_1.$$

Then, (5.20) and (5.23) imply (5.27). If $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$, we obtain (5.28) by using also (5.24) . □

If $B_1(\lambda)^{-1}$ exists in $S_1L^2(\mathbb{R}^2)$, then Lemma 5.3 implies that

$$M(\lambda)^{-1} = \mathcal{A}_0(\lambda) + \mathcal{A}_1(\lambda),\ \mathcal{A}_1(\lambda) \stackrel{\text{def}}{=} \mathcal{A}_0(\lambda)S_1B_1(\lambda)^{-1}S_1\mathcal{A}_0(\lambda). \tag{5.29}$$

5.5. The case H has singularities of the first kind at zero

THEOREM 5.12. *Suppose $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$ for a $\gamma > 4$ and H has singularities of the first kind at zero. Then, $\Omega_{\text{low},2a}$ is a good operator.*

Theorem 5.12 together with Theorem 4.1 proves that $W_\pm$ are good operators if $V \in \langle x\rangle^{-2}L^{\frac{4}{3}}(\mathbb{R}^2)\cap\langle x\rangle^{-\gamma}V \in L^1(\mathbb{R}^2)$ for a $\gamma > 4$ and if H has singularities of the first kind at zero. This slightly improves Theorem 1.1 (i) of [10] which assumes $|V(x)| \le C\langle x\rangle^{-6-\varepsilon}$, $\varepsilon > 0$.

Proof. We prove $v\mathcal{A}_1(\lambda)v$ is a good producer for small $\lambda > 0$. Then, Lemma 5.10 and (5.29) imply the same for $vM(\lambda)^{-1}v$ and Theorem 5.12 follows. We have $\operatorname{rank} S_1 = 1$ and $S_1 = \zeta\otimes\zeta$ for a normalized $\zeta \in S_1L^2(\mathbb{R}^2)$, $T_1 = c_3\zeta\otimes\zeta$ with $c_3 = \|PT_0\zeta\|^2 > 0$ and

$$B_1(\lambda)^{-1} = h_2(\lambda)^{-1}(\zeta\otimes\zeta),\quad h_2(\lambda) = -c_3h_1(\lambda)(1 + \mathcal{O}_2(g(\lambda)^2\lambda^2)) \tag{5.30}$$

from (5.27). Then, (5.23) and (5.20) imply that modulo $\mathcal{O}_2(g^2(\lambda)\lambda^2)$,

$$\mathcal{A}_1(\lambda) \equiv N(\lambda)S_1B_1(\lambda)^{-1}S_1N(\lambda) = -c_3^{-1}h_1(\lambda)^{-1}N(\lambda)(\zeta\otimes\zeta) + \mathcal{O}_2(g(\lambda)^3\lambda^2).$$

Since $QD_0Q\zeta = \zeta$, this simplifies to

$$\mathcal{A}_1(\lambda) \equiv -c_3^{-1}(h_1(\lambda)^{-1}\zeta\otimes\zeta + (L_1\zeta\otimes\zeta + \zeta\otimes L_1\zeta) + h_1(\lambda)(L_1\zeta)\otimes(L_1\zeta))).$$

Recall that $v(x)\zeta(x) \in L^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} v(x)\zeta(x)dx = 0$. Thus, $v\mathcal{A}_1(\lambda)v$ is a good producer by virtue of Proposition 1.4 (4) and (7). □

5.6. The case H has singularities of the second kind at zero

THEOREM 5.13. *Suppose that $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$ for a $\gamma > 8$ and that H has singularities of the second kind at zero. Then, for sufficently small $a > 0$, $\Omega_{\text{low},2a}$ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p \le 2$ but is unbounded in $L^p(\mathbb{R}^2)$ for $2 < p < \infty$.*

We assume in the rest of the paper $\langle x\rangle^\gamma V \in L^1(\mathbb{R}^2)$ for a $\gamma > 8$. The proof of Theorem 5.13 is long and is given by a series of lemmas. It is well known that $W_\pm$ are isometries of $L^2(\mathbb{R}^2)$ and we assume $p \ne 2$. Recall that (5.4) implies $1 \le \operatorname{rank} T_2 = \operatorname{rank} S_2 \le 2$ and T_2 is negative. In what follows we assume $\operatorname{rank} S_2 = 2$. Modification for the case $\operatorname{rank} S_2 = 1$ is obvious. Then, $n = \operatorname{rank} S_1$ is two or three depending on the absence or the presence of s-wave resonances. We take the orthonormal basis $\{\zeta_1,\dots,\zeta_n\}$ of $S_1L^2(\mathbb{R}^2)$ such that $\zeta_1,\zeta_2 \in S_2L^2(\mathbb{R}^2)$ are (real) eigenfunctions of T_2: $T_2\zeta_j = -\kappa_j^2\zeta_j$ for $\kappa_j > 0$, $j = 1, 2$.

5.6.1. Threshold analysis 2

We study $M(\lambda)^{-1}$ as $\lambda \to 0$ when T_1 is singular in $S_1L^2(\mathbb{R}^2)$ but $T_2 = S_2(vG_1v)S_2|_{S_2L^2(\mathbb{R}^2)}$ is non-singular. We consider S_1 and S_2 are projections also in $L^2(\mathbb{R}^2)$ as previously. Recall that we are omitting "for small $\lambda > 0$".

Lemma 5.14. *The projection S_2 annihilates T_0 and L_1:*

$$T_0 S_2 = S_2 T_0 = 0, \quad S_2 L_1 = L_1 S_2 = 0. \tag{5.31}$$

Proof. Since $QT_0QS_1 = 0$, we have $PT_0QS_1 = T_0QS_1$ and $T_1 = (T_0QS_1)^*(T_0QS_1)$. It follows that and $\mathrm{Ker}_{S_1L^2(\mathbb{R}^2)}T_1 = \mathrm{Ker}_{S_1L^2(\mathbb{R}^2)}T_0QS_1$. Thus, $T_0QS_1S_2 = 0$ or $T_0S_2 = 0$ and $S_2T_0 = 0$ by the duality. We have $PS_2 = PQS_2 = 0$ and likewise $S_2P = 0$; $D_0S_2 = D_0S_1S_2 = S_2$ and $S_2D_0 = S_2$. It follows

$$S_2L_1 = S_2(P - PT_0QD_0 - D_0QT_0P + D_0QT_0PT_0QD_0) = 0$$

and $L_1S_2 = (S_2L)^* = 0$. □

Recall that $B_1(\lambda) = -h_1(\lambda)(T_1 - \lambda^2 X(\lambda))$. We define for simplicity

$$A_1(\lambda) = T_1 - \lambda^2 X(\lambda) \ \text{ so that } \ B_1(\lambda) = -h_1(\lambda)A_1(\lambda)\,. \tag{5.32}$$

We study $A_1(\lambda)^{-1}$ via Lemma 5.3. Define

$$\tilde{\mathcal{R}}_1(\lambda) = S_2\mathcal{R}_1(\lambda)S_2 = S_2v(G_1 + g(\lambda)^{-1}G_2)vS_2. \tag{5.33}$$

and let $C(\lambda) = (c_{jk}(\lambda))$ be the representation matrix of $\tilde{\mathcal{R}}_1(\lambda)$ with respect to the basis $\{\zeta_1, \zeta_2\}$. We have

$$c_{jk}(\lambda) = -\kappa_j^2\delta_{jk} + g(\lambda)^{-1}\langle G_2v\zeta_j, v\zeta_k\rangle \quad j,k = 1,2. \tag{5.34}$$

$C(\lambda)$ is clearly invertible. If we write $D(\lambda) = (d_{jk}(\lambda))$ for $C(\lambda)^{-1}$,

$$S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2 = \sum_{j,k=1}^{2} d_{jk}(\lambda)\zeta_j \otimes \zeta_k. \ \text{ and } \ d_{jk}(\lambda)\chi_{\le 2a}(\lambda) \in \mathcal{M}(\mathbb{R}^2)\,,\ j,k = 1,2\,. \tag{5.35}$$

for small $a > 0$. $T_1 + S_2$ is clearly invertible in $S_1L^2(\mathbb{R}^2)$.

Lemma 5.15. (1) *$A_1(\lambda) + S_2$ on $S_1L^2(\mathbb{R}^2)$ is invertible and*

$$(A_1(\lambda) + S_2)^{-1} = (T_1 + S_2)^{-1} + \lambda^2(T_1 + S_2)^{-1}X(\lambda)(T_1 + S_2)^{-1} + \mathcal{O}_2(g(\lambda)^4\lambda^4). \tag{5.36}$$

With λ-independent operators $F_j^{(k)}$ on $S_1L^2(\mathbb{R}^2)$ and modulo $\mathcal{O}_2(g(\lambda)^3\lambda^4)$

$$(A_1(\lambda) + S_2)^{-1}S_2 \equiv S_2 + \lambda^2 h_1(\lambda)^{-1}g(\lambda)(F_0^{(1)} + h_1(\lambda)F_1^{(1)}), \tag{5.37}$$

$$S_2(A_1(\lambda) + S_2)^{-1} \equiv S_2 + \lambda^2 h_1(\lambda)^{-1}g(\lambda)(F_0^{(2)} + h_1(\lambda)F_1^{(2)}), \tag{5.38}$$

$$S_2(A_1(\lambda) + S_2)^{-1}S_2 \equiv S_2 - \lambda^2 h_1(\lambda)^{-1}g(\lambda)\tilde{\mathcal{R}}_1(\lambda)\,. \tag{5.39}$$

(2) *Define $B_2(\lambda) = S_2 - S_2(A_1(\lambda) + S_2)^{-1}S_2$ on $S_2L^2(\mathbb{R}^2)$. Then, $B_2(\lambda)$ is invertible and, as $\lambda \to 0$,*

$$B_2(\lambda)^{-1} = \lambda^{-2}h_1(\lambda)g(\lambda)^{-1}\tilde{\mathcal{R}}_1(\lambda)^{-1} + S_2\mathcal{O}_2(1)S_2 + S_2\mathcal{O}_2(g(\lambda)^2\lambda^2)S_2. \tag{5.40}$$

Proof. (1) Since $A_1(\lambda) + S_2 = (T_1 + S_2)(1 - \lambda^2(T + S_2)^{-1}X(\lambda))$ and $X(\lambda) \in \mathcal{O}_2(g(\lambda)^2)$,

$(A_1(\lambda)+S_2)^{-1}$ exists and (5.36) is satisfied. Since $S_2(T_1+S_2)^{-1}=(T_1+S_2)^{-1}S_2=S_2$ and $S_2L_1=L_1S_2=0$, the second half of (1) follows from (5.36), (5.31) and (5.28).

(2) By virtue of (5.39), we have as $\lambda\to 0$,

$$\begin{aligned} B_2(\lambda)&=\lambda^2 h_1(\lambda)^{-1}g(\lambda)\{\tilde{\mathcal{R}}_1(\lambda)+S_2\mathcal{O}_2(g(\lambda)^2\lambda^2)S_2\} \\ &=\lambda^2 h_1(\lambda)^{-1}g(\lambda)(1+S_2\mathcal{O}_2(g(\lambda)^2\lambda^2)S_2\tilde{\mathcal{R}}_1(\lambda)^{-1})\tilde{\mathcal{R}}_1(\lambda)\,. \end{aligned} \tag{5.41}$$

Thus, $B_2(\lambda)$ is invertible and (5.12) implies (5.40). □

Remark 5.16. (a) $\tilde{\mathcal{R}}_1(\lambda)^{-1}=\mathcal{O}_2(1)$ *is not assumed in* Lemma 5.15 (1).

(b) *If* $\tilde{\mathcal{R}}_1(\lambda)^{-1}=\mathcal{O}_2(g(\lambda))$ *as in* Lemma 5.28 *of §5.7,* (5.40) *remains to holds if the last two terms are replaced respectively by* $S_2\mathcal{O}_2(g(\lambda)^2)$ *and* $\mathcal{O}_2(g(\lambda)^5\lambda^2)S_2$

Proposition 5.17. *Modulo a good producer we have that*

$$vM(\lambda)^{-1}v\equiv -g(\lambda)^{-1}\lambda^{-2}vS_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2v\,. \tag{5.42}$$

Proof. Since $v\mathcal{A}_0(\lambda)v$ is a good producer, it suffices to show (5.42) with $\mathcal{A}_1(\lambda)$ replacing $M(\lambda)^{-1}$. Recall $\mathcal{A}_1(\lambda)=\mathcal{A}_0(\lambda)S_1B_1(\lambda)^{-1}S_1\mathcal{A}_0(\lambda)$. Lemmas 5.3 and 5.15 imply

$$A_1(\lambda)^{-1}=(A_1(\lambda)+S_2)^{-1}+(A_1(\lambda)+S_2)^{-1}B_2(\lambda)^{-1}(A_1(\lambda)+S_2)^{-1}.$$

and substituting $-h_1(\lambda)^{-1}A_1(\lambda)^{-1}$ for $B_1(\lambda)^{-1}$ yields $\mathcal{A}_1(\lambda)=-(\mathcal{A}_{11}(\lambda)+\mathcal{A}_{12}(\lambda))$:

$$\mathcal{A}_{11}(\lambda)=h_1(\lambda)^{-1}\mathcal{A}_0(\lambda)S_1(A_1(\lambda)+S_2)^{-1}S_1\mathcal{A}_0(\lambda), \tag{5.43}$$

$$\mathcal{A}_{12}(\lambda)=h_1(\lambda)^{-1}\mathcal{A}_0(\lambda)S_1(A_1(\lambda)+S_2)^{-1}S_2B_2(\lambda)^{-1}S_2(A_1(\lambda)+S_2)^{-1}S_1\mathcal{A}_0(\lambda). \tag{5.44}$$

We first show that $v\mathcal{A}_{11}(\lambda)v$ is a good producer. The following lemma also does not assume $\tilde{\mathcal{R}}_1(\lambda)^{-1}=\mathcal{O}_2(1)$. Recall $N(\lambda)=h_1(\lambda)L_1+QD_0Q$ (see (5.20)).

Lemma 5.18. (1) *We have that*

$$\begin{cases}\mathcal{A}_0(\lambda)S_1=S_1+h_1(\lambda)L_1S_1+\mathcal{O}_2(\lambda^2g(\lambda))S_1,\\ S_1\mathcal{A}_0(\lambda)=S_1+h_1(\lambda)S_1L_1+S_1\mathcal{O}_2(\lambda^2g(\lambda))\end{cases} \tag{5.45}$$

and

$$\begin{cases}\mathcal{A}_0(\lambda)S_2=S_2+g(\lambda)\lambda^2N(\lambda)\mathcal{R}_1(\lambda)S_2+\mathcal{O}_2(\lambda^4g(\lambda)^2),\\ S_2\mathcal{A}_0(\lambda)=S_2+g(\lambda)\lambda^2S_2\mathcal{R}_1(\lambda)N(\lambda)+\mathcal{O}_2(\lambda^4g(\lambda)^2).\end{cases} \tag{5.46}$$

(2) $v\mathcal{A}_{11}(\lambda)v$ *is a good producer.*

Proof. (1) If we multiply (5.23) by S_1 from the right or the left and use that $S_1QD_0Q=QD_0QS_1=S_1$, (5.45) follows. If we multiply (5.24) by S_2 from the right or from the left and apply (5.31), we obtain (5.46).
(2) Since $X(\lambda)=\mathcal{O}_2(g(\lambda)^2)$, we have from (5.36) that

$$S_1(A_1(\lambda)+S_2)^{-1}S_1=S_1(T_1+S_2)^{-1}S_1+\mathcal{O}_2(g(\lambda)^2\lambda^2).$$

Then, (5.43) and (5.45) imply that modulo $\mathcal{O}_2(g(\lambda)^3\lambda^2)$

$$\begin{aligned}\mathcal{A}_{11}(\lambda) &\equiv h_1(\lambda)^{-1}(S_1+h_1(\lambda)L_1S_1)(T_1+S_2)^{-1}(S_1+h_1(\lambda)S_1L_1)\\ &= h_1(\lambda)^{-1}S_1(T_1+S_2)^{-1}S_1+F_3+h_1(\lambda)F_4 \end{aligned} \tag{5.47}$$

with λ-independent finite rank operators F_3 and F_4 and, Proposition 1.4 implies $v\mathcal{A}_{11}v$ is a good producer. Note that $S_1(T_1+S_2)^{-1}S_1=\sum_{j,k=1}^n a_{jk}(\zeta_j\otimes\zeta_k)$ with $\zeta_1,\dots,\zeta_n\in S_1L^2(\mathbb{R}^2)$, hence $v\zeta_j\in L^1(\mathbb{R}^2)$, $\langle\zeta_j,v\rangle=0$ for $j=1,\dots,n$. □

Plugging (5.40) and (5.44) implies $\mathcal{A}_{12}(\lambda)\equiv\mathcal{A}_{12}^{(1)}(\lambda)+\mathcal{A}_{12}^{(2)}(\lambda)$ modulo $\mathcal{O}_2(g(\lambda)^3\lambda^2)$:

$$\begin{aligned}\mathcal{A}_{12}^{(1)}(\lambda) &= g(\lambda)^{-1}\lambda^{-2}\mathcal{A}_0(\lambda)S_1(A_1(\lambda)+S_2)^{-1}S_2\\ &\quad\times\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2(A_1(\lambda)+S_2)^{-1}S_1\mathcal{A}_0(\lambda),\end{aligned} \tag{5.48}$$

$$\begin{aligned}\mathcal{A}_{12}^{(2)}(\lambda) &= \mathcal{A}_0(\lambda)S_1(A_1(\lambda)+S_2)^{-1}S_2\mathcal{O}_2(g(\lambda))S_2\\ &\quad\times(A_1(\lambda)+S_2)^{-1}S_1\mathcal{A}_0(\lambda)+\mathcal{O}_2(g(\lambda)^3\lambda^2).\end{aligned} \tag{5.49}$$

We first show

Lemma 5.19. *$v\mathcal{A}_{12}^{(2)}(\lambda)v$ is a good producer for small $\lambda>0$.*

Proof. $\mathcal{A}_{12}^{(2)}(\lambda)=\mathcal{A}_0(\lambda)S_2\mathcal{O}_2(g(\lambda))S_2\mathcal{A}_0(\lambda)+\mathcal{O}_2(\lambda^2g(\lambda)^3)$ by virtue of (5.37) and (5.38) and, further applying (5.45) and (5.31), we obtain that $\mathcal{A}_{12}^{(2)}(\lambda)=S_2\mathcal{O}_2(g(\lambda))S_2+\mathcal{O}_2(\lambda^2g(\lambda)^3)$. In the basis $\{\zeta_1,\zeta_2\}$ of $S_2L^2(\mathbb{R}^2)$, we have that

$$S_2\mathcal{O}_2(g(\lambda))S_2=g(\lambda)\sum b_{jk}(\lambda)\zeta_j\otimes\zeta_k,\quad b_{jk}(\lambda)\in\mathcal{M}(\mathbb{R}^2). \tag{5.50}$$

The lemma follows from Proposition 1.4 (5) and (7). □

Remark 5.20. Lemma 5.19 *holds if $\tilde{\mathcal{R}}_1(\lambda)=\mathcal{O}_2(g(\lambda))$ as in* Lemma 5.28 *because this only changes $d_{jk}(\lambda)$ in* (5.35) *by $g(\lambda)\tilde{d}_{jk}(\lambda)$ with another $\tilde{d}_{jk}(\lambda)\in\mathcal{M}(\mathbb{R}^2)$ (see* (5.81) *below) and, by virtue of* Remark 5.16, *$\mathcal{O}_2(g(\lambda))$ and $\mathcal{O}_2(\lambda^2g(\lambda)^3)$ in the proof above to $\mathcal{O}_2(g(\lambda)^3)$ and $\mathcal{O}_2(\lambda^2g(\lambda)^5)$ respectively and because the power k of $g(\lambda)^k$ is arbitrary in* Proposition 1.4 (7).

Completion of the proof of Proposition 5.17. We have only to show that $v\mathcal{A}_{12}^{(1)}(\lambda)v\equiv-g(\lambda)^{-1}\lambda^{-2}vS_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2v$ modulo a good producer. Define

$$\mathcal{P}(\lambda)=\lambda^{-2}g(\lambda)^{-1}(A_1(\lambda)+S_2)^{-1}S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2(A_1(\lambda)+S_2)^{-1} \tag{5.51}$$

so that

$$\mathcal{A}_{12}^{(1)}(\lambda)=\mathcal{A}_0(\lambda)S_1\mathcal{P}(\lambda)S_1\mathcal{A}_0(\lambda). \tag{5.52}$$

We substitute $(A_1(\lambda)+S_2)^{-1}S_2$ and $S_2(A_1(\lambda)+S_2)^{-1}$ by right sides of (5.37) and (5.38) respectively and expand the sum. Then,

$$\begin{aligned}\mathcal{P}(\lambda) &= \lambda^{-2}g(\lambda)^{-1}S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2+h_1(\lambda)^{-1}L_3(\lambda)+\mathcal{O}_2(g(\lambda)^3\lambda^2),\\ L_3(\lambda) &= (F_0^{(1)}+h_1(\lambda)F_1^{(1)})\tilde{\mathcal{R}}_1(\lambda)^{-1}+\tilde{\mathcal{R}}_1(\lambda)^{-1}(F_0^{(2)}+h_1(\lambda)F_k^{(2)}),\end{aligned}$$

which we insert for $\mathcal{P}(\lambda)$ of (5.52). This produces three operators for $\mathcal{A}_{12}^{(1)}(\lambda)$. After sandwitched by v, the one produced by $\mathcal{O}_2(g(\lambda)^3\lambda^2)$ is a good producer by Proposition 1.4 (5). The definition of $L_3(\lambda)$ and (5.45) imply

$$\mathcal{A}_0(\lambda)S_1(h^{-1}L_3(\lambda))S_1\mathcal{A}_0(\lambda) = h^{-1}L_3(\lambda) + L_1S_1L_3(\lambda) + L_3(\lambda)S_1L_1 + \mathcal{O}_2(g(\lambda)^2\lambda^2). \tag{5.53}$$

Again $v\mathcal{O}_2(g(\lambda)^2\lambda^2)v$ is a good producer and the same is true for first three terms by virtue of Proposition 1.4. This is because (5.35) implies $L_3(\lambda) = \sum_{j,k=1}^2 b_{jk}(\lambda)\zeta_j \otimes \zeta_k$ with good multipliers $b_{jk}(\lambda)$ and they are the sum of the operators which appear in Proposition 1.4 (7). By virtue of (5.46) the one produced by $g(\lambda)^{-1}\lambda^{-2}S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2$ may be expressed in the form

$$g(\lambda)^{-1}\lambda^{-2}S_2\tilde{\mathcal{R}}_1(\lambda)S_2 + N(\lambda)\mathcal{R}_1(\lambda)S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2 + S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2\mathcal{R}_1(\lambda)N(\lambda) + \mathcal{O}_2(g(\lambda)\lambda^2). \tag{5.54}$$

Here the last three terms are good producers by the same reason as above and we obtain $vM(\lambda)v \equiv g(\lambda)^{-1}\lambda^{-2}vS_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2v$. This proves the proposition □

Remark 5.21. Proposition 5.17 *holds if* $\tilde{\mathcal{R}}_1(\lambda) = \mathcal{O}_2(g(\lambda))$ *as in* Lemma 5.28 *by the same reason as in* Remark 5.20.

5.6.2. Good and bad parts.

By virtue of Proposition 5.17 the proof of Theorem 5.13 is reduced to proving that

$$\tilde{\Omega}_{\mathrm{low},2a}u = -\int_0^\infty g(\lambda)^{-1}\lambda^{-2}G_0(-\lambda)vS_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2v\Pi(\lambda)u\lambda\chi_{\le 2a}(\lambda)d\lambda \tag{5.55}$$

defined by (1.34) is bounded in $L^p(\mathbb{R}^2)$ for $1 < p \le 2$ and unbounded for $2 < p < \infty$. Substituting (5.35) for $S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2$ shows that $\tilde{\Omega}_{\mathrm{low},2a}u$ is the sum over $j,k = 1,2$ of

$$\tilde{\Omega}_{\mathrm{low},2a}^{(j,k)}u = -\int_0^\infty g(\lambda)^{-1}\lambda^{-2}d_{jk}(\lambda)G_0(-\lambda)|v\zeta_j\rangle\langle\zeta_k v, \Pi(\lambda)u\rangle\lambda\chi_{\le 2a}(\lambda)d\lambda\,. \tag{5.56}$$

Recall that for $u \in \mathcal{D}_*$, $\Pi(\lambda)u = 0$ for $\lambda \notin (\alpha,\beta)$ for $0 < \alpha < \beta < \infty$

Since $\langle v, \zeta_j\rangle = 0$, $j = 1,2$, we may replace $\Pi(\lambda)u(z)$ by $\Pi(\lambda)u(z) - \Pi(\lambda)u(0)$ which we decompose into the sum of the good part (1.35) and the bad part (1.36):

$$\Pi(\lambda)u(z) - \Pi(\lambda)u(0) = \tilde{g}(\lambda,z) + \tilde{b}(\lambda,z). \tag{5.57}$$

Define $\tilde{\Omega}_{(g)}^{(j,k)}$ and $\tilde{\Omega}_{(b)}^{(j,k)}$ by (5.56) by replacing $\Pi(\lambda)u(z)$ by $\tilde{g}(\lambda,z)$ and $\tilde{b}(\lambda,z)$ respectively and

$$\tilde{\Omega}_{(g)}u(x) = \sum_{j,k=1}^2 \tilde{\Omega}_{(g)}^{(j,k)}u(x), \quad \tilde{\Omega}_{(b)}u(x) = \sum_{j,k=1}^2 \tilde{\Omega}_{(b)}^{(j,k)}u(x) \tag{5.58}$$

so that $\tilde{\Omega}_{\text{low},2a}u(x) = \tilde{\Omega}_{(g)}u(x) + \tilde{\Omega}_{(b)}u(x)$. We recall

$$\tilde{g}(\lambda, z) = -\sum_{l,m=1}^{2} z_l z_m \lambda^2 \int_0^1 (1-\theta)\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} \mathcal{F}(\tau_{-\theta z}R_l R_m u)(\lambda\omega)d\omega\right) d\theta.$$

$$\tilde{b}(\lambda, z) = \frac{i\lambda}{2\pi}\int_{\mathbb{S}^1}(z\omega)\hat{u}(\lambda\omega)d\omega = \sum_{l=1}^{2}\frac{i\lambda z_l}{2\pi}\int_{\mathbb{S}^1}(\mathcal{F}R_l u)(\lambda\omega)d\omega\,. \tag{5.59}$$

PROPOSITION 5.22. *Te good part $\tilde{\Omega}_{(g)}u(x)$ is a good operator.*

Proof. Define $\mu_{jk}(\lambda) = d_{jk}(\lambda)g(\lambda)^{-1}\chi_{\le 2a}(\lambda)$, $j,k = 1,2$. $\mu_{jk}(\lambda)$ is a good multiplier. Then, after changing the order of integrations, we can express $\tilde{\Omega}^{(j,k)}_{(g)}u(x)$ as

$$\tilde{\Omega}^{(j,k)}_{(g)}u(x) = \sum_{l,m=1}^{2}\int_0^1 (1-\theta)d\theta\int_{\mathbb{R}^4} dydz\, v(y)\zeta_j(y)v(z)\zeta_k(z)z_l z_m$$
$$\times\left\{\int_0^\infty \mu_{jk}(\lambda)\mathcal{G}_{-\lambda}(x-y)\left(\frac{1}{2\pi}\int_{\mathbb{S}^1}\mathcal{F}(\tau_{-\theta z}R_l R_m u)(\lambda\omega)d\omega\right)\lambda d\lambda\right\}.$$

The identity (1.16) and the definition (1.19) of K imply that the second line is equal to

$$\begin{aligned}&\tau_y\int_0^\infty \mathcal{G}_{-\lambda}(x)\left(\frac{1}{2\pi}\int_{\mathbb{S}^1}\mathcal{F}(\mu_{jk}(|D|)\tau_{-\theta z}R_l R_m u)(\lambda\omega)d\omega\right)\lambda d\lambda\\ &\qquad = (\tau_y K\mu_{jk}(|D|)\tau_{-\theta z}R_l R_m u)(x).\end{aligned} \tag{5.60}$$

Since $\|\tau_y K\mu_{jk}(|D|)\tau_{-\theta z}R_l R_m u\|_p \le C\|u\|_p$ with C independent of $y, z \in \mathbb{R}^2$, $0<\theta<1$ and $u \in \mathcal{D}_*$, Minkowki's inequality implies

$$\|\tilde{\Omega}^{(j,k)}_{(g)}u\|_p \le C_p\left(\sum \|z_l z_m v\zeta_k\|_1\|v\zeta_j\|_1\right)\|u\|_p. \tag{5.61}$$

Since p-wave resonances ζ_j satisfies $\langle x\rangle^{-\delta}\zeta_j \in L^2(\mathbb{R}^2)$ for any $0<\delta<1$, Schwarz inequality implies $\|\langle z\rangle^2\zeta_j v\|_1^2 \le C\|\langle x\rangle^{4+\delta}V\|_1$. Thus, $\tilde{\Omega}_{(g)}$ is a good operator. □

5.6.3. Estimate of bad part 1, Positive result

The following proposition completes the proof of the positive part of Theorem 5.13.

PROPOSITION 5.23. *The operator $\tilde{\Omega}_{(b)}$ is bounded in $L^p(\mathbb{R}^2)$ for $1<p\le 2$.*

Proof. Let $\mu_{jk}(\lambda) = g(\lambda)^{-1}d_{jk}(\lambda)\chi_{\le 2a}(\lambda)$ for $j,k=1,2$ as previously. We have

$$\tilde{\Omega}^{(j,k)}_{(b)}u(x) = -\int_0^\infty \lambda^{-1}\mu_{jk}(\lambda)G_0(-\lambda)(v\zeta_j)(x)\langle\zeta_k, \tilde{b}(\lambda)\rangle d\lambda. \tag{5.62}$$

We have $\tilde{\Omega}^{(j,k)}_{(b)}u = \chi_{>4a}(|D|)\tilde{\Omega}^{(j,k)}_{(b)}u + \chi_{\le 4a}(|D|)\tilde{\Omega}^{(j,k)}_{(b)}u$ and the following two lemmas prove the proposition. □

LEMMA 5.24. *For $1<p\le 2$, $\chi_{>4a}(|D|)\tilde{\Omega}_{(b)} \in \mathbf{B}(L^p(\mathbb{R}^2))$.*

Proof. For $j,k,l=1,2$, define $X_{jkl}u(x)$ by

$$\int_0^\infty \chi_{>4a}(|D|)G_0(-\lambda)(v\zeta_j)(x)\left(\int_{\mathbb{S}^1}(\mathcal{F}\mu_{jk}(|D|)R_l u)(\lambda\omega)d\omega\right)d\lambda. \tag{5.63}$$

Substituting (5.59) for $\tilde b(\lambda,z)$ in (5.62), we obtain

$$\chi_{>4a}(|D|)\tilde\Omega_{(b)}^{(j,k)}u(x) = -\sum_{l=1}^2 \frac{i}{2\pi}\langle z_l v,\zeta_k\rangle X_{jkl}u(x) \tag{5.64}$$

and we prove $X_{jkl}u(x)$, $j,k,l=1,2$ are bounded in $L^p(\mathbb{R}^2)$ for $1<p\le 2$. Let $\mu(\xi)=\chi_{>4a}(\xi)|\xi|^{-2}$ as in the proof of Lemma 3.7. Then, we have (see (3.23))

$$\chi_{>4a}(|D|)G_0(-\lambda)v\zeta_j(x) = \frac{1}{2\pi}\hat\mu*(v\zeta_j)(x)+\lambda^2\int_{\mathbb{R}^2}\mu(|D|)\mathcal{G}_{-\lambda}(x-y)(v\zeta_j)(y)dy. \tag{5.65}$$

Plugging (5.65) and (5.63) yields

$$X_{jkl}u(x) = X_{jkl}^{(b)}u(x)+X_{jkl}^{(g)}u(x), \tag{5.66}$$

where $X_{jkl}^{(b)}u(x)$ and $X_{jkl}^{(g)}u(x)$ are the functions produced by the first and the second terms on the right of of (5.65) respectively. Denote $\rho_{jk}(\lambda)=\lambda\mu_{jk}(\lambda)$. Then, by integrating with respect to $d\lambda$ first and by recalling (1.19), we obtain that $X_{jkl}^{(g)}u(x)$ is equal to

$$\begin{aligned}&\int_{\mathbb{R}^2}(v\zeta_j)(y)\left\{\int_0^\infty \mu(|D|)\tau_y\mathcal{G}_{-\lambda}(x)\left(\int_{\mathbb{S}^1}(\mathcal{F}\rho_{jk}(|D|)R_l u)(\lambda)d\omega\right)\lambda d\lambda\right\}dy\\ &=2\pi\int_{\mathbb{R}^2}(v\zeta_j)(y)(\mu(|D|)K\rho_{jk}(|D|)R_l u)(x-y)dy.\end{aligned}$$

Minkowski's inequality, (2.13) and the multiplier theory then imply

$$\|X_{jkl}^{(g)}u\|_p \le C_p\|v\zeta_j\|_1\|u\|_p,\quad 1<p<\infty. \tag{5.67}$$

and $X_{jkl}^{(g)}u(x)$ is a good operator, $1\le j,k,l\le 2$.

By observing that $\hat\mu*(v\zeta_j)(x)$ is independent of λ and by using the polar coordinates $\xi=\lambda\omega$, $\lambda>0$, $\omega\in\mathbb{S}^1$, we represent

$$\begin{aligned}X_{jkl}^{(b)}u(x) &= \frac{1}{2\pi}(\hat\mu*v\zeta_j)(x)\int_0^\infty\left(\int_{\mathbb{S}^1}(\mathcal{F}\mu_{jk}(|D|)R_l u)(\lambda)d\omega\right)d\lambda\\ &=\frac{1}{2\pi}(\hat\mu*v\zeta_j)(x)\int_{\mathbb{R}^2}(\mathcal{F}\mu_{jk}(|D|)R_l u)(\xi)|\xi|^{-1}d\xi.\end{aligned} \tag{5.68}$$

Since $\hat\mu\in L^p(\mathbb{R}^2)$, $1\le p<\infty$ and $v\zeta_j\in L^1(\mathbb{R}^2)$, $\hat\mu*v\zeta_j\in L^p(\mathbb{R}^2)$, $1\le p<\infty$. Since $\mu_{jk}(\lambda)|\xi|^{-1}\in L^p(\mathbb{R}^2)$ for $1\le p\le 2$, Hausdorff-Young's inequality implies that

$$\int_{\mathbb{R}^2}(\mathcal{F}\mu_{jk}(|D|)R_l u)(\xi)|\xi|^{-1}d\xi = \int_{\mathbb{R}^2}u(x)\mathcal{F}(\mu_{jk}(|\xi|)\xi_l|\xi|^{-2})(x)dx \tag{5.69}$$

are bounded linear functionals on $L^p(\mathbb{R}^2)$ for $1\le p\le 2$. Thus, $X_{jkl}^{(b)}$ are bounded in

$L^p(\mathbb{R}^2)$ for $1 < p \le 2$. This together with (5.67) proves the lemma. □

LEMMA 5.25. *Let $1 < p < 2$. The low energy part $\chi_{\le 4a}(|D|)\tilde{\Omega}_{(b)}$ is bounded from $L^p(\mathbb{R}^2)$ to itself and, hence the same holds for $\tilde{\Omega}_{(b)}$.*

Proof. The proof is a slight modification of that of Lemma 3.8 and we shall be a little sketchy in some places. As in the proof of Lemma 5.24 it suffices to show the lemma for $\tilde{X}_{jkl}u(x)$, $j,k,l = 1,2$ defined by replacing $\chi_{>4a}(|D|)$ by $\chi_{\le 4a}(|D|)$ in (5.63):

$$\tilde{X}_{jkl}u(x) = \int_0^\infty \chi_{\le 4a}(|D|)G_0(-\lambda)v\zeta_j(x)\left(\int_{\mathbb{S}^1}(\mathcal{F}\mu_{jk}(|D|)R_l u)(\lambda\omega)d\omega\right)d\lambda\,. \tag{5.70}$$

We denote $u_{jkl} = \mu_{jk}(|D|)R_l u$. We use (3.27) for $\psi = v\zeta_j$ that

$$\chi_{\le 4a}(|D|)G_0(-\lambda)v\zeta_j(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)\mathcal{F}(v\zeta_j)(\xi)}{\xi^2-(-\lambda+i0)^2}d\xi, \tag{5.71}$$

which, by virtue of (3.28) adapted for $v\zeta_j(x)$, is equal to

$$\frac{-i}{2\pi}\sum_{m=1}^2\int_0^1\int_{\mathbb{R}^2} z_m(v\zeta_j)(z)\tau_{\theta z}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{ix\xi}\frac{\xi_m\chi_{\le 4a}(|\xi|)}{\xi^2-(-\lambda+i0)^2}d\xi\right)dzd\theta$$

(see (3.29)). The inner integral is computed in (3.30) and is equal to

$$2\pi R_m\lambda\chi_{\le 4a}(|D|)\mathcal{G}_{-\lambda}(x) + R_m\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{ix\xi}\frac{\chi_{\le 4a}(|\xi|)}{|\xi|+\lambda}d\xi\right). \tag{5.72}$$

The contribution of the first term for $\tilde{X}_{jkl}u(x)$ is given as in (3.31) by

$$-2\pi i\sum_{m=1}^2\int_0^1 d\theta\int_{\mathbb{R}^2} z_m(v\zeta_j)(z)(R_m\chi_{\le 4a}(|D|)Ku_{jkl})(x-\theta z)dzd\theta$$

and is a good operator being bounded by

$$C\int_{\mathbb{R}^2}|z_m(v\zeta_j)(z)|\|u_{jkl}\|_p \le C\|z_m v(z)\zeta_j(z)\|_1\|u\|_p\,, \quad 1 < p < \infty. \tag{5.73}$$

The contribution of the second term of (5.72) for $\tilde{X}_{jkl}u(x)$ is given after changing the order of integrations by the $\sum_{l=m}^2\int_0^1 d\theta$ of

$$\frac{-iR_m}{(2\pi)^2}\int_{\mathbb{R}^2} z_m(v\zeta_j)(z)\tau_{\theta z}\left(\int_{\mathbb{R}^2}\frac{e^{ix\xi}\chi_{\le 4a}(|\xi|)}{(|\xi|+|\eta|)|\eta|}\widehat{u_{jkl}}(\eta)d\xi d\eta\right)dz. \tag{5.74}$$

If we define the integral operator L_1 by the integral kernel

$$L_1(x,y) = \int_{\mathbb{R}^4}\frac{e^{ix\xi-iy\eta}\chi_{\le 4a}(|\xi|)\chi_{\le 2a}(|\eta|)}{(|\xi|+|\eta|)|\eta|g(|\eta|)}d\xi d\eta$$

then, the inner integral of (5.74) may be written in the form $L_1 d_{jk}(|D|)R_l u(x)$ (recall that $u_{jkl} = \mu_{jk}(|D|)R_l u)$ and

$$(5.74) = \frac{-iR_m}{(2\pi)^2}\int_{\mathbb{R}^2} z_m(v\zeta_j)(z)(Ld_{jk}(|D|)R_m u)(x-\theta z)dz\,.$$

We have shown in Lemma 4.7 of [43] that L_1 is a bounded operator of $L^p(\mathbb{R}^2)$ for $1<p\le 2$. Thus, Minkowski's inequality implies

$$\|(5.74)\|_p \le C\|z_m(v\zeta_j)(z)\|_1\|u\|_p, \quad 1<p\le 2.$$

This to gether with (5.73) proves Lemma 5.25. □

5.6.4. Estimate of bad part 2, Negative result

PROPOSITION 5.26. *The operator $\chi_{>4a}(|D|)\tilde\Omega_{(b)}$ is unbounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$ if $a>0$ is sufficiently small.*

Since $\chi_{>4a}(|D|)$ is a good operator, the proposition implies $\tilde\Omega_{(b)}$ is unbounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$ for small $a>0$. It follow, since $\tilde\Omega_{(g)}$ is a good operator for any $a>0$ by virtue of Proposition 5.22, W_+ is unbounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$ if H has a singularity of the second kind at zero.

Proof. By virtue of (5.64), (5.66) and (5.67), it suffices to show that the operator $\tilde\Omega_{\text{low},*}u(x) \overset{\text{def}}{=} \sum_{j,k,l=1}^2 i\langle z_l v|\zeta_k\rangle X_{jkl}^{(b)}u(x)$ is unbounded in $L^p(\mathbb{R}^2)$ if $2<p<\infty$ and if $a>0$ is small enough. By virtue of (5.68) and (5.69)

$$\tilde\Omega_{\text{low},*}u(x) = \sum_{j=1}^2 (\hat\mu*(v\zeta_j))(x)\ell_j(u) \tag{5.75}$$

where $\ell_1(u)$ and $\ell_2(u)$ are linear functionals defined by

$$\ell_j(u) = \sum_{k,l=1}^2 i\langle z_l v|\zeta_k\rangle \int_{\mathbb{R}^2} u(x)\mathcal{F}(\mu_{jk}(|\xi|)\xi_l|\xi|^{-2})(x)dx, \; j=1,2\,. \tag{5.76}$$

It is obvious that $\hat\mu*(v\zeta_1)$ and $\hat\mu*(v\zeta_2)\in L^p(\mathbb{R}^2)$ for $2<p<\infty$ and, for small $a>0$, they are linearly independent in $L^p(\mathbb{R}^2)$ as will be shown in the next subsection where we shall prove a more general result. Then, if $\tilde\Omega_{\text{low}*}$ were bounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$, it must be that both ℓ_1 and ℓ_2 were continuous functionals on $L^p(\mathbb{R}^2)$ by the Hahn-Banach theorem, hence that for $q=p/(p-1)$, $1<q<2$

$$\sum_{k,l=1}^2 \langle z_l v|\zeta_k\rangle \mathcal{F}(\mu_{jk}(|\xi|)\xi_l|\xi|^{-2}) \in L^q(\mathbb{R}^2), \quad j=1,2$$

by the Riesz representation theorem and

$$\sum_{k=1}^2 d_{jk}(|\xi|)\sum_{l=1}^2 \langle z_l v|\zeta_k\rangle \chi_{\le 2a}(\xi)\xi_l g(|\xi|)^{-1}|\xi|^{-2} \in L^p(\mathbb{R}^2) \tag{5.77}$$

by Haudorff-Young's inequality, where we restored $\mu_{jk}(\lambda)=g(\lambda)^{-1}d_{jk}(\lambda)\chi_{\le 2a}(\lambda)$. (5.77) means in the matrix notation that

$$\frac{\chi_{\le 2a}(|\xi|)}{g(|\xi|)|\xi|^2} D(|\xi|) \begin{pmatrix} \langle z_1 v|\zeta_1\rangle \xi_1 + \langle z_2 v|\zeta_1\rangle \xi_2 \\ \langle z_1 v|\zeta_2\rangle \xi_1 + \langle z_2 v|\zeta_2\rangle \xi_2 \end{pmatrix} \in L^p(\mathbb{R}^2, \mathbb{C}^2)\,. \tag{5.78}$$

Then, since $D(\lambda) = C(\lambda)^{-1}$, (5.34) and (5.78) imply that

$$\frac{\chi_{\le 2a}(|\xi|)}{g(|\xi|)|\xi|^2} (\langle z_1 v|\zeta_k\rangle \xi_1 + \langle z_2 v|\zeta_k\rangle \xi_2) \in L^p(\mathbb{R}^2), \quad k = 1, 2\,. \tag{5.79}$$

But for $p > 2$ this can happen only when $\langle z_j v|\zeta_k\rangle = 0$ for $j, k = 1, 2$. However, $-\kappa_j^2 = \langle v G_1 v \zeta_k, \zeta_k\rangle = -\frac{1}{2}\sum_{j=1}^2 |\langle z_j v, \zeta_k\rangle|^2 < 0$, $k = 1, 2$ and this is impossible. Thus, $\tilde\Omega_{\text{low}*}$ must be unbounded in $L^p(\mathbb{R}^2)$ for any $2 < p < \infty$ if $a > 0$ is small enough. □

5.7. The case H has sigularities of the third kind at zero

In this case T_2 is singular in $S_2 L^2(\mathbb{R}^2)$; S_3 is the projection in $S_2 L^2(\mathbb{R}^2)$ onto $\operatorname{Ker} T_2$ and $T_3 = S_3 v G_2 v S_3$ is non-singular in $S_3 L^2(\mathbb{R}^2)$. We first assume that $S_2 \ominus S_3 \neq 0$. As previously we shall often omit the phrase "for small $\lambda > 0$".

Theorem 5.27. *Suppose that H has singularities of the third kind at zero and $S_3 \subsetneq S_2$. Then W_+ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p \le 2$ and is unbounded in $L^p(\mathbb{R}^2)$ for $2 < p < \infty$.*

5.7.1. Threshold anaysis 3

For shortening formulas, define

$$T = S_2 v G_1 v S_2 (= T_2), \quad \tilde T = S_2 v G_2 v S_2,$$

so that $\tilde{\mathcal{R}}_1(\lambda) = T + g(\lambda)^{-1}\tilde T$ and $T_3 = S_3 \tilde T S_3$. We define $\mathcal{X}_2 = (S_2 \ominus S_3)L^2(\mathbb{R}^2)$ and $\mathcal{X}_3 = S_3 L^2(\mathbb{R}^2)$ and denote by P_2 and P_3 the projections in $S_2 L^2(\mathbb{R}^2)$ onto $\mathcal{X}_2$ and $\mathcal{X}_3$ respectively. We express $\tilde{\mathcal{R}}_1(\lambda)$ in $S_2 L^2(\mathbb{R}^2)$ as the operator matrix in the decomposition $S_2 L^2(\mathbb{R}^2) = \mathcal{X}_2 \oplus \mathcal{X}_3$:

$$\tilde{\mathcal{R}}_1(\lambda) = \begin{pmatrix} T_{22} + g(\lambda)^{-1}\tilde T_{22} & g(\lambda)^{-1}\tilde T_{23} \\ g(\lambda)^{-1}\tilde T_{32} & g(\lambda)^{-1}\tilde T_{33} \end{pmatrix} \tag{5.80}$$

where $T_{jk} = P_j T P_k$, $\tilde T_{jk} = P_j \tilde T P_k$ for $j, k = 2, 3$ and we have used that $T_{23} = 0, T_{32} = 0$ and $T_{33} = 0$. Note that T_{22} and $\tilde T_{33} = T_3$ are invertible in $\mathcal{X}_2$ and in $\mathcal{X}_3$ respectively. It follows that $T_{22} + g(\lambda)^{-1}(\tilde T_{22} - \tilde T_{23}\tilde T_{33}^{-1}\tilde T_{32})$ is invertible in $\mathcal{X}_2$ for small $\lambda > 0$ and we define

$$\tilde d(\lambda) \overset{\text{def}}{=} T_{22}^{-1}(1_{\mathcal{X}_2} + g(\lambda)^{-1}(\tilde T_{22} - \tilde T_{23}\tilde T_{33}^{-1}\tilde T_{32})T_{22}^{-1})^{-1}. \tag{5.81}$$

We have $\tilde d(\lambda) = T_{22}^{-1} + \mathcal{O}_2(g(\lambda)^{-1})$ and entries of $\tilde d(\lambda)$ are good multipliers. The following lemma follows by virtue of the Feshbach formula.

Lemma 5.28. *$\tilde{\mathcal{R}}_1(\lambda)$ is invertible in $S_2 L^2(\mathbb{R}^2)$ and*

$$\tilde{\mathcal{R}}_1(\lambda)^{-1} = g(\lambda) S_3 T_3^{-1} S_3 + L_4(\lambda), \tag{5.82}$$

$$L_4(\lambda) = \begin{pmatrix} \tilde d(\lambda) & -\tilde d(\lambda)\tilde T_{23}\tilde T_{33}^{-1} \\ -\tilde T_{33}^{-1}\tilde T_{32}\tilde d(\lambda) & \tilde T_{33}^{-1}\tilde T_{32}\tilde d(\lambda)\tilde T_{23}\tilde T_{33}^{-1} \end{pmatrix}. \tag{5.83}$$

The entries of $L_4(\lambda)$ are good multipliers.

By virtue of Lemma 5.28 and Remarks 5.16, 5.20 and 5.21 the result of Proposition 5.17 that $vM(\lambda)^{-1}v \equiv -g(\lambda)^{-1}\lambda^{-2}vS_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2v$ modulo a good produre is likewise satisfied if H has singularities of the third kind at zero. Thus, it suffices to show that $\tilde\Omega_{\text{low},2a}$ defined by (5.55) satisfies the statement of Theorem 5.13.

We take the orthonormal basis $\{\zeta_1,\dots,\zeta_m\}$ of $S_2L^2(\mathbb{R}^2)$ such that $\zeta_1,\zeta_2 \in \mathcal{X}_2$ and

$$T_2\zeta_j = -\kappa_j^2\zeta_j, \quad \kappa_j > 0,\ j = 1,2$$

and $\{\zeta_3,\dots,\zeta_m\}$ is the basis of $\mathcal{X}_3$ such that

$$T_3\zeta_k = \alpha_k\zeta_k, \quad k = l+1,\dots,m.$$

Since $T_2\zeta_k = 0$, we have the extra cancellation property:

$$\int_{\mathbb{R}^2} x_1\zeta_k(x)v(x)dx = \int_{\mathbb{R}^2} x_2\zeta_k(x)v(x)dx = 0. \quad k = l+1,\dots,m. \tag{5.84}$$

The following lemma proves the positive part of Theorem 5.27.

Lemma 5.29. *For any $a>0$, $\tilde\Omega_{\text{low},2a}$ is bounded in $L^p(\mathbb{R}^2)$ for $1<p\le 2$.*

Proof. Let $D(\lambda) = (d_{jk}(\lambda))$ be the representation matrix of $\tilde{\mathcal{R}}_1^{-1} = g(\lambda)S_3T_3^{-1}S_3 + L_4(\lambda)$ with respect to the basis $\{\zeta_1,\dots,\zeta_m\}$. $D(\lambda)$ is the sum of $\tilde D(\lambda)$ for $g(\lambda)S_3T_3^{-1}S_3$ and $B(\lambda) = (\beta_{jk}(\lambda))$ for $L_4(\lambda)$. By virtue of (5.83) and (5.81), $\beta_{jk}(\lambda)\chi_{\le 2a}(\lambda)$ are good multipliers. Recall that the moment property that $\langle\zeta_j, v\rangle = 0$, $j=1,\dots,m$ is satisfied.

Let $\tilde\Omega^B$ and $\tilde\Omega^D$ be defined by (5.55) respectively with $L_4(\lambda)$ and $g(\lambda)S_3T_3^{-1}S_3$ replacing $S_2\tilde{\mathcal{R}}_1(\lambda)^{-1}S_2$ so that $\tilde\Omega_{\text{low},2a} = \tilde\Omega^B + \tilde\Omega^D$. Then, $\tilde\Omega^B u$ is the sum over $1\le j,k\le m$ of

$$\tilde\Omega^B_{jk}u = \int_0^\infty g(\lambda)^{-1}\lambda^{-2}\beta_{jk}(\lambda)|G_0(-\lambda)v\zeta_j\rangle\langle\zeta_k v, \Pi(\lambda)u\rangle\chi_{\le 2a}(\lambda)\lambda d\lambda. \tag{5.85}$$

This is the same as (5.56) with $\beta_{jk}(\lambda)$ replacing by $d_{jk}(\lambda)$ and the former function can play the role of the latter in the proof of the positive part of Theorem 5.13. Thus, $\tilde\Omega^B$ is bounded in $L^p(\mathbb{R}^2)$ for $1<p<2$.

Since $\tilde D(\lambda) = g(\lambda)\mathrm{diag}(\alpha_1,\dots,\alpha_m)$ is diagonal and $\alpha_1=\alpha_2=0$,

$$\tilde\Omega^D u(x) = \sum_{j=3}^m \alpha_j\int_0^\infty \lambda^{-1}(G_0(-\lambda)\zeta_j v)(x)\langle\zeta_j v, \Pi(\lambda)u\rangle\chi_{\le 2a}(\lambda)d\lambda.$$

This is of the form (3.8) but with the singular factor λ^{-1} in place of λ. However, for $j = 3,\dots,m$ the extra cancellation property (5.84) implies that $\langle\zeta_j v, \tilde b(\lambda,\cdot)\rangle = 0$ and

$\langle \zeta_j v, \Pi(\lambda)u\rangle = \langle \zeta_j v, \tilde{g}(\lambda,\cdot)\rangle$. It follows that $\tilde{\Omega}^D u(x)$ is equal to $\sum_{l,k=1}^2 \int_0^1 (1-\theta)d\theta$ of

$$-\int_{\mathbb{R}^4} z_l z_k (\zeta_j v)(y)(\zeta_j v)(z)(\tau_y K \chi_{\le 2a}(|D|)R_k R_l \tau_{-\theta z} u)(x)dydz.$$

Then Minkowski's inequality, (2.13) and the multiplier theorem imply that for any $\gamma > 4$

$$\|\tilde{\Omega}^D u\|_p \le C\sum_{j=3}^m \|z^2\zeta_j v\|_1 \|\zeta_j v\|_1 \|u\|_p \le C\|\langle z\rangle^\gamma V\|_1 \|u\|_p, \quad 1<p<\infty$$

and $\tilde{\Omega}^D$ is a good operator. This proves the lemma. □

If $T_2 = 0$, then $\tilde{\mathcal{R}}_1(\lambda) = g(\lambda)S_3 T_3^{-1} S_3$ does not contain $L_4(\lambda)$ and the proof of Lemma 5.29 above implies the following theorem.

THEOREM 5.30. *Suppose that H has singularities of the third kind at zero and $T_2 = 0$, viz. $S_2 = S_3$. Then W_+ is a good operator.*

The following lemma completes the proof of Theorem 5.27

LEMMA 5.31. *Suppose that H has singularities of the third kind at zero and $S_3 \subsetneq S_2$. Then W_+ is unbounded in $L^p(\mathbb{R}^2)$ if $2 < p < \infty$.*

Proof. In view of the proof of Lemma 5.29, it suffices to prove that $\tilde{\Omega}^B \stackrel{\text{def}}{=} \sum_{j,k=1}^m \tilde{\Omega}^B_{jk}$ is unbounded in $L^p(\mathbb{R}^2)$ if $2<p<\infty$. Since $\langle \zeta_j, v\rangle = 0$, we may replace $\Pi(\lambda)u(x)$ of (5.85) by $\Pi(\lambda)u(x) - \Pi(\lambda)u(0) = \tilde{g}(\lambda,x) + \tilde{b}(\lambda,x)$ as previously, which produces $\tilde{\Omega}^B u = \tilde{\Omega}^B_g u + \tilde{\Omega}^B_b u$, where the definition of $\tilde{\Omega}^B_g$ and $\tilde{\Omega}^B_b$ should be obvious. Then, the factor λ^2 in $\tilde{g}(\lambda,x)$ cancels the singularity in the integrand of (5.85) and $\tilde{\Omega}^B_g$ becomes a good operator as in Proposition 5.22. Then, since $\chi_{>4a}(|D|)$ is a good operator, it suffices to show that $\chi_{>4a}(|D|)\tilde{\Omega}^B_b$ is unbounded in $L^p(\mathbb{R}^2)$ if $2<p<\infty$. If we use (5.65) for $\chi_{>4a}(|D|)G_0(-\lambda)(v\zeta_j)$, then the proof of Lemma 5.24 shows that the seccond term on the right of (5.65) produces a good operator and we have only to prove that

$$Z_a u(x) = \sum_{j,k=1}^m (\hat{\mu}_a * (v\zeta_j)(x) \int_0^\infty \beta_{jk}(\lambda)\langle \zeta_k v, \tilde{b}(\lambda,\cdot)\rangle \rho_a(\lambda)d\lambda \tag{5.86}$$

is unbounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$, where $\rho_a(\lambda) = g(\lambda)^{-1}\lambda^{-1}\chi_{2a}(\lambda)$ and we wrote $\mu_a(\xi)$ for $\mu(\xi)$ to make a dependence of $\mu(\xi)$ explicit. Recall that $\hat{\mu}_a(x) \in L^p(\mathbb{R}^2)$ for $1<p<\infty$ To make the argument transparent, we introduce the vector notation

$$\mathcal{Z} = \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{pmatrix}, \quad \mathcal{Z}_1 = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} \zeta_3 \\ \vdots \\ \zeta_m \end{pmatrix}, \quad \mathcal{Z}^* = (\mathcal{Z}_1^*, \mathcal{Z}_2^*)$$

and express $Z_a u(x)$ in the form

$$Z_a u(x) = \Big\langle \hat{\mu}_a * v(x)\mathcal{Z}(x), \int_0^\infty B(\lambda)\langle \mathcal{Z}v, \tilde{b}(\lambda,\cdot)\rangle \rho_a(\lambda)d\lambda \Big\rangle_{\mathbb{C}^2}. \tag{5.87}$$

Here (5.59) and the extra moment condition (5.84) imply that $\langle \mathcal{Z}_2 v, \tilde{b}(\lambda,\cdot)\rangle = 0$. It follows that

$$B(\lambda)\langle \mathcal{Z}v, \tilde{b}(\lambda,\cdot)\rangle = \begin{pmatrix} \mathbf{1}_2 \\ A \end{pmatrix} \tilde{d}(\lambda)\langle v\mathcal{Z}_1, \tilde{b}(\lambda,\cdot)\rangle\,,$$

where $\mathbf{1}_2$ stands for the 2×2 identity matrix, $A = (a_{jk})_{3\le j\le m, 1\le k\le 2}$ is the representation matrix for $\tilde{T}_{33}^{-1}\tilde{T}_{32}$ and $\tilde{d}(\lambda)$ is defined in (5.81). Since the matrix $\begin{pmatrix} \mathbf{1}_2 \\ A \end{pmatrix}$ does not depend on λ, we can transpose it in front of $\hat{\mu}_a * v(x)\mathcal{Z}(x)$ in (5.87) and express $Z_a u(x)$ in the form

$$Z_a u(x) = f_1(x)\tilde{\ell}_1(u) + f_2(x)\tilde{\ell}_2(u) \tag{5.88}$$

where $f_j(x)$ and $\tilde{\ell}_j(u)$, $j = 1,2$ are defined by

$$\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \hat{\mu}_a * \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}, \quad \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} = v(x)\mathcal{Z}_1(x) + A^* v(x)\mathcal{Z}_2(x)$$

and

$$\begin{pmatrix} \tilde{\ell}_1(u) \\ \tilde{\ell}_2(u) \end{pmatrix} = \int_0^\infty \tilde{d}(\lambda)\langle \mathcal{Z}_1 v, \tilde{b}(\lambda,\cdot)\rangle \rho_a(\lambda) d\lambda$$

and both depend on $a > 0$. It is obvious that $f_1, f_2 \in L^p(\mathbb{R}^2)$ for $1 < p < \infty$ and we show that they are linearly independent if $a > 0$ is sufficiently small. Indeed if otherwise, there exists for any $a > 0$ a point $\begin{pmatrix} C_{1a} \\ C_{2a} \end{pmatrix} \in \mathbb{S}^1$ such that $C_{1a} f_1(x) + C_{2a} f_2(x) = 0$ and via Fourier transform

$$\mathcal{F}(C_{1a} w_1 + C_{2a} w_2)(\xi) = 0 \quad \text{for } |\xi| \ge 4a \tag{5.89}$$

Then the subset $\mathcal{C}_a \subset \mathbb{S}^1$ of points $\begin{pmatrix} C_{1a} \\ C_{2a} \end{pmatrix}$ which satisfy (5.89) is compact and non-empty and $\mathcal{C}_a \subset \mathcal{C}_b$ if $a < b$. Thus, $\mathcal{C} = \cap_{a>0}\mathcal{C}_a \neq \varnothing$ and, for a $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \in \mathcal{C}$, we must have

$$\mathcal{F}(C_1 w_1 + C_2 w_2)(\xi) = 0, \quad \xi \in \mathbb{R}^2$$

and hence $C_1 w_1(x) + C_2 w_2(x) = 0$ or

$$v(C_1\zeta_1 + C_2\zeta_2 + (C_1 a_{13} + C_2 a_{23})\zeta_3 + \cdots + (C_1 a_{1m} + C_2 a_{2m})\zeta_m) = 0$$

However, $-\kappa_j^2\zeta_j(x) = v(x)(G_1 v\zeta_j)(x)$ for $j = 1,2$ and $\alpha_j\zeta_j(x) = v(x)(G_2 v\zeta_j)(x)$ for $j = 3,\ldots,m$. Hence $\zeta_j(x) = 0$, $j = 1,\ldots,m$, if $v(x) = 0$. It follows that

$$C_1\zeta_1 + C_2\zeta_2 + (C_1 a_{13} + C_2 a_{23})\zeta_3 + \cdots + (C_1 a_{1m} + C_2 a_{2m})\zeta_m = 0$$

and it must be that $C_1 = C_2 = 0$ which is a contradiction. Thus, f_1 andd f_2 must be linearly independent for small $a > 0$. Incidentally, this simultaneously proves the corresponding statement in the proof of Proposition 5.26.

The rest of the proof is the repetition of that of Proposition 5.26. If $Z_a u$ were bounded in $L^p(\mathbb{R}^2)$ for $2<p<\infty$, then $\ell_1(u)$ and $\ell_2(u)$ must be bounded functionals in $L^p(\mathbb{R}^2)$ by the Hahn-Banach theorem and we must have (5.78) with $\tilde{d}(|\xi|)$ in place of $D(|\xi|)$, viz

$$\frac{\chi_{\leq 2a}(|\xi|)}{g(|\xi|)|\xi|^2}\tilde{d}(|\xi|)\mathcal{N}(\xi) \in L^p(\mathbb{R}^2), \quad \mathcal{N}(\xi)=\begin{pmatrix} \langle z_1 v, \zeta_1\rangle \xi_1 + \langle z_2 v, \zeta_1\rangle \xi_2 \\ \langle z_1 v, \zeta_2\rangle \xi_1 + \langle z_2 v, \zeta_2\rangle \xi_2 \end{pmatrix}. \tag{5.90}$$

Since $\tilde{d}(|\xi|)^{-1}$ is bounded (see (5.81)), (5.90) would lead to $\langle z_1 v, \zeta_k\rangle = \langle z_2 v, \zeta_k\rangle = 0$ for $k=1,2$ which contradicts to (5.4). The lemma follows. □

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