A CARTAN DECOMPOSITION FOR GELFAND PAIRS AND INDUCTION OF SPHERICAL FUNCTIONS

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Abstract. In this article we show a Cartan decomposition for reductive Riemannian Gelfand pairs and an induction of spherical functions for Riemannian Gelfand pairs. With the induction we find that the property of the symmetry of spherical functions, which is known for Riemannian symmetric pairs, can also be induced from the corresponding property of smaller dimension. A Fourier transform of a positive function for a Riemannian Gelfand pair with abelian unipotent radical is also given under some condition on its support by using the symmetry of spherical function.

0. Introduction

In this article we prove a Cartan decomposition for reductive Riemannian Gelfand pairs and show an application to spherical functions for Riemannian Gelfand pairs. A pair (G, H) of a real Lie group G and its compact subgroup H with G/H connected is a Riemannian Gelfand pair if the algebra (under convolution) of H-biinvariant finite complex Radon measures on G is commutative. A reductive Riemannian symmetric pair is a typical example of Riemannian Gelfand pairs. The reader is referred to [Wo07] for the general theory (G is not necessarily a Lie group) of Gelfand pair and [Ya05] for the classification. Our first result is a Cartan decomposition (Theorem 2.5) of the form $G = HAH$ with A an abelian Lie subgroup of G for a reductive Riemannian Gelfand pair (G, H), which is proved in Section 2. The proof uses the induction on the dimension of G. We find all the reductive Riemannian Gelfand pairs for which we cannot reduce a Cartan decomposition to more smaller dimensional cases with the Cartan decomposition for reductive Riemannian symmetric pairs [He78] in Section 1 by inspecting Krämer's classification of reductive spherical subalgebras [Kr79]. In Section 3 we show an induction of spherical functions (Theorem 3.1) for a Riemannian Gelfand pair (G, H). The induction is given as the integration on H, whose integral kernel is provided from the Iwasawa projection on the reductive part. In Section 4 we show that the property of the symmetry of spherical functions, which is known for reductive Riemannian symmetric pairs, can also be induced from the corresponding property of smaller dimension by using the induction of spherical functions (Lemma 4.8), and that the property holds in the case when the unipotent radical of G is abelian (Theorem 4.19). As an application of this property we find that the convolution product of a compactly supported function and a spherical function takes a simple

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form (Corollary 4.20). Also we see that the symmetry property does not hold for pairs with non-abelian unipotent radicals (Proposition 4.21). In Section 5 we show a Fourier transform for Riemannian Gelfand pairs with abelian unipotent radicals by using the convolution product (Theorem 5.1). See [Wo07] for the general theory of spherical functions in the framework of locally compact groups and [He84, He94] for spherical functions on Riemannian symmetric spaces from the analytic viewpoint.

A reductive Riemannian weakly symmetric space G/H is a special case of a real spherical space. Here a real G-manifold of a real reductive group G is real spherical if a minimal parabolic subgroup has an open orbit on it. The reader is referred to [Ko95, KO13] for representation theoretic properties of real spherical spaces. Recently the Plancherel theory for a real spherical space has been developed extensively [DKKS18].

In the following, we denote Lie groups by capital Latin letters and their Lie algebras by corresponding small German letters. The identity component of a Lie group G is denoted by G_e. For a closed linear Lie group G we take a complexification $G_{\mathbb{C}}$ as $G_{\mathbb{C}} = G \cdot G_{\mathbb{C},e}$ for the analytic subgroup $G_{\mathbb{C},e}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $GL(n, \mathbb{C})$ for which we have a closed embedding $G \hookrightarrow GL(n, \mathbb{R})$ unless otherwise specified. A Lie group G is reductive if it is closed linear with finitely many connected components and stable under transpose, and G is of inner type if the adjoint group $\mathrm{Ad}(G)$ is a subgroup of the inner automorphism group $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$. A Riemannian Gelfand pair (G, H) is reductive if G is a real reductive Lie group, and compact if G is a compact Lie group. A reductive Lie algebra $\mathfrak{g}$ is always assumed to be embedded into some $\mathfrak{gl}(n, \mathbb{R})$ and stable under transpose, and $V^{\perp}$ stands for the orthogonal complement of a vector subspace V of $\mathfrak{g}$ with respect to the bilinear form $B(X, Y) = \mathrm{Trace}(XY)$. A reductive Lie algebra is compact if it is fixed under the negative transpose. A measure on a unimodular Lie group G is always a Haar measure dg, and the total measure $\int_K 1dk$ of a compact Lie group K is normalized to be 1. For a group or an algebra A and for a set B, we denote by $Z_A(B)$ and $N_A(B)$ the centralizer and the normalizer of B in A, respectively if they make sense. We denote by G_x the stabilizer at a point x for an action of G on a set X.

In this article, for a pair (G, H) of a Lie group G and a closed subgroup H, we assume that G can be expressed as a semi-direct product $G = U \cdot L$ for the largest connected unipotent closed normal subgroup U and a maximal reductive subgroup L, and that all reductive stabilizers of the L-action on $\mathfrak{u}^*$ are of inner type.

1. Non-symmetric polar pair

In this section we find the reductive Riemannian Gelfand pairs $(\mathfrak{g}, \mathfrak{h})$ such that there are no reductive Riemannian symmetric pairs $(\mathfrak{g}, \tilde{\mathfrak{h}})$ with $\tilde{\mathfrak{h}} \supset \mathfrak{h}$ by using Krämer's classification.

We introduce some terminology. A pair $(\mathfrak{g}, \mathfrak{h})$ of a Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$ is non-trivial if $\mathfrak{g} \neq \mathfrak{h}$. A pair $(\mathfrak{g}, \mathfrak{h})$ is a reductive Riemannian Gelfand pair if it arises as a pair of Lie algebras of some reductive Riemannian Gelfand pair (G, H). A reductive Riemannian Gelfand pair $(\mathfrak{g}, \mathfrak{h})$ is indecomposable if it is non-trivial, $\mathfrak{g}$ is semisimple and $(\mathfrak{g}, \mathfrak{h})$ cannot be represented non-trivially as the direct product $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, (\mathfrak{h} \cap \mathfrak{g}_1) \oplus (\mathfrak{h} \cap \mathfrak{g}_2))$

of reductive Riemannian Gelfand pairs $(\mathfrak{g}_1, \mathfrak{h} \cap \mathfrak{g}_1)$ and $(\mathfrak{g}_2, \mathfrak{h} \cap \mathfrak{g}_2)$. For a decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ as the sum of ideals $\mathfrak{g}_i$ of $\mathfrak{g}$, we denote by $p_i : \mathfrak{g} \to \mathfrak{g}_i$ the projection to $\mathfrak{g}_i$.

PROPOSITION 1.1 (see [KKPS19, Lemma 1.4] for a general result on spherical pairs). *Let $(\mathfrak{g}, \mathfrak{h})$ be a non-trivial and indecomposable reductive Riemannian Gelfand pair. If there is no symmetric subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$, which contains $\mathfrak{h}$, then the pair $(\mathfrak{g}, \mathfrak{h})$ is either $(\mathfrak{so}(7), \mathfrak{g}_{2(-14)})$ or $(\mathfrak{g}_{2(-14)}, \mathfrak{su}(3))$.*

For the proof of the proposition, we prepare some lemmas. The following is an observation on the above two pairs.

LEMMA 1.2. *Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive Riemannian Gelfand pair with $\mathfrak{g}$ semisimple. Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be the decomposition as the sum of simple ideals $\mathfrak{g}_i$ with the corresponding projection $p_i : \mathfrak{g} \to \mathfrak{g}_i$. Suppose that there is an index k such that either the restriction $p_k|_{\mathfrak{h}}$ is surjective or the pair $(\mathfrak{g}_k, p_k(\mathfrak{h}))$ is isomorphic to one of the two pairs in Proposition 1.1. Then the image of the center of $\mathfrak{h}$ under the projection p_k is the zero vector space, and there is only one simple ideal $\mathfrak{h}_k$ of $\mathfrak{h}$ such that $p_k(\mathfrak{h}_k) \neq 0$.*

Proof. We can see this directly for each of the two pairs when $(\mathfrak{g}_k, p_k(\mathfrak{h}))$ is isomorphic to one of them. If $p_k|_{\mathfrak{h}}$ is surjective, then we have $\mathfrak{h} = \ker(p_k|_{\mathfrak{h}}) \oplus \ker(p_k|_{\mathfrak{h}})^{\perp}$ and $\ker(p_k|_{\mathfrak{h}})^{\perp} \simeq \mathfrak{g}_k$. Hence the assertion follows. □

The following is an observation on a reductive Riemannian Gelfand pair.

LEMMA 1.3. *Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive Riemannian Gelfand pair and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ a decomposition as the sum of two ideals $\mathfrak{g}_1$, $\mathfrak{g}_2$ of $\mathfrak{g}$ with $p_i : \mathfrak{g} \to \mathfrak{g}_i$ the corresponding projection $(i = 1, 2)$. Then $(\mathfrak{g}_i, p_i(\mathfrak{h}))$ is a reductive Riemannian Gelfand pair $(i = 1, 2)$.*

Proof. Let $\mathfrak{b}_i$ be a Borel subalgebra of $\mathfrak{g}_{i,\mathbb{C}}$ $(i = 1, 2)$ such that $\mathfrak{g}_{\mathbb{C}} = (\mathfrak{b}_1 + \mathfrak{b}_2) + \mathfrak{h}_{\mathbb{C}}$. Then we obtain $\mathfrak{g}_{i,\mathbb{C}} = p_i(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{b}_i + p_i(\mathfrak{h}_{\mathbb{C}})$ for $i = 1, 2$. □

Let us show Proposition 1.1 using the above two lemmas.

Proof of Proposition 1.1. We retain the setting of Proposition 1.1. We may assume $\mathfrak{g}$ is compact. Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be the decomposition as the sum of simple ideals $\mathfrak{g}_i$ with $p_i : \mathfrak{g} \to \mathfrak{g}_i$ the projection to the i-th ideal $\mathfrak{g}_i$.

Suppose that there is some index k such that the restriction of p_k to $\mathfrak{h}$ is not surjective and the pair $(\mathfrak{g}_k, p_k(\mathfrak{h}))$ is isomorphic to none of the two pairs. Then by the classification of compact Gelfand pairs by Krämer [Kr79, Tabelle 1] we can find that there exists a symmetric pair $(\mathfrak{g}_k, \tilde{\mathfrak{h}}_k)$ such that $\tilde{\mathfrak{h}}_k \supset p_k(\mathfrak{h})$ (see [Ya05, Theorem 2.3]). By using such an $\tilde{\mathfrak{h}}_k$ we obtain a symmetric subalgebra $\bigoplus_{i \neq k} \mathfrak{g}_i \oplus \tilde{\mathfrak{h}}_k$ of $\mathfrak{g}$, which contains $\mathfrak{h}$.

Suppose that for any index i the restriction of p_i to $\mathfrak{h}$ is surjective. Then $(\mathfrak{g}, \mathfrak{h})$ is of diagonal type, that is, $\mathfrak{h}$ is simple and $p_i|_{\mathfrak{h}}$ is an isomorphism for any i by the indecomposability of $(\mathfrak{g}, \mathfrak{h})$. Let $\operatorname{diag}(\mathfrak{g})_{1,2}$ be the diagonal subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then we obtain a symmetric subalgebra $\tilde{\mathfrak{h}} = \operatorname{diag}(\mathfrak{g})_{1,2} \oplus \bigoplus_{i \neq 1,2} \mathfrak{g}_i$ of $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, which contains $\mathfrak{h}$.

By the above two arguments, we only need to consider the case where there is some index k such that the pair $(\mathfrak{g}_k, p_k(\mathfrak{h}))$ is isomorphic to one of the two pairs, and for any other index j either the restriction $p_j|_{\mathfrak{h}}$ is surjective or the pair $(\mathfrak{g}_j, p_j(\mathfrak{h}))$ is isomorphic to one of the above two pairs. Then by Lemma 1.2 the image of the center of $\mathfrak{h}$ under the projection p_i is the zero vector space, and there is only one simple ideal $\mathfrak{h}_i$ of $\mathfrak{h}$ such that $p_i(\mathfrak{h}_i) \neq 0$ for each index i. By contradiction, let us see that $p_j(\mathfrak{h}_k) = 0$, that is, $\mathfrak{h}_j \neq \mathfrak{h}_k$ for any index j not equal to k. Assume that there is some index j not equal to k such that $\mathfrak{h}_j = \mathfrak{h}_k$. Then we find that $(\mathfrak{g}_k \oplus \mathfrak{g}_j, \mathrm{diag}(\mathfrak{h}_k))$ is a compact Gelfand pair with Lemma 1.3 by taking $\mathfrak{g}_1$ and $\mathfrak{g}_2$ to be $\mathfrak{g}_k \oplus \mathfrak{g}_j$ and $\bigoplus_{i \neq j,k} \mathfrak{g}_i$, respectively in the statement of Lemma 1.3. On the other hand the pair $(\mathfrak{g}_k \oplus \mathfrak{g}_j, \mathrm{diag}(\mathfrak{h}_k))$ is given as either $(\mathfrak{so}(7) \oplus \mathfrak{so}(7), \mathrm{diag}(\mathfrak{g}_{2(-14)}))$, $(\mathfrak{so}(7) \oplus \mathfrak{g}_{2(-14)}, \mathrm{diag}(\mathfrak{g}_{2(-14)}))$, $(\mathfrak{g}_{2(-14)} \oplus \mathfrak{g}_{2(-14)}, \mathrm{diag}(\mathfrak{su}(3)))$ or $(\mathfrak{g}_{2(-14)} \oplus \mathfrak{su}(3), \mathrm{diag}(\mathfrak{su}(3)))$, and hence we can see that $\dim(\mathfrak{g}_{k,\mathbb{C}} \oplus \mathfrak{g}_{j,\mathbb{C}}) > \dim(\mathfrak{b}_k \oplus \mathfrak{b}_j) + \dim(\mathfrak{h}_{k,\mathbb{C}})$ for Borel subalgebras $\mathfrak{b}_k$ and $\mathfrak{b}_j$ of $\mathfrak{g}_{k,\mathbb{C}}$ and $\mathfrak{g}_{j,\mathbb{C}}$, respectively by counting the dimensions. Therefore $\mathfrak{h}_j \neq \mathfrak{h}_k$, that is, $p_j(\mathfrak{h}_k) = 0$ for any index j not equal to k. Thus $(\mathfrak{g}, \mathfrak{h})$ contains $(\mathfrak{g}_k, \mathfrak{h}_k)$ as a direct factor. By the indecomposability of $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}, \mathfrak{h})$ coincides with $(\mathfrak{g}_k, \mathfrak{h}_k)$, that is, one of the two pairs. □

We end this section with a Cartan decomposition for the non-symmetric polar cases.

PROPOSITION 1.4. *Let (G, H) be a reductive Riemannian Gelfand pair whose Lie algebras are either $(\mathfrak{g}_{2(-14)}, \mathfrak{su}(3))$ or $(\mathfrak{so}(7), \mathfrak{g}_{2(-14)})$. Then we have $G = H \exp(\mathfrak{a}) H$ for some semisimple abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$.*

This should be known but we show a proof for the convenience.

Proof. There is a one-dimensional subspace $\mathfrak{a}$ of $\mathfrak{h}^\perp$, which satisfies $\mathfrak{h}^\perp = \bigcup_{h \in H} \mathrm{Ad}(h)\mathfrak{a}$ since the action of H on $\mathfrak{h}^\perp$ is polar with a one-dimensional Cartan subspace [Da85]. Thus we have $G = H \exp(\mathfrak{a}) H$. □

2. CARTAN DECOMPOSITION

In this section we show a Cartan decomposition for reductive Riemannian Gelfand pairs. For the later purpose we consider pairs that are not necessarily reductive. We let G be a real Lie group and fix a semidirect decomposition $G = U \cdot L$ for the unipotent radical U and a reductive subgroup L. Also let $H \subset L$ be a compact subgroup with L/H connected. We denote by $M(G)$ the algebra (under convolution) of finite complex Radon measures on G and by $M(G)^{H_1 \times H_2}$ the subalgebra of $H_1 \times H_2$-invariant ones for closed subgroups H_1 and H_2 of G. We recall the following theorem.

THEOREM 2.1 ([He84, Th84, AV99, VK78]). *The following are equivalent:*

(1) *(G, H) is a Riemannian Gelfand pair, that is, the algebra $M(G)^{H \times H}$ is commutative.*
(2) *The algebra $D(G/H)$ of G-invariant differential operators on G/H is commutative.*

Further, the condition (1) (equivalently, the condition (2)) is equivalent to each of the following if G is reductive:

(3) *G/H is absolutely spherical, that is, a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/H_{\mathbb{C}}$.*

(4) *(Multiplicity at most one) For all irreducible finite-dimensional representations V of G one has $\dim(V^H) \leq 1$ for the subspace V^H of H-fixed vectors.*

We also recall the following criterion.

Theorem 2.2 ([Ya05, Theorem 1.3]). *A pair (G, H) is a Riemannian Gelfand pair if and only if the following conditions hold.*

(A) $\mathbb{R}[\mathfrak{u}^*]^L = \mathbb{R}[\mathfrak{u}^*]^H$.

(B) *(L_δ, H_δ) is a (reductive) Riemannian Gelfand pair for any $\delta \in \mathfrak{u}^*$.*

(C) *$(U \cdot H_\beta, H_\beta)$ is a Riemannian Gelfand pair for any $\beta \in (\mathfrak{l}/\mathfrak{h})^*$.*

We note that the condition (A) implies that the non-compact semisimple part of L acts trivially on $\mathfrak{u}^*$.

We then let (G, H) be a Riemannian Gelfand pair. Let $\delta \in \mathfrak{u}^*$ and K be a maximal compact subgroup of L, which contains H. By the condition (3) of Theorem 2.1 and the condition (B) of Theorem 2.2, (L_δ, H_δ) is absolutely spherical. We fix a minimal parabolic subgroup P_0 of L and write $P_0 = M_0 A_0 N_0$, where $A_0 = \exp(\mathfrak{a}_0)$ for a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{k}^\perp \cap \mathfrak{l}$, M_0 is the centralizer of $\mathfrak{a}_0$ in K and N_0 is the unipotent radical of P_0. Let $P = M_P A_P N_P$ be a parabolic subgroup of L such that $A_P \subset A_0$ and $P_0 \subset P$. Then $P_\delta = M_{P,\delta} A_P N_P$ is a parabolic subgroup of L_δ.

Lemma 2.3. *$K_\delta = (M_{P,\delta} \cap K_\delta) H_\delta$.*

Proof. Since (L_δ, H_δ) is absolutely spherical, we have

$$\mathfrak{l}_{\delta,\mathbb{C}} = \mathfrak{h}_{\delta,\mathbb{C}} + \mathfrak{m}_{P,\delta,\mathbb{C}} + \mathfrak{a}_{P,\mathbb{C}} + \mathfrak{n}_{P,\mathbb{C}}.$$

Thus we obtain

$$\mathfrak{k}_{\delta,\mathbb{C}} = \mathfrak{h}_{\delta,\mathbb{C}} + (\mathfrak{m}_{P,\delta,\mathbb{C}} \cap \mathfrak{k}_{\delta,\mathbb{C}}).$$

Then we find $K_\delta = (M_{P,\delta} \cap K_\delta) H_\delta$ since $K_\delta/(M_{P,\delta} \cap K_\delta)$ is connected. □

Lemma 2.4. *$(M_{P,\delta}, M_{P,\delta} \cap H_\delta)$ is a reductive Riemannian Gelfand pair.*

Proof. For a Borel subalgebra $\mathfrak{b}_{\mathfrak{m}_{\delta,\mathbb{C}}}$ of $\mathfrak{m}_{P,\delta,\mathbb{C}}$ we have

$$\mathfrak{l}_{\delta,\mathbb{C}} = \mathfrak{h}_{\delta,\mathbb{C}} + \mathfrak{b}_{\mathfrak{m}_{\delta,\mathbb{C}}} + \mathfrak{a}_{P,\mathbb{C}} + \mathfrak{n}_{P,\mathbb{C}}.$$

Let $E \in \mathfrak{a}_P$ a generic element, i.e., any restricted root that is non-trivial on $\mathfrak{a}_P$ has non-zero value on E. Suppose that $X \in \mathfrak{h}_{\delta,\mathbb{C}}$ and $Y \in \mathfrak{n}_{P,\mathbb{C}}$ satisfy $[X + Y, E] = 0$. This implies that $[X, E] \in \mathfrak{n}_{P,\mathbb{C}}$. We take $X' \in \mathfrak{m}_{P,\delta,\mathbb{C}}$, $Z \in \mathfrak{n}_{P,\mathbb{C}}$ and $Z' \in \bar{\mathfrak{n}}_{P,\mathbb{C}}$ such that $X = X' + Z + Z'$. Here $\bar{\mathfrak{n}}_P$ is the opposite nilpotent subalgebra of $\mathfrak{l}_\delta$. Since $[E, Z] \in \mathfrak{n}_{P,\mathbb{C}}$ and $[E, Z'] \in \bar{\mathfrak{n}}_{P,\mathbb{C}}$, we have $[E, Z'] = 0$ and thus $Z' = 0$. So $X - X' = Z \in (\mathfrak{h}_{\delta,\mathbb{C}} + \mathfrak{m}_{P,\delta,\mathbb{C}}) \cap \mathfrak{n}_{P,\mathbb{C}} = 0$. Therefore we have $X = X' \in \mathfrak{h}_{\delta,\mathbb{C}} \cap \mathfrak{m}_{P,\delta,\mathbb{C}}$. We thus have $\mathfrak{m}_{P,\delta,\mathbb{C}} = \mathfrak{h}_{\delta,\mathbb{C}} \cap \mathfrak{m}_{P,\delta,\mathbb{C}} + \mathfrak{b}_{\mathfrak{m}_{\delta,\mathbb{C}}}$. Since

$(M_{P,\delta}\cap K_\delta)/(M_{P,\delta}\cap H_\delta) \simeq K_\delta/H_\delta$ by Lemma 2.3 and K_δ/H_δ is connected, $M_{P,\delta}/(M_{P,\delta}\cap H_\delta)$ is also connected. □

THEOREM 2.5. *Suppose that G is reductive. Then $G = H\exp(\mathfrak{a})H$ holds for some semisimple abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$.*

Proof. We prove this claim by the induction on the dimension of G. We may assume that both G and H are connected, that $\mathfrak{g}$ is semisimple and that $(\mathfrak{g}, \mathfrak{h})$ is indecomposable.

Let us suppose that there is no symmetric subalgebra containing $\mathfrak{h}$. Then by Proposition 1.1 (G, H) is either $(\mathrm{SO}(7), \mathrm{G}_{2(-14)})$ or $(\mathrm{G}_{2(-14)}, \mathrm{SU}(3))$ up to coverings. We can apply Proposition 1.4 to these pairs.

Let us suppose that there is a symmetric subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, which contains $\mathfrak{h}$. We denote by S the analytic subgroup of G with Lie algebra $\mathfrak{s}$. If G is non-compact, then we can take $\mathfrak{s}$ to be $\mathfrak{k}$ and apply Lemmas 2.3 and 2.4. If G is compact, then we can apply Lemmas 2.3 and 2.4 to the Riemannian non-compact dual (G^d, S) of the compact symmetric pair (G, S). Hence in either case we obtain a reductive Riemannian Gelfand pair $(G^1, G^1 \cap H)$ with $G^1/(G^1 \cap H)$ connected such that $G^1 \subset S$ and $S = G^1 \cdot H$ hold, where G^1 is the centralizer of a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{s}^\perp$ in S. Since $\dim(G^1)$ is smaller than $\dim(G)$, we can apply the induction hypothesis, and obtain a Cartan decomposition

$$G^1 = (G^1 \cap H)\exp(\mathfrak{a}^1)(G^1 \cap H)$$

for some semisimple abelian subspace $\mathfrak{a}^1$ of $\mathfrak{g}^1$. We put $\mathfrak{a} = \mathfrak{a}^1 + \mathfrak{b}$. Then we have

$$\begin{aligned} G &= S\exp(\mathfrak{b})S = HG_1\exp(\mathfrak{b})G_1H \\ &= HG_1\exp(\mathfrak{b})H = H(G^1\cap H)\exp(\mathfrak{a}^1)(G^1\cap H)\exp(\mathfrak{b})H \\ &= H\exp(\mathfrak{a})H. \end{aligned}$$

□

REMARK 2.6. 1. *Akhiezer* [Ak93] *proved that for a connected reductive linear algebraic group G over an algebraically closed field k of characteristic zero and its spherical subgroups H and L there exists a torus A such that $\mathfrak{h} + \mathfrak{a} + \mathrm{Ad}(g)\mathfrak{l} = \mathfrak{g}$ holds for g in a Zariski open subset.*

2. *In the local structure theorem of Brion, Luna and Vust* [BLV86] *the above "$\exp(\mathfrak{a})$" is not a subgroup but a quotient C_X of a Cartan subgroup. One has $C_X = A_{0,\mathbb{C}} \cdot C_{X_{M_0}}$, $\dim(C_X) = \mathrm{rank}(X)$ and $\dim(C_{X_{M_0}}) = \mathrm{rank}(X_{M_0})$ for $X_{M_0} = M_0/(M_0\cap H)$.*

EXAMPLE 2.7. *Let us see one example of computation of an abelian subspace $\mathfrak{a}$. We let $\mathfrak{g} = \mathfrak{g}_0 = \mathfrak{so}(2n, 1)$ and $\mathfrak{h} = \mathfrak{h}_0 = \mathfrak{u}(n)$, and take $\tilde{\mathfrak{h}}_0 = \mathfrak{so}(2n)$ containing $\mathfrak{h}_0$. We recursively define $\mathfrak{g}_i$ to be $Z_{\tilde{\mathfrak{h}}_{i-1}}(\mathfrak{a}_{i-1})$ for a one dimensional subspace $\mathfrak{a}_{i-1}$ of $\tilde{\mathfrak{h}}_{i-1}^\perp$ and $\mathfrak{h}_{i-1} = \mathfrak{h}_0 \cap \mathfrak{g}_{i-1}$, where $\tilde{\mathfrak{h}}_{i-1}$ is a symmetric subalgebra of $\mathfrak{g}_{i-1}$ containing $\mathfrak{h}_{i-1}$ $(1 \le i \le n)$. Then we can take $\mathfrak{a}$ as $\sum_{i=0}^{n-1} \mathfrak{a}_i$.*

Let us see a matrix form of $\mathfrak{a}$. We realize $\mathfrak{g}_0 = \mathfrak{so}(2n, 1)$ as

$$\mathfrak{g}_0 = \{X \in \mathfrak{sl}(2n+1, \mathbb{C}) : {}^tXJ_{2n+1} + J_{2n+1}X = O, {}^t\overline{X} + I_{n,1,n}XI_{n,1,n} = O\}, \tag{2.1}$$

where ^{t} *stands for the transpose,* $\bar{\cdot}$ *for the complex conjugate,* $J_{2n+1} = \sum_{i=1}^{2n+1} E_{i,2n+2-i}$ *and* $I_{n,1,n} = \sum_{i=1}^{2n+1} E_{ii} - 2E_{n+1,n+1}$ *with* $E_{i,j}$ *the* (i,j)*-th matrix unit. We set* $\mathfrak{h}_0 = \mathfrak{u}(n)$ *as the fixed points subspace of* μ *defined by* $X \overset{\mu}{\mapsto} I_{\sqrt{-1}} X I_{-\sqrt{-1}}$ *for* $X \in \mathfrak{g}_0$*, where* $I_{\pm\sqrt{-1}} = (\pm \sum_{i=1}^{n} \sqrt{-1}E_{ii}) + E_{n+1,n+1} + (\mp \sum_{i=n+2}^{2n+1} \sqrt{-1}E_{ii})$*. We take* $\tilde{\mathfrak{h}}_0 = \mathfrak{so}(2n)$ *as the fixed points subspace of* τ_0 *defined by* $X \overset{\tau_0}{\mapsto} I_{n,1,n} X I_{n,1,n}$ *for* $X \in \mathfrak{g}_0$ *and a one-dimensional subspace* $\mathfrak{a}_0 = \mathbb{R}(E_{1,n+1} + E_{n+1,1} - E_{n+1,2n+1} - E_{2n+1,n+1})$ *of* $\tilde{\mathfrak{h}}_0^{\perp}$*. We also define* $(1 \leq i \leq [\frac{n}{2}], 1 \leq j \leq [\frac{n-1}{2}])$

$$\begin{aligned}
\mathfrak{a}_{2i-1} = \mathbb{R}(&E_{i,n+1-i} - E_{n+1-i,i} + E_{i,n+1+i} - E_{n+1+i,i} \\
&- E_{n+1+i,2n+2-i} + E_{2n+2-i,n+1+i} - E_{n+1-i,2n+2-i} + E_{2n+2-i,n+1-i}), \\
\mathfrak{a}_{2j} = \mathbb{R}(&E_{j+1,n+1-j} - E_{n+1-j,j+1} - E_{j+1,n+1+j} + E_{n+1+j,j+1} \\
&- E_{n+1+j,2n+1-j} + E_{2n+1-j,n+1+j} + E_{n+1-j,2n+1-j} - E_{2n+1-j,n+1-j}).
\end{aligned}$$

Then we can take $\mathfrak{a}$ *as* $\sum_{i=0}^{n-1} \mathfrak{a}_i$*.*

3. Induction of spherical function

In this section we show an induction of spherical functions for Riemannian Gelfand pairs. Let $G = U \cdot L$ be a real Lie group and $H \subset L$ a compact subgroup. A smooth function f on G under the normalization $f(e) = 1$ is an elementary H-spherical function if it is an H-biinvariant joint eigenfunction of the ring of G-invariant differential operators on G/H.

We then suppose that (G, H) is a Riemannian Gelfand pair and let $\delta \in \mathfrak{u}^*$. We denote by ρ_0 the half sum of the roots of $\mathfrak{a}_0$ in $\mathfrak{n}_0$. We write $l = n(l)a(l)\kappa(l)$ according to the Iwasawa decomposition $L = N_0 \exp(\mathfrak{a}_0)K$. By Lemma 2.3 we can express G as $G = UL = U(N_0 \exp(\mathfrak{a}_0)K) = UN_0 \exp(\mathfrak{a}_0)M_{0,\delta}H = N_0 \exp(\mathfrak{a}_0)(UM_{0,\delta})H$ and thus write $g = u(g)n(g)a(g)m_\delta(g)\eta(g) = n(g)a(g)um_\delta(g)\eta(g)$ with $u(g) \in U$, $m_\delta(g) \in M_{0,\delta}$, $\eta(g) \in H$ and $um_\delta(g) \in U \cdot M_{0,\delta}$. Here we note that this expression is not unique and hence especially $m_\delta(g) \in M_{0,\delta}$ and $um_\delta(g) \in U \cdot M_{0,\delta}$ are only defined up to the right action of $M_{0,\delta} \cap H$ on $M_{0,\delta}$. We put $a^\lambda(\cdot) = e^{\lambda \log a(\cdot)}$ ($\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$) for simplicity. The following is an induction of spherical functions.

Theorem 3.1. *Let* ϕ *be an elementary* $(M_{0,\delta} \cap H)$*-spherical function on* $U \cdot M_{0,\delta}/(M_{0,\delta} \cap H)$*. Then the following integral*

$$E(\phi, \lambda)(g) = \int_H a^{\lambda+\rho_0}(hg)\phi(um_\delta(hg))dh \tag{3.1}$$

is an elementary H*-spherical function on* G/H *for* $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$*.*

Proof. The H-biinvariance of $E(\phi, \lambda)$ is clear, and $E(\phi, \lambda)(e) = 1$ since $\phi(e) = 1$. Let us show that $E(\phi, \lambda)$ is a joint eigenfunction. We identify the ring of left invariant differential operators on the identity component G_e with the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We also identify a right H-invariant function on G with a function on G/H. Since $G_e/(H \cap G_e) \simeq G/H$, it suffices to show that $E(\phi, \lambda)$ is a joint eigenfunction of the H-invariant part $U(\mathfrak{g})^H$ of $U(\mathfrak{g})$. We put $\Phi_{\phi,\lambda}(g) = a^{\lambda+\rho_0}(g)\phi(um_\delta(g))$ and take any $D \in U(\mathfrak{g})$.

Since $\mathfrak{g} = \mathfrak{n}_0 + (\mathfrak{a}_0 + \mathfrak{m}_{0,\delta} + \mathfrak{u}) + \mathfrak{h}$, we have $U(\mathfrak{g}) = \mathfrak{n}_0 U(\mathfrak{g}) + U(\mathfrak{a}_0) \cdot U(\mathfrak{m}_{0,\delta} + \mathfrak{u}) + U(\mathfrak{g})\mathfrak{h}$. Thus we can write D as $D = X + Y + Z$ with $X \in \mathfrak{n}_0 U(\mathfrak{g})$, $Y \in U(\mathfrak{g})\mathfrak{h}$, $Z \in U(\mathfrak{a}_0) \cdot U(\mathfrak{m}_{0,\delta} + \mathfrak{u})$. Let us show that $(D\Phi_{\phi,\lambda})|_{UA_0M_{0,\delta}} = Z(\Phi_{\phi,\lambda}|_{UA_0M_{0,\delta}})$. For the proof, we check $(DT\Phi_{\phi,\lambda})(uam) = (XD\Phi_{\phi,\lambda})(uam) = 0$ for any $T \in \mathfrak{h}$, $X \in \mathfrak{n}_0$, $u \in U$, $a \in A_0$ and $m \in M_{0,\delta}$. First, $(DT\Phi_{\phi,\lambda})(uam) = 0$ is clear since $T\Phi_{\phi,\lambda} \equiv 0$. Next, we have

$$\begin{aligned}
(XD\Phi_{\phi,\lambda})(uam) &= \lim_{t\to 0} \frac{1}{t}(D\Phi_{\phi,\lambda}(uam \exp(tX)) - D\Phi_{\phi,\lambda}(uam)) \\
&= \lim_{t\to 0} \frac{1}{t}(D\Phi_{\phi,\lambda}(\exp(t \operatorname{Ad}(am)X)uam) - D\Phi_{\phi,\lambda}(uam)) \\
&= \lim_{t\to 0} \frac{1}{t}(D\Phi_{\phi,\lambda}(uam) - D\Phi_{\phi,\lambda}(uam)) \\
&= 0.
\end{aligned}$$

Now we suppose that D belongs to $U(\mathfrak{g})^H$. We can take Z from $U(\mathfrak{a}_0) \cdot U(\mathfrak{m}_{0,\delta} + \mathfrak{u})^{M_{0,\delta}\cap H}$ in the above since $\mathfrak{n}_0 U(\mathfrak{g})$, $U(\mathfrak{a}_0) \cdot U(\mathfrak{m}_{0,\delta} + \mathfrak{u})$ and $U(\mathfrak{g})\mathfrak{h}$ are all $(M_{0,\delta} \cap H)$-stable. Since the $U(\mathfrak{m}_{0,\delta} + \mathfrak{u})^{M_{0,\delta}\cap H}$-action on $a^\lambda(\cdot)|_{UA_0M_{0,\delta}}$ and the $U(\mathfrak{a}_0)$-action on $\phi(um_\delta(\cdot))|_{UA_0M_{0,\delta}}$ are trivial, there is a constant $c_D \in \mathbb{C}$ such that $(D\Phi_{\phi,\lambda})|_{UA_0M_{0,\delta}} = Z(\Phi_{\phi,\lambda}|_{UA_0M_{0,\delta}}) = c_D \Phi_{\phi,\lambda}|_{UA_0M_{0,\delta}}$. Noting that both $D\Phi_{\phi,\lambda}$ and $c_D\Phi_{\phi,\lambda}$ are left N_0-invariant and right H-invariant, we have $D\Phi_{\phi,\lambda} \equiv c_D\Phi_{\phi,\lambda}$. Further,

$$D\int_H \Phi_{\phi,\lambda}(h\cdot)dh = \int_H D\Phi_{\phi,\lambda}(h\cdot)dh = c_D \int_H \Phi_{\phi,\lambda}(h\cdot)dh.$$

Therefore $E(\phi, \lambda)$ is an elementary spherical function. □

REMARK 3.2.
1. *By Lemma 4.13 the elementary spherical function associated to an irreducible admissible H-spherical Hilbert representation of a real reductive group G takes the form (3.1). See also Remark 3.3 below.*
2. *We have $\dim(V^H) \leq 1$ for all irreducible admissible Hilbert representations V of a real reductive Lie group G (see Lemma 4.4).*
3. *The above induction formula can also be deduced from* [Wo07, Proposition 8.5.3].

The following remark is suggested by the referee.

REMARK 3.3. *Any elementary spherical function ϕ is of the form (3.1) if G is reductive. Let f be an H-biinvariant compactly supported continuous function such that $f * \phi$ is non-zero. We take a left K-finite sequence $\{f_i\}_{i\in\mathbb{N}} \subset C_c(H\backslash G/H)$, which converges to f and then $f_i * \phi$ is non-zero for some index i. Hence ϕ is also left K-finite and this implies that the finitely generated module $V = dL(U(\mathfrak{g}))\phi \subset C^\infty(G)$ is left K and $Z(\mathfrak{g})$-finite, and so a Harish-Chandra module, where L stands for the left regular representation of G and $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. We take a $(\mathfrak{g}, K)$-homomorphism $T\colon V \to (\pi_{\sigma,\lambda})_K$ such that $T(\phi) \neq 0$ for some irreducible unitary representation σ of M_0 and $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$. By the argument as in Lemma 4.13 we have $\phi(g) = \int_H a^{\rho_0+\bar{\lambda}}(hg)\phi_{\sigma^*}(m(hg))dh$.*

4. Symmetry of spherical function

In this section, we consider the property of the symmetry of spherical functions, which is known for reductive Riemannian symmetric spaces [He94, Chapter III Theorem 1.1 and Lemma 9.2]. Our argument is based on the induction on the dimension. We firstly recall some basic facts for spherical functions.

4.1. **Preliminaries.** Let G be a real Lie group, H a compact subgroup and (π, V) a continuous representation of G on a topological vector space V, that is, the map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. We use the notation $\pi(\cdot)$ also for the action $\pi(f)v = \int_G f(g)\pi(g)v dg$ of the convolution algebra $C_c(G)$ of compactly supported continuous functions on G ($f \in C_c(G)$, $v \in V$). (π, V) is a Hilbert representation if V is a Hilbert space. A Hilbert representation (π, V) is H-spherical if the subspace V^H of H-fixed vectors in V is one-dimensional. A vector $v \in V$ is smooth if the map $G \to V$, $g \mapsto \pi(g)v$ is smooth and we write V^∞ for the space of smooth vectors of V. We denote by $V(\tau)$ the τ-isotopic component of V, that is, the sum of all the H-stable, finite-dimensional subspaces of V, which are in the class $\tau \in \hat{H}$, where $\hat{H}$ is the set of the equivalence classes of irreducible unitary representations of H. If G is reductive, H is maximal compact and $V(\tau)$ is finite-dimensional for each $\tau \in \hat{H}$, then V is admissible.

In the following we let (G, H) be a Riemannian Gelfand pair and K a maximal compact subgroup of the reductive part L of G, which contains H. We write $C_c(H\backslash G/H)$ for the convolution algebra of compactly supported continuous H-biinvariant functions on G. We denote by $\langle \cdot, \cdot \rangle$ the inner product of a Hilbert representation, which is always assumed to be H-invariant.

Lemma 4.1. *Let (π, V) be a Hilbert representation of G. If the H-invariant subspace V^H is at most one-dimensional, then $C_c(H\backslash G/H)$ acts on any H-fixed vector v as scalars.*

Proof. For any $f \in C_c(H\backslash G/H)$, $\pi(f)v$ is H-invariant and thus a scalar multiple of v. □

Lemma 4.2. *Let (π, V) be a Hilbert representation of G such that V^H is one-dimensional and v a non-zero H-fixed vector. The $C_c(H\backslash G/H)$-eigenvalues $\lambda_v(\cdot)$ of v are the same as those of $\langle \pi(g^{-1})v, v\rangle$.*

Here $\langle \pi(g)v, v\rangle$ under the normalization $\langle v, v\rangle = 1$ is the elementary H-spherical function on G/H, associated to V.

Proof. We have for any $f \in C_c(H\backslash G/H)$

$$\begin{aligned}
(f * \langle \pi(\cdot)^{-1}v, v\rangle)(g) &= \int_G f(x)\langle \pi(g^{-1}x)v, v\rangle dx \\
&= \langle \pi(g^{-1}) \int_G f(x)\pi(x)v dx, v\rangle \\
&= \langle \pi(g^{-1})\lambda_v(f)v, v\rangle \\
&= \lambda_v(f)\langle \pi(g^{-1})v, v\rangle.
\end{aligned}$$

□

We note that an H-biinvariant continuous function is an elementary H-spherical function up to scalar multiples if and only if it is a joint eigenfunction of the convolution algebra of H-biinvariant compactly supported continuous functions [Wo07, Theorem 8.3.3].

LEMMA 4.3. *An elementary H-spherical function f on G/H is determined by the eigenvalues of $C_c(H\backslash G/H)$.*

Proof. Suppose that the $C_c(H\backslash G/H)$-eigenvalues $\lambda_{f_1}(\cdot)$ and $\lambda_{f_2}(\cdot)$ of f_1 and f_2 are the same. For any compactly supported continuous function f' on G, we have

$$\begin{aligned}
&\int_G (f_1 - f_2)(g^{-1})f'(g)dg \\
&= \int_G \int_H \int_H (f_1 - f_2)(kg^{-1}h)f'(g)dkdhdg \\
&= \int_G (f_1 - f_2)(g^{-1}) \left(\int_H \int_H f'(hgk)dkdh \right) dg \\
&= \left(\left(\int_H \int_H f'(h\cdot k)dkdh \right) * f_1 \right)(e) - \left(\left(\int_H \int_H f'(h\cdot k)dkdh \right) * f_2 \right)(e) \\
&= \lambda_{f_1} \left(\int_H \int_H f'(h\cdot k)dkdh \right) f_1(e) - \lambda_{f_2} \left(\int_H \int_H f'(h\cdot k)dkdh \right) f_2(e) = 0.
\end{aligned}$$

Hence $f_1 = f_2$. □

4.2. **Induction of the symmetry of spherical function.** In this subsection we show that the symmetry of spherical functions can be induced from those of smaller dimensional cases. Let (G,H) be a Riemannian Gelfand pair and $\delta \in \mathfrak{u}^*$. We let $P = M_PA_PN_P$ be the Langlands decomposition of a parabolic subgroup P of the reductive part L of G such that $P_0 \subset P$ and $A_P \subset A_0$. We denote by ρ_P the half sum of the roots of $\mathfrak{a}_P$ in $\mathfrak{n}_P$. For a Hilbert representation (σ, W_σ) of $U \cdot M_{P,\delta}$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ we denote by $I_{U\cdot P_\delta}(\sigma,\lambda) = \{f \in C^\infty(G, W_\sigma) | f(unamg) = a^{\rho_P+\lambda}\sigma(um)f(g), u \in U, n \in N_P, a \in A_P, m \in M_{P,\delta}\}$ the representation space of the corresponding induced representation $(\pi_{\sigma,\lambda}, I_{U\cdot P_\delta}(\sigma,\lambda))$, where $P_\delta = M_{P,\delta}A_PN_P$. We write $g = n_P(g)a_P(g)um_{P,\delta}(g)\eta(g)$ for $g \in G$ accordingly to the decomposition $G = N_PA_PUM_{P,\delta}H$. We note that $um_{P,\delta}(\cdot)$ and $\eta(\cdot)$ are only defined up to the right $(M_{P,\delta} \cap H)$-action and the left $(M_{P,\delta} \cap H)$-action, respectively. Let $H_{U\cdot P_\delta}(\sigma,\lambda)$ be the Hilbert completion (see [Wa88, Section 1.5.2]) of $I_{U\cdot P_\delta}(\sigma,\lambda)$. We note that $G = UP_\delta H$.

LEMMA 4.4. *Let (π, V) be a Hilbert representation of G. Suppose that we have a G-equivariant continuous embedding of V^∞ into $I_{U\cdot P_\delta}(\sigma,\lambda)$ for some (σ, W_σ) and some λ and that* $\dim(W_\sigma^{H\cap M_{P,\delta}}) \leq 1$. *Then V^H is at most one-dimensional.*

Proof. Since H acts on $(UP_\delta)\backslash G$ transitively and the invariant subspace $W_\sigma^{H\cap M_{P,\delta}}$ is at most one-dimensional, we have $\dim(I_{U\cdot P_\delta}(\sigma,\lambda)^H) \leq 1$ and thus $\dim(V^H) \leq 1$. □

Let us see an induction of spherical functions associated to induced representations.

Lemma 4.5. *We let F be the elementary H-spherical function on G/H, which is associated to $H_{U\cdot P_\delta}(\sigma,\lambda)$ for $\lambda\in\mathfrak{a}_{P,\mathbb{C}}^*$ and an $(M_{P,\delta}\cap H)$-spherical Hilbert representation (σ,W_σ) of $U\cdot M_{P,\delta}$ with $\langle\cdot,\cdot\rangle_\sigma$ its inner product. Then F takes the following form for the $(M_{P,\delta}\cap H)$-spherical function ϕ_σ on $U\cdot M_{P,\delta}/(M_{P,\delta}\cap H)$, associated to (σ,W_σ).*

$$F(g)=\int_H a_P^{\rho_P+\lambda}(hg)\phi_\sigma(um_{P,\delta}(hg))dh.$$

Proof. We let w_σ be an $(M_{P,\delta}\cap H)$-invariant vector of norm 1 in W_σ, and take an H-invariant element $\Phi(g)=a_P^{\rho_P+\lambda}(g)\sigma(um_{P,\delta}(g))w_\sigma$ of $I_{U\cdot P_\delta}(\sigma,\lambda)$. We have

$$\begin{aligned}F(g)&=\int_H\langle\Phi(hg),\Phi(h)\rangle_\sigma dh\\&=\int_H\langle a_P^{\rho_P+\lambda}(hg)\sigma(um_{P,\delta}(hg))w_\sigma,\sigma(um_{P,\delta}(h))w_\sigma\rangle_\sigma dh\\&=\int_H a_P^{\rho_P+\lambda}(hg)\langle\sigma(um_{P,\delta}(hg))w_\sigma,w_\sigma\rangle_\sigma dh\\&=\int_H a_P^{\rho_P+\lambda}(hg)\phi_\sigma(um_{P,\delta}(hg))dh.\end{aligned}$$

□

To obtain the symmetry of spherical functions we use the following.

Lemma 4.6. *Let F be a left $(M_{P,\delta}\cap H)$-invariant continuous function on H. For any $g\in G$ we have*

$$\int_H F(\eta(hg^{-1}))dh=\int_H F(h)a_P^{2\rho_P}(hg)dh.$$

Proof. We obtain this by applying the proof of [Wa88, Lemma 2.4.1]. Here we note that $G=(UP_\delta)H$ and $(UP_\delta)\cap H$ is compact ([Wa88, Lemma 0.1.4]) and that $U\times N_P\times M_{P,\delta}\times\exp(\mathfrak{a}_P)\to UP_\delta$ is a diffeomorphism ([Wa88, Lemma 2.2.7]). □

Let us take an abelian subgroup A^δ of L_δ such that $L_\delta=H_\delta A^\delta H_\delta$. For this we note that by Proposition 1.1 a compact Gelfand pair $(\mathfrak{g}_1,\mathfrak{g}_2)$ is either associated with a symmetric pair $(\mathfrak{g}_1,\mathfrak{s})$ with $\mathfrak{g}_2\subset\mathfrak{s}$ or given as a direct sum of a trivial pair, non-symmetric compact polar pairs of rank-one and a pair of compact abelian Lie algebras (see Remark 4.7). Here a pair of a real Lie algebra $\mathfrak{g}$ and a compact Lie subalgebra $\mathfrak{h}$ is polar if the H-action on the orthogonal complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathfrak{g}$ is polar for a connected compact Lie group H with Lie algebra $\mathfrak{h}$. A polar pair $(\mathfrak{g},\mathfrak{h})$ is of rank r if $\mathfrak{h}^\perp$ has an r-dimensional Cartan subspace.

Remark 4.7. *Let $(\mathfrak{g}_1,\mathfrak{g}_2)$ be a compact Gelfand pair. We let $\mathfrak{z}$ be the center of $\mathfrak{g}_1$ and $p:\mathfrak{g}_1\to\mathfrak{g}_1/\mathfrak{z}$ the corresponding projection. If every non-trivial indecomposable factor of $(p(\mathfrak{g}_1),p(\mathfrak{g}_2))$ is a non-symmetric compact polar pair of rank-one, then $(\mathfrak{g}_1,\mathfrak{g}_2)$ is the direct sum of a trivial pair, non-symmetric compact polar pairs of rank-one and a pair of compact abelian Lie algebras. If $(p(\mathfrak{g}_1),p(\mathfrak{g}_2))$ contains a non-trivial indecomposable factor $(\mathfrak{f},\mathfrak{e})$ that is not a non-symmetric compact polar pair of rank-one, then by Proposition 1.1* *there exists a symmetric subalgebra* $\tilde{\mathfrak{e}}$ *of* $\mathfrak{f}$ *containing* $\mathfrak{e}$*. We obtain a symmetric subalgebra* $\mathfrak{s} = p^{-1}(\tilde{\mathfrak{e}}) \oplus (p^{-1}(\mathfrak{f}))^{\perp}$ *of* $\mathfrak{g}_1 = p^{-1}(\mathfrak{f}) \oplus (p^{-1}(\mathfrak{f}))^{\perp}$*, which contains* $\mathfrak{g}_2$*.*

Hence we can take a sequence of quadruples $(\mathfrak{m}_j^\delta, \mathfrak{s}_{j+1}^\delta, \mathfrak{a}_{j+1}^\delta, \mathfrak{n}_{j+1,\mathbb{C}}^\delta)$ $(0 \leq j \leq p)$ such that $\mathfrak{m}_0^\delta = \mathfrak{m}_{0,\delta}$, $\mathfrak{s}_{i+1}^\delta$ is a symmetric or polar subalgebra of $\mathfrak{m}_i^\delta$, which contains both $\mathfrak{m}_i^\delta \cap \mathfrak{h}$ and the center of $\mathfrak{m}_i^\delta$, $\mathfrak{a}_{i+1}^\delta$ is a Cartan subspace of $(\mathfrak{s}_{i+1}^\delta)^\perp \cap \mathfrak{m}_i^\delta$, $\mathfrak{n}_{i+1,\mathbb{C}}^\delta$ is the sum of positive root spaces in $\mathfrak{m}_{i,\mathbb{C}}^\delta$ with respect to $\mathfrak{a}_{i+1,\mathbb{C}}^\delta$, $\mathfrak{m}_{i+1}^\delta = Z_{\mathfrak{s}_{i+1}^\delta}(\mathfrak{a}_{i+1}^\delta)$ $(0 \leq i \leq p)$ and $\mathfrak{s}_{p+1}^\delta = \mathfrak{m}_p^\delta \cap \mathfrak{h}$. Here we can see that $(\mathfrak{m}_{i+1}^\delta, \mathfrak{m}_{i+1}^\delta \cap \mathfrak{h})$ is a Gelfand pair by applying Lemma 2.4 to the Riemannian non-compact dual $(\exp(\mathfrak{s}_{i+1}^\delta) \cdot \exp(\sqrt{-1}((\mathfrak{s}_{i+1}^\delta)^\perp \cap \mathfrak{m}_i^\delta)), \exp(\mathfrak{s}_{i+1}^\delta))$ of $(\exp(\mathfrak{m}_i^\delta), \exp(\mathfrak{s}_{i+1}^\delta))$ $(0 \leq i \leq p-1)$. Then we let ρ_{i+1}^δ be the half sum of roots of $\mathfrak{a}_{i+1,\mathbb{C}}^\delta$ in $\mathfrak{n}_{i+1,\mathbb{C}}^\delta$ $(0 \leq i \leq p)$ and put

$$\mathfrak{a}^\delta = \mathfrak{a}_0 + \sum_{j=1}^{p+1} \mathfrak{a}_j^\delta, M^\delta = Z_{H_\delta}(\mathfrak{a}^\delta), \mathfrak{n}_\mathbb{C}^\delta = \mathfrak{n}_{0,\mathbb{C}} + \sum_{j=1}^{p+1} \mathfrak{n}_{j,\mathbb{C}}^\delta \text{ and } \rho^\delta = \rho_0 + \sum_{j=1}^{p+1} \rho_j^\delta. \tag{4.1}$$

By a decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{u}_\mathbb{C} + \mathfrak{l}_\mathbb{C} = (\mathfrak{u}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^\delta + \mathfrak{a}_\mathbb{C}^\delta) + \mathfrak{h}_\mathbb{C}$, we have a projection $p\colon U(\mathfrak{g}_\mathbb{C}) = U(\mathfrak{u}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^\delta + \mathfrak{a}_\mathbb{C}^\delta) + U(\mathfrak{g}_\mathbb{C})\mathfrak{h}_\mathbb{C} \to U(\mathfrak{u}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^\delta + \mathfrak{a}_\mathbb{C}^\delta)$. We extend a character on $\mathfrak{a}_\mathbb{C}^\delta$ to $\mathfrak{n}_\mathbb{C}^\delta + \mathfrak{a}_\mathbb{C}^\delta$ by letting it act on $\mathfrak{n}_\mathbb{C}^\delta$ trivially, and if δ is a character on $\mathfrak{u}$ then regard $\delta + \lambda$ with $\lambda \in (\mathfrak{a}_\mathbb{C}^\delta)^*$ as a character on $D(G/H)$ through

$$D(G/H) \simeq U(\mathfrak{g}_\mathbb{C})^{H_\mathbb{C}}/U(\mathfrak{g}_\mathbb{C})\mathfrak{h}_\mathbb{C} \xrightarrow{p} U(\mathfrak{u}_\mathbb{C} + \mathfrak{n}_\mathbb{C}^\delta + \mathfrak{a}_\mathbb{C}^\delta) \xrightarrow{\delta+\lambda+\rho^\delta} \mathbb{C}, \tag{4.2}$$

where the last map is shifted from $\delta + \lambda$ by ρ^δ.

We can induce the symmetry of spherical functions on G/H from that on $M_{P,\delta}/(M_{P,\delta} \cap H)$.

Lemma 4.8. *We let* ϕ *be an* $(M_{P,\delta} \cap H)$*-spherical function on* $U \cdot M_{P,\delta}/(M_{P,\delta} \cap H)$ *and set* $F(g) = \int_H a_P^{\rho_P+\lambda}(hg)\phi(um_{P,\delta}(hg))dh$ *for* $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$*. We assume that there exist left* M^δ *and right* $(M_{P,\delta} \cap H)$*-invariant functions* $\mathcal{K}_\pm^\phi$ *defined on some* $(M_{P,\delta} \cap H)$*-biinvariant neighborhood* $O_{U \cdot M_{P,\delta}}$ *of the identity of* $U \cdot M_{P,\delta}$ *such that* $\mathcal{K}_\pm^\phi(e) = 1$*,* $\phi(x^{-1}y) = \int_{M_{P,\delta} \cap H} \mathcal{K}_+^\phi(ly)\mathcal{K}_-^\phi(lx)dl$ *holds for* $x, y \in O_{U \cdot M_{P,\delta}}$*. Then for any elements* $x, y \in G$ *such that* $um_{P,\delta}(Hx)$*,* $um_{P,\delta}(Hy) \subset O_{U \cdot M_{P,\delta}}$*, we have*

$$F(x^{-1}y) = \int_H a_P^{\rho_P+\lambda}(hy)\mathcal{K}_+^\phi(um_{P,\delta}(hy))a_P^{\rho_P-\lambda}(hx)\mathcal{K}_-^\phi(um_{P,\delta}(hx))dh.$$

The kernel functions $\mathcal{K}_\pm^F(\cdot) = a_P^{\rho_P \pm \lambda}(\cdot)\mathcal{K}_\pm^\phi(um_{P,\delta}(\cdot))$ *are left* M^δ *and right* H*-invariant.*

Proof. For any $h \in H$ and $x \in G$, we write $hx = n_P(hx)a_P(hx)um_{P,\delta}(hx)\eta(hx)$ correspondingly to the decomposition $G = N_P A_P U M_{P,\delta} H$. Then for any $y \in G$, we have $\eta(hx)y = n_P(\eta(hx)y)a_P(\eta(hx)y)um_{P,\delta}(\eta(hx)y)\eta(\eta(hx)y)$ and hence obtain

$$\begin{aligned} hxy &= n_P(hx)a_P(hx)um_{P,\delta}(hx)n_P(\eta(hx)y)a_P(\eta(hx)y)um_{P,\delta}(\eta(hx)y)\eta(\eta(hx)y) \\ &= (n_P(hx)(a_P(hx)um_{P,\delta}(hx))n_P(\eta(hx)y)(a_P(hx)um_{P,\delta}(hx))^{-1}) \\ &\quad \times (a_P(hx)a_P((\eta(hx)y))(um_{P,\delta}(hx)um_{P,\delta}(\eta(hx)y))\eta(\eta(hx)y). \end{aligned}$$

We note that $um_{P,\delta}(\cdot)$ and $\eta(\cdot)$ are only defined up to the right $(M_{P,\delta}\cap H)$-action on $M_{P,\delta}$ and left $(M_{P,\delta}\cap H)$-action on H, respectively. From the above expression of hxy we have

$$\begin{aligned}
(4.3)\qquad a_P(hx^{-1}y) &= a_P(hx^{-1})a_P(\eta(hx^{-1})y)\\
&= a_P(\eta(hx^{-1})x)^{-1}a_P(\eta(hx^{-1})y),\\
(4.4)\qquad um_{P,\delta}(hx^{-1}y) &= um_{P,\delta}(hx^{-1})um_{P,\delta}(\eta(hx^{-1})y)\\
&= um_{P,\delta}(\eta(hx^{-1})x)^{-1}um_{P,\delta}(\eta(hx^{-1})y).
\end{aligned}$$

Here we used

$$\begin{aligned}
\eta(hx^{-1})x &= (n_P(hx^{-1})a_P(hx^{-1})um_{P,\delta}(hx^{-1}))^{-1}h\\
&= ((a_P(hx^{-1})um_{P,\delta}(hx^{-1}))^{-1}n_P(hx^{-1})^{-1}a_P(hx^{-1})um_{P,\delta}(hx^{-1}))\\
&\quad\times a_P(hx^{-1})^{-1}um_{P,\delta}(hx^{-1})^{-1}h.
\end{aligned}$$

By substituting the above two identities, we obtain

$$\begin{aligned}
F(x^{-1}y) &= \int_H a_P^{\rho_P+\lambda}(hx^{-1}y)\phi(um_{P,\delta}(hx^{-1}y))dh\\
(4.5)\qquad &= \int_H a_P^{-\rho_P-\lambda}(\eta(hx^{-1})x)a_P^{\rho_P+\lambda}(\eta(hx^{-1})y)\\
&\quad\times\phi(um_{P,\delta}(\eta(hx^{-1})x)^{-1}um_{P,\delta}(\eta(hx^{-1})y))dh.
\end{aligned}$$

Here we note that both $\phi(um_{P,\delta}(hx^{-1}y))$ and $\phi(um_{P,\delta}(\eta(hx^{-1})x)^{-1}um_{P,\delta}(\eta(hx^{-1}y)))$ are well-defined. We put $S'=(M_{P,\delta}\cap H)$ for simplicity and continue as follows.

$$\begin{aligned}
(4.5) &= \int_H a_P^{-\rho_P-\lambda}(\eta(hx^{-1})x)a_P^{\rho_P+\lambda}(\eta(hx^{-1})y)\\
&\quad\times\int_{S'}\mathcal{K}_-^{\phi}(s'um_{P,\delta}(\eta(hx^{-1})x))\mathcal{K}_+^{\phi}(s'um_{P,\delta}(\eta(hx^{-1})y))ds'dh\\
&= \int_H a_P^{-\rho_p-\lambda}(hx)a_P^{\rho_P+\lambda}(hy)a_P^{2\rho_P}(hx)\\
&\quad\times\int_{S'}\mathcal{K}_-^{\phi}(s'um_{P,\delta}(hx))\mathcal{K}_+^{\phi}(s'um_{P,\delta}(hy))ds'dh\\
&= \int_H a_P^{\rho_P-\lambda}(hx)a_P^{\rho_P+\lambda}(hy)\\
&\quad\times\int_{S'}\mathcal{K}_-^{\phi}(um_{P,\delta}(s'hx))\mathcal{K}_+^{\phi}(um_{P,\delta}(s'hy))ds'dh\\
&= \int_H a_P^{\rho_P-\lambda}(hx)a_P^{\rho_P+\lambda}(hy)\mathcal{K}_-^{\phi}(um_{P,\delta}(hx))\mathcal{K}_+^{\phi}(um_{P,\delta}(hy))dh.
\end{aligned}$$

Here we used $\phi(x^{-1}y)=\int_{S'}\mathcal{K}_-^{\phi}(s'x)\mathcal{K}_+^{\phi}(s'y)ds'$ for the first identity, Lemma 4.6 for the second one, the left S'-equivariance of $m_P(\cdot)$ in the third one and the left S'-invariance of $a_P(\cdot)$ and dh for the last. □

4.3. **Non-symmetric polar space.** In this subsection we show the symmetry of spherical functions for non-symmetric compact polar pairs. Let (G, H) be a compact Gelfand pair such that G is connected and $(\mathfrak{g}, \mathfrak{h})$ is either $(\mathfrak{spin}(7), \mathfrak{g}_{2(-14)})$ or $(\mathfrak{g}_{2(-14)}, \mathfrak{su}(3))$. Then accordingly we let (G', H') be a compact Gelfand pair such that G' is connected, $(\mathfrak{g}', \mathfrak{h}')$ is either $(\mathfrak{so}(8), \mathfrak{so}(7))$ or $(\mathfrak{so}(7), \mathfrak{so}(6))$, $G' = GH'$ holds and H coincides with $G \cap H'$. We let $\mathfrak{a}$ be a one-dimensional vector subspace of $\mathfrak{h}^\perp \subset \mathfrak{g}$. We write $M' \subset H'$ for the subgroup stabilizing $X + \mathfrak{h}' \in \mathfrak{g}'/\mathfrak{h}'$ for any $X \in \mathfrak{a}$, and denote by $C(H\backslash G/H)$ and $C(H'\backslash G'/H')$ the convolution algebras of biinvariant continuous functions on G and G', respectively.

LEMMA 4.9. *The map $C(H'\backslash G'/H') \to C(H\backslash G/H)$ induced from the natural map $G/H \to G'/H'$ is an algebra isomorphism.*

Proof. Let us firstly see that the natural map $H\backslash G/H \to H'\backslash G'/H'$ is bijective. Since $G/H \simeq G'/H'$, it suffices to show that the double coset $Hx'H'$ coincides with $H'x'H'$ for any $x' \in G'$. By the Cartan decomposition $G' = H' \exp(\mathfrak{a})H'$, there exist $a \in \exp(\mathfrak{a})$ and $h'_1, h'_2 \in H'$ such that $x' = h'_1 a h'_2$. Also by $H' = HM'$ there exist $h_1 \in H$ and $m'_1 \in M'$ such that $h'_1 = h_1 m'_1$. Hence we have $H'x'H' = H'aH' = (Hm'_1M')aH' = Hm'_1aH' = Hh_1m'_1ah'_2H' = Hh'_1ah'_2H' = Hx'H'$. Thus the map $H\backslash G/H \to H'\backslash G'/H'$ is bijective. This implies that the induced linear map $C(H'\backslash G'/H') \to C(H\backslash G/H)$ is an isomorphism, which we denote by $f \mapsto f'$. Let us see that this is an algebra homomorphism. Let $f_1, f_2 \in C(H'\backslash G'/H')$ and $x \in G$. By $G' = GH'$ we have $(f_1 * f_2)'(x) = \int_{G'} f_1(g')f_2((g')^{-1}x)dg' = \int_G \int_{H'} f_1(gh')f_2((gh')^{-1}x)dgdh' = \int_G f_1(g)f_2(g^{-1}x)dg = ((f_1)' * (f_2)')(x)$. ☐

We can see the following from this lemma and [Wo07, Theorem 8.2.6].

LEMMA 4.10. *Let ϕ be a continuous function on G/H. Then through the isomorphism $G/H \to G'/H'$, ϕ is an elementary H-spherical function on G/H if and only if it is an elementary H'-spherical function on G'/H'.*

We put $M = Z_H(\mathfrak{a})$. The symmetry of spherical functions on G/H comes from that on G'/H'.

COROLLARY 4.11. *There exist a left H and right $H_{\mathbb{C}}$-invariant neighborhood O of $G_{\mathbb{C}}$ and a function $\mathcal{K}_\#(\bullet)$ that is left M and right $H_{\mathbb{C}}$-invariant with respect to $\bullet \in O$ with $\mathcal{K}_\#(e) = 1$, and holomorphic with respect to $\# \in \mathfrak{a}^*_{\mathbb{C}}$, satisfying the following property: For any elementary H-spherical function ϕ on G/H there exists $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, which gives the $D(G/H)$-eigencharacter of ϕ, such that $\phi(x^{-1}y) = \int_H \mathcal{K}_\lambda(hy)\mathcal{K}_{-\lambda}(hx)dh$ holds for x, $y \in G \cap O$.*

Proof. Let ϕ' be an elementary H'-spherical function on G'/H'. Since ϕ' can be expressed as a matrix coefficient of a finite-dimensional representation of G', we can extend ϕ' holomorphically to $G'_{\mathbb{C}}$. We consider the restriction of ϕ' to the non-compact real form $(G')^d \subset G'_{\mathbb{C}}$ such that $((G')^d, H')$ is the Riemannian non-compact dual of (G', H'). Then by the symmetry of spherical functions for the Riemannian symmetric space $(G')^d/H'$ [He94, Chapter III Theorem 1.1], $\phi'(x^{-1}y) = \int_{H'} a^{\lambda+\rho}(h'y)a^{-\lambda+\rho}(h'x)dh'$ $(x, y \in (G')^d)$ for some $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, where $a^\lambda(x) = \exp(\lambda \log a(x))$ for the Iwasawa projection $(G')^d = N'A^dH' \to A^d$, $x \mapsto a(x)$ with $A^d = \exp(\sqrt{-1}\mathfrak{a})$, $x \in (G')^d$ and ρ is the half sum of roots of the nilpotent radical $\mathfrak{n}'$ of a minimal parabolic subalgebra $\mathfrak{p}' = \mathfrak{m}' + \sqrt{-1}\mathfrak{a} + \mathfrak{n}'$ of $(\mathfrak{g}')^d$.

For the analytic subgroups $A_\mathbb{C}$ and $N'_\mathbb{C}$ of $G'_\mathbb{C}$ with Lie algebras $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{n}'_\mathbb{C}$, we take a neighborhood O_A of the identity of $A_\mathbb{C}$ such that the complexified Iwasawa decomposition $N'_\mathbb{C}O_AH'_\mathbb{C} \to N'_\mathbb{C} \times O_A \times H'_\mathbb{C}$, $g \mapsto (n'(g), a(g), h'(g))$ is well-defined and biholomorphic [KS04, Proposition 1.3]. We then take an H'-conjugation invariant connected neighborhood O' of the identity of $G'_\mathbb{C}$, which is contained in $N'_\mathbb{C}O_AH'_\mathbb{C}$, and put $O_{G'} = O' \cdot H'_\mathbb{C}$. The kernel functions $a^{\lambda+\rho}(\cdot)$ and $a^{-\lambda+\rho}(\cdot)$ are extended holomorphically to this left H' and right $H'_\mathbb{C}$-invariant neighborhood $O_{G'}$.

Since $a(\cdot)$ is left M'-invariant and H' coincides with $M' \cdot H$, the equality $\phi'(x^{-1}y) = \int_{H'} a^{\lambda+\rho}(h'y)a^{-\lambda+\rho}(h'x)dh'$ can be rewritten as $\phi'(x^{-1}y) = \int_H a^{\lambda+\rho}(hy)a^{-\lambda+\rho}(hx)dh$ for $x, y \in O_{G'}$. We put $O = G_\mathbb{C} \cap O_{G'}$. Then the restriction $\phi = \phi'|_G$ is an elementary H-spherical function on G/H by Lemma 4.10, and ϕ has a kernel function $\mathcal{K}_\lambda = a^{\lambda+\rho}(\cdot)|_O$ such that $\phi(x^{-1}y) = \int_H \mathcal{K}_\lambda(hy)\mathcal{K}_{-\lambda}(hx)dh$ holds for any x, $y \in G \cap O$. The kernel is a left M and right $H_\mathbb{C}$-invariant $D(X)$-eigenfunction. By Lemma 4.10 this implies that any elementary H-spherical function on G/H has such a kernel function. □

Remark 4.12. *Harmonic analysis on a homogeneous space G/H that admits an action of a larger Lie group (an "overgroup") G' is studied in* [Ko17, KK19] *in the context of "hidden symmetry". The above triples $(G, H, G') = (\mathrm{G}_{2(-14)}, \mathrm{SU}(3), \mathrm{SO}(7))$ and $(\mathrm{SO}(7), \mathrm{G}_{2(-14)}, \mathrm{SO}(8))$ are examples of hidden symmetry and Lemma 4.10 also follows from* [KK19, Propositions 6.10.1 and 6.11.1].

4.4. **H-spherical functions for G with U trivial.** In this subsection we show an integral expression of spherical functions for reductive Riemannian Gelfand pairs and see the symmetry of spherical functions for compact pairs. Let (G, H) be a reductive Riemannian Gelfand pair.

Lemma 4.13. *Let F be the elementary H-spherical function on G/H, associated to an irreducible admissible H-spherical Hilbert representation (π, V) of G. Then F takes the following form for some elementary $(M_0 \cap H)$-spherical function ϕ on $M_0/(M_0 \cap H)$ and some $\lambda \in \mathfrak{a}^*_{0,\mathbb{C}}$.*

$$F(g) = \int_H a^{\rho_0+\lambda}(hg)\phi(m(hg))dh.$$

Proof. Let $v \in V^H$ be a unit vector. Let $I_{P_0}(\sigma, \lambda)$ be a non-unitary principal series such that V_K is embedded into $I_{P_0}(\sigma, \lambda)_K$ for some irreducible unitary representation (σ, W_σ) of M_0 and some $\lambda \in \mathfrak{a}^*_{0,\mathbb{C}}$ by the subrepresentation theorem [Wa88, Section 3.8]. The embedding extends to a homomorphism from V^∞ to $I_{P_0}(\sigma, \lambda)$, which we denote by T [Wa92, Corollary 11.5.4]. We take an H-invariant element of $I_{P_0}(\sigma, \lambda)$ as $\Phi(g) = a^{\rho_0+\lambda}(g)\sigma(m(g))w_\sigma$ for an $(M_0 \cap H)$-invariant vector w_σ of norm 1 in W_σ. Since $I_{P_0}(\sigma, \lambda)^H$ is at most one-dimensional, $T(v) = c\Phi$ for some constant c with $|c| = 1$. We write $\phi_\sigma(\cdot) = \langle \sigma(\cdot)w_\sigma, w_\sigma \rangle$. We have by

Lemmas 4.2 and 4.3

$$F(g) = \langle \pi(g)v, v\rangle = \langle \pi_{\sigma,\lambda}(g)c\Phi, c\Phi\rangle = \int_H a^{\rho_0+\lambda}(hg)\phi_\sigma(m(hg))dh. \tag{4.6}$$

□

We write $\phi_{\sigma,\lambda}$ for the right hand side of (4.6). Let $W = W(\mathfrak{g}, \mathfrak{a}_0) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$ be the little Weyl group. For $w \in W$ we define $w\lambda(H) = \lambda(\mathrm{Ad}(w^{-1})Y)$ for $Y \in \mathfrak{a}_{0,\mathbb{C}}$ and $\sigma_w(m) = \sigma(w^{-1}mw)$ for $m \in M_0$. The following Lemma 4.14 and Remark 4.15 are suggested by the referee.

LEMMA 4.14. *We have $\phi_{\sigma_w,w\lambda} = \phi_{\sigma,\lambda}$ for $w \in W$.*

Proof. By the decomposition $K = HM_0$ we take representatives of elements w of W from H and use the same letters to denote them. We take an H-invariant element of $I_{P_0}(\sigma,\lambda)$ as $\Phi_{\lambda,\sigma}(\cdot) = a^{\rho_0+\lambda}(\cdot)\sigma(m(\cdot))v_\sigma$ for an $(M_0 \cap H)$-invariant vector v_σ of norm 1 in W_σ. By the analyticity of $\phi_{\sigma,\lambda}$ in λ we may assume that the real part of λ is sufficiently positive. Then we have the intertwining operator $A_{P_0}(w,\sigma,\lambda)\colon I_{P_0}(\sigma,\lambda) \to I_{P_0}(\sigma_w, w\lambda)$, $F(\cdot) \mapsto \int_{\overline{N_0}\cap w^{-1}N_0 w} F(\bar{n}w^{-1}\cdot)d\bar{n}$ ([Wa92, Lemma 10.1.11]). Since $I_{P_0}(\sigma,\lambda)^H$ and $I_{P_0}(\sigma_w, w\lambda)^H$ are at most one-dimensional by Lemma 4.4, we find $A_{P_0}(w,\sigma,\lambda)(\Phi_{\lambda,\sigma})(\cdot) = c_w(\sigma,\lambda)\Phi_{w\lambda,\sigma_w}(\cdot)$ for some constant $c_w(\sigma,\lambda)$. Integrating on H we find that

$$\begin{aligned} c_w(\sigma,\lambda)\phi_{\sigma_w,w\lambda}(\cdot) &= c_w(\sigma,\lambda)\int_H a^{\rho_0+w\lambda}(h\cdot)\langle \sigma_w(m(h\cdot))v_\sigma, v_\sigma\rangle dh \\ &= \int_H \langle c_w(\sigma,\nu)\Phi_{w\lambda,\sigma_w}(h\cdot), v_\sigma\rangle dh \\ &= \int_H \langle \int_{\overline{N_0}\cap w^{-1}N_0 w} a^{\rho_0+\lambda}(\bar{n}w^{-1}h\cdot)\sigma(m(\bar{n}w^{-1}h\cdot))v_\sigma d\bar{n}, v_\sigma\rangle dh. \end{aligned} \tag{4.7}$$

Here we used (4.6) with $\phi_\sigma(\cdot) = \langle \sigma(\cdot)v_\sigma, v_\sigma\rangle$ for the first identity, $\Phi_{\lambda,\sigma}(\cdot) = a^{\rho_0+\lambda}(\cdot)\sigma(m(\cdot))v_\sigma$ for the second one and $A_{P_0}(w,\sigma,\lambda)(\Phi_{\lambda,\sigma})(\cdot) = c_w(\sigma,\lambda)\Phi_{w\lambda,\sigma_w}(\cdot)$ for the third one. Then we continue as follows.

$$\begin{aligned} &(4.7) \\ &= \int_H \langle \int_{\overline{N_0}\cap w^{-1}N_0 w} a^{\rho_0+\lambda}(\bar{n}w^{-1})\sigma(m(\bar{n}w^{-1}))a^{\rho_0+\lambda}(\eta(\bar{n}w^{-1})h\cdot)\sigma(m(\eta(\bar{n}w^{-1})h\cdot))v_\sigma d\bar{n}, v_\sigma\rangle dh \\ &= \int_H a^{\rho_0+\lambda}(h\cdot)\langle \sigma(m(h\cdot))v_\sigma, \int_{\overline{N_0}\cap w^{-1}N_0 w} a^{\rho_0+\bar{\lambda}}(\bar{n}w^{-1})\sigma(m(\bar{n}w^{-1}))^{-1}v_\sigma d\bar{n}\rangle dh \\ &= \int_H a^{\rho_0+\lambda}(h\cdot)\langle \sigma(m(h\cdot))v_\sigma, d_w(\sigma,\bar{\lambda})v_\sigma\rangle dh \\ &= \overline{d_w(\sigma,\bar{\lambda})}\phi_{\sigma,\lambda}(\cdot) \end{aligned}$$

for some constant $d_w(\sigma,\bar{\lambda})$. Here we used (4.3) and (4.4) for the first one, the unitarity of σ and the left H-invariance of dh for the second one, the fact that $W_\sigma^{M_0\cap H}$ is at most one-dimensional for the third one and the definition of $\phi_{\sigma,\lambda}$ for the last. Since $c_w(\sigma,\lambda)$ is non-zero for generic λ ([Wa92, Corollary 10.4.7]) and $\phi_{\sigma_w,w\lambda}(e) = \phi_{\sigma,\lambda}(e) = 1$, we obtain $\phi_{\sigma_w,w\lambda} = \phi_{\sigma,\lambda}$. □

Remark 4.15. *The Knop homomorphism* $D(X) \simeq S(\mathfrak{c}_X)^{W_X}$ *([Kn94]) suggests that there are more symmetries beyond the ones in the above lemma, where* W_X *is the little Weyl group of* $X_{\mathbb{C}}$.

Thanks to Lemmas 4.8 and 4.13 the induction argument works for the symmetry of spherical functions on compact weakly symmetric spaces.

Lemma 4.16. *Let us suppose that* G *is compact. There exist a left* H *and right* $H_{\mathbb{C}}$*-invariant neighborhood* O *of* $G_{\mathbb{C}}$ *and a* $D(G/H)$*-eigenfunction* $\mathcal{K}_{\#}(\bullet)$ *that is left* M *and right* $H_{\mathbb{C}}$*-invariant with respect to* $\bullet \in O$ *with* $\mathcal{K}_{\#}(e) = 1$*, and holomorphic with respect to* $\# \in \mathfrak{a}_{\mathbb{C}}^*$*, satisfying the following property: For any elementary* H*-spherical function* ϕ *on* G/H *there exists* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$*, which gives the* $D(G/H)$*-eigencharacter of* ϕ*, such that* $\phi(x^{-1}y) = \int_H \mathcal{K}_\lambda(hy)\mathcal{K}_{-\lambda}(hx)dh$ *holds for* x, $y \in G \cap O$.

Proof. We extend F holomorphically to $G_{\mathbb{C}} = G \cdot \exp(\sqrt{-1}\mathfrak{g})$. We use the induction on the length $p+1$ of the sequence of quadruples $(\mathfrak{m}_j, \mathfrak{s}_{j+1}, \mathfrak{a}_{j+1}, \mathfrak{n}_{j+1,\mathbb{C}})$ $(0 \leq j \leq p)$. If $p = 0$, then the lemma follows from Corollary 4.11 for the case of non-symmetric polar pairs of rank-one and from [He94, Chapter III Theorem 1.1] combined with the holomorphy of F and the Iwasawa projection for the case of symmetric pairs (see the proof of Corollary 4.11). Hence we may assume $p \geq 1$.

We put $S_1 = (M_0 \cap H) \cdot \exp(\mathfrak{s}_1)$, $M_1 = Z_{S_1}(\mathfrak{a}_1)$ and $\mathfrak{a}_{M_1} = \sum_{j=2}^{p+1} \mathfrak{a}_j$. We note that $(M_0 \cap H)$ normalizes $\mathfrak{s}_1$ (see Remark 4.17). We take the Riemannian non-compact dual $(G^d, S_1) = (S_1 \cdot \exp(\sqrt{-1}\mathfrak{s}_1^{\perp}), S_1)$ of (G, S_1) and write $F = E(\phi, \lambda_1)$ on G^d for some elementary $(M_1 \cap H)$-spherical function ϕ on $M_1/(M_1 \cap H)$ and some $\lambda_1 \in \mathfrak{a}_{1,\mathbb{C}}^*$ by Lemma 4.13.

We apply the induction hypothesis to M_1. We note that $M = Z_H(\mathfrak{a}_1) \cap Z_H(\mathfrak{a}_{M_1}) = Z_{M_1 \cap H}(\mathfrak{a}_{M_1})$. Then there exist a left $(M_1 \cap H)$ and right $(M_1 \cap H)_{\mathbb{C}}$-invariant neighborhood $O_{M_{1,\mathbb{C}}}$ of the identity of $M_{1,\mathbb{C}}$, and a left M and right $(M_1 \cap H)_{\mathbb{C}}$-invariant $D(M_1/(M_1 \cap H))$-eigenfunction $\mathcal{K}_{\#}^{M_1}(\bullet)$ with $\# \in \mathfrak{a}_{M_1,\mathbb{C}}^*$, which is defined on $O_{M_{1,\mathbb{C}}}$ and depending holomorphically on $\#$, such that $\phi(x^{-1}y) = \int_{M_1 \cap H} \mathcal{K}_{\mu}^{M_1}(ly)\mathcal{K}_{-\mu}^{M_1}(lx)dl$ holds for $\mu \in \mathfrak{a}_{M_1,\mathbb{C}}^*$ giving the $D(M_1/(M_1 \cap H))$-eigencharacter of ϕ $(x, y \in M_1 \cap O_{M_{1,\mathbb{C}}})$. Using $O_{M_{1,\mathbb{C}}}$ let us find a suitable neighborhood of the identity of the complexification $G_{\mathbb{C}}$ of G, on which kernel functions can be defined.

We take a neighborhood $O_{A_{1,\mathbb{C}}}$ of the identity of $A_{1,\mathbb{C}}$ such that the complexified Iwasawa decomposition $N_{1,\mathbb{C}} O_{A_{1,\mathbb{C}}} S_{1,\mathbb{C}} \to N_{1,\mathbb{C}} \times O_{A_{1,\mathbb{C}}} \times S_{1,\mathbb{C}}$ given as $g \mapsto (n_1(g), a_1(g), s_1(g))$ is well-defined and biholomorphic [KS04, Proposition 1.3]. Here $A_{1,\mathbb{C}}$ and $N_{1,\mathbb{C}}$ are the analytic subgroups of $G_{\mathbb{C}}$ whose Lie algebras are $\mathfrak{a}_{1,\mathbb{C}}$ and $\mathfrak{n}_{1,\mathbb{C}}$, respectively. Then we take an H-conjugation invariant connected neighborhood $O_{0,G_{\mathbb{C}}}$ of the identity of $G_{\mathbb{C}}$, which is contained in $N_{1,\mathbb{C}} O_{A_{1,\mathbb{C}}} (O_{M_{1,\mathbb{C}}} \cdot H_{\mathbb{C}})$. Here we note $S_{1,\mathbb{C}} = M_{1,\mathbb{C}} H_{\mathbb{C}}$. Since $a_1(\cdot)$ and $m_1(\cdot)$ can be defined on $O_{G_{\mathbb{C}}} = O_{0,G_{\mathbb{C}}} \cdot H_{\mathbb{C}}$, the function $\mathcal{K}_\lambda(\cdot) = a_1^{\rho_1+\lambda_1}(\cdot)\mathcal{K}_{\mu}^{M_1}(m_1(\cdot))$ with $\lambda = \lambda_1 + \mu$ is holomorphically defined on $O_{G_{\mathbb{C}}}$. Here we note that $m_1(\cdot)$ is only defined up to the right action of $(M_1 \cap H)_{\mathbb{C}}$ on $M_{1,\mathbb{C}}$. The function satisfies $F(x^{-1}y) = \int_H \mathcal{K}_\lambda(hy)\mathcal{K}_{-\lambda}(hx)dh$ for $x, y \in G^d \cap O_{G_{\mathbb{C}}}$ by Lemma 4.8, and thus for $x, y \in O_{G_{\mathbb{C}}}$ by the holomorphy. □

REMARK 4.17. *Let (G,H) be a compact Gelfand pair. Suppose that there exists a symmetric subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, which contains $\mathfrak{h}$. Then H normalizes $\mathfrak{s}$. Indeed, let $u \in N_G(\mathfrak{h})$ and write $u = s\exp(X)$ for some $s \in N_G(\mathfrak{s})$ and $X \in \mathfrak{s}^\perp$ by the polar decomposition with respect to the compact symmetric pair (G_e, S), where S is the analytic subgroup of G with Lie algebra $\mathfrak{s}$. Here we note that $N_G(\mathfrak{s})$ contains the kernel $\mathrm{Ker}(\mathrm{Ad})$ of the adjoint representation Ad of G on $\mathfrak{g}$ and that being of inner type implies $G/\mathrm{Ker}(\mathrm{Ad})$ is connected and hence $G/N_G(\mathfrak{s})$ is connected. We denote by $\mathfrak{l}$ the centralizer of X in $\mathfrak{s}$ and note that $\mathfrak{s} = \mathfrak{l} + \mathfrak{h}$ by applying Lemma 2.3 to the Riemannian non-compact dual $(S \cdot \exp(\sqrt{-1}\mathfrak{s}^\perp), S)$ of (G_e, S). Then we have $\mathrm{Ad}(u)\mathfrak{s} = \mathrm{Ad}(u)(\mathfrak{h} + \mathfrak{l}) = \mathrm{Ad}(u)\mathfrak{h} + \mathrm{Ad}(s\exp(X))\mathfrak{l} = \mathfrak{h} + \mathrm{Ad}(s)\mathfrak{l} \subset \mathfrak{s}$.*

4.5. **H-spherical functions for G with U abelian.** In this subsection we show an integral expression of spherical functions and find the symmetry of spherical functions for Riemannian Gelfand pairs with abelian unipotent radicals. Let (G,H) be a Riemannian Gelfand pair and assume that the unipotent radical U of G is abelian.

LEMMA 4.18. *Let F be the elementary H-spherical function on G/H, associated to an irreducible unitary H-spherical representation (π, V) of G. Then F takes the following form for some $\delta \in \sqrt{-1}\mathfrak{u}^*$ with χ_δ the corresponding character on U, some elementary $(M_{0,\delta} \cap H)$-spherical function ψ on $M_{0,\delta}/(M_{0,\delta} \cap H)$ and some $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$.*

$$F(g) = \int_H \chi_\delta(u(hg)) a^{\rho_0+\lambda}(hg)\psi(m(hg))dh.$$

Proof. By [Ma49, Theorem 3], (π, V) is equivalent to $(\pi_{\delta,\tau}, H_{UL_\delta}(\chi_\delta \otimes \tau))$ for some $\delta \in \sqrt{-1}\mathfrak{u}^*$ and some irreducible unitary representation (τ, W) of L_δ. Since $\dim(V^H) = 1$, W^{H_δ} is non-zero and we have an H_δ-fixed vector $w_\tau \in W$ of norm one. We then take an H-invariant element $\Phi(g) = \chi_\delta(u(g))\tau(l_\delta(g))w_\tau$ of $I_{UL_\delta}(\chi_\delta \otimes \tau)$, where we write $g = u(g)l_\delta(g)\eta(g)$ accordingly to $G = UL_\delta H$. Since Φ is of norm one, we have

$$\begin{aligned} F(g) = \langle \pi_{\delta,\tau}(g)\Phi, \Phi\rangle_{\pi_{\delta,\tau}} &= \int_H \langle \Phi(hg), \Phi(h)\rangle_\tau dh \\ &= \int_H \langle \chi_\delta(u(hg))\tau(l_\delta(hg))w_\tau, w_\tau\rangle_\tau dh. \end{aligned} \tag{4.8}$$

Since (L_δ, H_δ) is a reductive Riemannian Gelfand pair by the condition (B) of Theorem 2.2, we can apply Lemma 4.13 and find $\langle \tau(x)w_\tau, w_\tau\rangle_\tau = \int_{H_\delta} a^{\rho_0+\lambda}(jx)\psi(m_\delta(jx))dj$ for some $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$ and some elementary $(M_{0,\delta} \cap H_\delta)$-spherical function ψ on $M_{0,\delta}/(M_{0,\delta} \cap H_\delta)$. Then we can continue as follows.

$$\begin{aligned} (4.8) &= \int_H \chi_\delta(u(hg)) \int_{H_\delta} a^{\rho_0+\lambda}(jhg)\psi(m_\delta(jhg))djdh \\ &= \int_{H_\delta}\int_H \chi_\delta(u(j^{-1}hg))a^{\rho_0+\lambda}(hg)\psi(m_\delta(hg))dhdj \\ &= \int_{H_\delta}\int_H \chi_\delta(u(hg))a^{\rho_0+\lambda}(hg)\psi(m_\delta(hg))dhdj \\ &= \int_H \chi_\delta(u(hg))a^{\rho_0+\lambda}(hg)\psi(m_\delta(hg))dh. \end{aligned}$$

□

Applying Lemma 4.8 with $M_{P,\delta} = M_{0,\delta}$ and $\phi(\cdot) = \chi_\delta(u(\cdot))\psi(m_\delta(\cdot))$ combined with Lemmas 4.16 and 4.18, we obtain the symmetry of spherical functions for elementary H-spherical functions on G/H as follows.

Let $L\cdot\delta_1, \ldots, L\cdot\delta_k$ be representatives of the orbit types of L on $\sqrt{-1}\mathfrak{u}^*$ [Mo57, Theorem]. An H-biinvariant neighborhood O_{δ_i} of the identity of G can be obtained as $O_{\delta_i} = O_{\delta_i,0} \cdot H$ from an H-conjugation invariant neighborhood $O_{\delta_i,0}$ that is contained in $N_0A_0(UO_{M_{0,\delta_i}})H$ for an $(M_{0,\delta_i} \cap H)$-biinvariant neighborhood $O_{M_{0,\delta_i}}$ of the identity of M_{0,δ_i}. Then $O = \cap_{1\le i\le k} O_{\delta_i}$ is also an H-biinvariant neighborhood of the identity of G. Here we note that any orbit of type $L\cdot\delta_i$ on $\sqrt{-1}\mathfrak{u}^*$ has the stabilizer of the form $h^{-1}L_{\delta_i}h$ for some $h \in H$ by Theorem 2.2 (A), and hence it suffices to consider representatives of the orbit types to find an H-invariant open neighborhood.

Theorem 4.19. *There exists an H-biinvariant neighborhood O of the identity of G such that for any $\delta \in \sqrt{-1}\mathfrak{u}^*$ there exists a function $\mathcal{K}_{\star\delta,\#}(\bullet)$ that are left M^δ and right H-invariant with respect to $\bullet \in O$ with $\mathcal{K}_{\star\delta,\#}(e) = 1$, and analytic with respect to $\star \in \mathbb{R}$ and $\# \in (\mathfrak{a}^\delta_{\mathbb{C}})^*$, satisfying the following property: Let F be the elementary H-spherical function on G/H, associated to an irreducible unitary H-spherical representation (π, V) of G, which corresponds to the orbit $G\cdot\delta$ on $\sqrt{-1}\mathfrak{u}^*$ through Mackey machine with respect to the abelian normal subgroup U. Then there exists $\lambda \in (\mathfrak{a}^\delta_{\mathbb{C}})^*$ such that $\delta+\lambda$ is the $D(G/H)$-eigencharacter of F and $F(x^{-1}y) = \int_H \mathcal{K}_{\delta,\lambda}(hy)\mathcal{K}_{-\delta,-\lambda}(hx)dh$ holds for any x, y in O.*

We show an application of the symmetry of spherical functions to the convolution products. Let F be an elementary H-spherical function associated with an irreducible unitary H-spherical representation (π, V) of G. Then there exist a left M^δ and right H-invariant function $\mathcal{K}_{\pm\delta,\pm\lambda}$ with $\delta \in \sqrt{-1}\mathfrak{u}^*$ and $\lambda \in (\mathfrak{a}^\delta_{\mathbb{C}})^*$, defined on an H-biinvariant neighborhood O of the identity of G such that $F(x^{-1}y) = \int_H \mathcal{K}_{\delta,\lambda}(hy)\mathcal{K}_{-\delta,-\lambda}(hx)dh$ holds for any $x, y \in O$ by Theorem 4.19. For a compactly supported continuous function f on G, whose support is contained in O, we define a function $\widetilde{f}$ on $B^\delta = H/M^\delta$ as follows.

$$\widetilde{f}(b) = \int_G f(x)\mathcal{K}_{-\delta,-\lambda}(b^{-1}x)dx \quad (b \in B^\delta).$$

Corollary 4.20. *For $g \in O$ we have*

$$(f * F)(g) = \int_{B^\delta} \widetilde{f}(b)\mathcal{K}_{\delta,\lambda}(b^{-1}g)db.$$

Proof. We have

$$\begin{aligned}(f*F)(g) &= \int_G f(x)F(x^{-1}g)dx \\ &= \int_G\int_H f(x)\mathcal{K}_{-\delta,-\lambda}(hx)\mathcal{K}_{\delta,\lambda}(hg)dhdx \\ &= \int_G\int_H f(x)\mathcal{K}_{-\delta,-\lambda}(h^{-1}x)\mathcal{K}_{\delta,\lambda}(h^{-1}g)dhdx \\ &= \int_{B^\delta} \widetilde{f}(b)\mathcal{K}_{\delta,\lambda}(b^{-1}g)db.\end{aligned}$$

Here we used $F(x^{-1}y)=\int_H \mathcal{K}_{\delta,\lambda}(hy)\mathcal{K}_{-\delta,-\lambda}(hx)dh$ for the second identity, the fact that H is unimodular for the third one, and the definition of $\widetilde{f}(b)$ and the fact that f is continuous with compact support for the last. □

4.6. **H-spherical functions for G with U non-abelian.** In this section we see that the symmetry property of a spherical function does not hold when the unipotent radical is non-abelian. Let (G,H) be a Riemannian Gelfand pair such that $L=H$ and the unipotent radical U of G is the Heisenberg group $H_n \simeq \mathbb{C}^n+\mathbb{R}$ with $(z,t)\cdot(w,u)=(z+w,u+t-\frac{1}{2}\mathrm{Im}(z\cdot w))$ its multiplication law. Here $\mathbb{C}^n$ is endowed with the standard Hermitian inner product $z\cdot w=\sum_{i=1}^n z_i\bar{w}_i$ for $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)\in\mathbb{C}^n$ and $\mathrm{Im}(\cdot)$ stands for the imaginary part of a complex number.

Proposition 4.21. *We let F be an elementary H-spherical function on G/H of the first type in the sense of* [BJR92, bottom of page 410]. *There do not exist right H-invariant, $D(G/H)$-eigenfunctions $\phi_\pm$ defined on an H-biinvariant neighborhood O of the identity of G such that $F(x^{-1}y)=\int_H \phi_-(hx)\phi_+(hy)dh$ holds for $x,y\in O$.*

Proof. We show the proposition by contradiction. By [BJR92, Remark after Proposition 4.2] F takes the form $F(z,t)=\exp(\sqrt{-1}\lambda t)\exp(-\frac{1}{4}|\lambda|\cdot\|z\|^2)p(|\lambda|^{\frac{1}{2}}z)$ $((z,t)\in H_n)$ for some $\lambda\in\mathbb{R}\setminus\{0\}$ and some polynomial p with $p(0)=1$. Let $x=(l,w,u)$, $y=(k,z,t)\in O\subset G=H\cdot U\simeq H\times(\mathbb{C}^n+\mathbb{R})$. By the H-biinvariance of F we see that

$$\begin{aligned}F(x^{-1}y) &= F(e,z-w,t-u+\frac{1}{2}\mathrm{Im}(w\cdot z)) \\ &= \exp(\sqrt{-1}\lambda(t-u+\frac{1}{2}\mathrm{Im}(w\cdot z)))\exp(-\frac{1}{4}|\lambda|\cdot\|z-w\|^2)p(|\lambda|^{\frac{1}{2}}(z-w)). \qquad (4.9)\end{aligned}$$

On the other hand we have $F(x^{-1}y)=\int_H\phi_-(e,hw,u)\phi_+(e,hz,t)dh$ by the right H-invariance of $\phi_\pm$. We fix (w,u) with $w\neq 0$ and let

$$L=2\sum_{i=1}^n\left((\frac{\partial}{\partial\bar{z}_i}+\frac{\sqrt{-1}z_i}{4}\frac{\partial}{\partial t})(\frac{\partial}{\partial z_i}-\frac{\sqrt{-1}\bar{z}_i}{4}\frac{\partial}{\partial t})+(\frac{\partial}{\partial z_i}-\frac{\sqrt{-1}\bar{z}_i}{4}\frac{\partial}{\partial t})(\frac{\partial}{\partial\bar{z}_i}+\frac{\sqrt{-1}z_i}{4}\frac{\partial}{\partial t})\right)$$

be an invariant differential operator as given in [BJR92, (6.5)]. Since ϕ_+ is an eigenfunction of $D(G/H)$, L acts on it as a scalar-multiplication. However, by applying L to (4.9) we obtain $\exp(\sqrt{-1}\lambda(t-u+\frac{1}{2}\mathrm{Im}(w\cdot z)))\exp(-\frac{1}{4}|\lambda|\cdot\|z-w\|^2)q(z)$ for some polynomial q whose degree is higher than p, a contradiction. □

5. Spherical transform

In this section we let (G, H) be a Riemannian Gelfand pair with abelian unipotent radical. The aim of this section is to show a Fourier transform for (G, H). We firstly recall some facts on the spherical transform. Let (π, V_π) be an irreducible unitary representation of trace class of G and $V_\pi = \bigoplus_i V_{\pi,i}$ the irreducible decomposition as a representation of H. Then its trace is given by

$$\mathrm{Trace}(\pi(F)) = \sum_i \int_G F(g)\mathrm{Trace}(E_{\pi,i}\pi(g)E_{\pi,i})dg$$

for any compactly supported smooth function F on G. Here $E_{\pi,i}$ is the projection onto $V_{\pi,i}$. If F is right H-invariant, we have

$$\begin{aligned}\mathrm{Trace}(\pi(F)) &= \sum_i \int_G \int_H F(gh)\mathrm{Trace}(E_{\pi,i}\pi(g)E_{\pi,i})dhdg \\ &= \sum_i \int_G F(g) \int_H \mathrm{Trace}(E_{\pi,i}\pi(g)\pi(h^{-1})E_{\pi,i})dhdg \\ &= \sum_i \int_G F(g)\mathrm{Trace}\left(E_{\pi,i}\pi(g) \int_H \pi(h^{-1})dhE_{\pi,i}\right) dg. \qquad (5.1)\end{aligned}$$

Since $E_{\pi,i}\pi(g) \int_H \pi(h^{-1})dhE_{\pi,i} \in \mathrm{End}(V_{\pi,i})$ factors through the H-invariant subspace $V_{\pi,i}^H$, V_π^H is non-zero if $\mathrm{Trace}(\pi(F))$ is non-zero.

Let us suppose that F is H-biinvariant and that $\mathrm{Trace}(\pi(F))$ non-zero. Then we have

$$\begin{aligned}(5.1) &= \int_G F(g)\mathrm{Trace}\left(E_{\pi,0} \int_H \pi(h^{-1})dh\pi(g) \int_H \pi(h^{-1})dhE_{\pi,0}\right) dg \\ &= \int_G F(g)\mathrm{Trace}\left(E_{\pi^H}\pi(g)E_{\pi^H}\right) dg \\ &= \int_G F(g)\phi_\pi(g)dg. \qquad (5.2)\end{aligned}$$

Here $V_{\pi,0}$ is the unique H-trivial representation in V_π, E_{π^H} the projection onto the space of H-invariant vectors and $\phi_\pi(g)$ the elementary H-spherical function associated to V_π. For the last identity we used $\mathrm{Trace}(E_{\pi^H}\pi(g)E_{\pi^H}) = \langle \pi(g)v_\pi, v_\pi\rangle_\pi$ for an H-fixed vector v_π of norm 1 in V_π, where $\langle\cdot,\cdot\rangle_\pi$ is the G-invariant inner product of V_π.

By the Plancherel formula for group extensions [KL73, Theorem 2.3] we have for $F = \phi * \phi^*$ with ϕ a compactly supported smooth function, where $\phi^*(x) = \bar{\phi}(x^{-1})$,

$$F(e) = \int_{\sqrt{-1}\mathfrak{u}^*/L} \int_{\widehat{L_\delta}} \mathrm{Trace}(\pi_{\delta,\sigma}(F))d\nu_\delta(\sigma)d\bar{\nu}_{\mathfrak{u}^*}(\bar{\delta}), \qquad (5.3)$$

where the hat $\hat{}$ stands for the unitary dual, $\pi_{\delta,\sigma}$ is the induced representation from $\delta \otimes \sigma$ of G_δ to G, ν_δ the Plancherel measure on $\widehat{L_\delta}$ and $\bar{\nu}_{\mathfrak{u}^*}$ the image of a Lebesgue measure $\nu_{\mathfrak{u}^*}$ on $\sqrt{-1}\mathfrak{u}^*$.

We take a minimal parabolic subgroup $P_0 = M_0A_0N_0$ of L with $\mathfrak{a}_0 \subset \mathfrak{k}^\perp \cap \mathfrak{l}$ a maximal abelian subspace. By Harish-Chandra Plancherel formula and [Wa92, Theorem 14.12.4], we have

$$\nu_\delta = \left(\sum_{(Q,A_Q)\succsim(P_0,A_0)} C(A_Q) \sum_{\omega\in\mathcal{E}_2(M_{Q_\delta})} d(\omega)\mu(\omega,\sqrt{-1}\lambda)\lambda \right) \circ \Psi^{-1} \tag{5.4}$$

outside a ν_δ-measure zero set. Here we use the notation as in [Wa92, Section 13], hence $C(A_Q)$ and $d(\omega)$ are constants, $\mathcal{E}_2(M_{Q_\delta})$ is the set of the equivalence classes of discrete series representations of M_{Q_δ}, $\mu(\omega,\sqrt{-1}\lambda)$ Harish-Chandra μ-function, λ is a (normalized) Lebesgue measure on $\mathfrak{a}_Q^*$, the relation $(Q,A_Q)\succsim(P_0,A_0)$ implies that Q contains the minimal parabolic subgroup P_0 and satisfies $A_Q\subset A_0$, and Ψ associates (ω,λ) for generic λ to the corresponding irreducible unitary representation $\sigma(\omega,\lambda)$ of L_δ.

We denote by $F_{\delta,\omega,\lambda}$ the elementary H-spherical function associated to $\pi_{\delta,\sigma(\omega,\lambda)}$. We put

$$\widehat{F}(\delta,\omega,\lambda) = \int_G F(x)F_{\delta,\omega,\lambda}(x)dx,$$

and by (5.2) we have $\mathrm{Trace}(\pi_{\delta,\omega,\lambda}(F)) = \widehat{F}(\delta,\omega,\lambda)$. Thus we can rewrite (5.3) as follows.

(5.5)

$$F(e) = \int_{\sqrt{-1}\mathfrak{u}^*/L} \sum_{(Q,A_Q)\succsim(P_0,A_0)} C(A_Q) \sum_{\omega\in\mathcal{E}_2^{M_{Q_\delta}\cap H}(M_{Q_\delta})} d(\omega) \int_{\mathfrak{a}_Q^*} \widehat{F}(\delta,\omega,\lambda)\mu(\omega,\sqrt{-1}\lambda)d\lambda d\bar{\nu}_{\mathfrak{u}^*}(\bar{\delta}).$$

Here $\mathcal{E}_2^{M_{Q_\delta}\cap H}(M_{Q_\delta})$ is the set of the equivalence classes of $(M_{Q_\delta}\cap H)$-spherical discrete series representations of M_{Q_δ}.

We let O be an H-biinvariant neighborhood of the identity of G as taken just before Theorem 4.19. By Lemma 4.13 with (G,π,F) taken to be $(M_{Q_\delta},\omega,\phi_\omega)$ and Lemma 4.8 with (G,M_{P_δ},F) taken to be $(M_{Q_\delta},M_{0,\delta},\phi_\omega)$, the elementary spherical function ϕ_ω associated to ω has kernel functions $\mathcal{K}_\pm^\omega(\cdot)$ defined on $m_{Q,\delta}(O)$ such that $\phi_\omega(x^{-1}y) = \int_{M_{Q_\delta}\cap H}\mathcal{K}_+^\omega(hy)\mathcal{K}_-^\omega(hx)dh$. We note that $m_{0,\delta}(m_{Q,\delta}(O)) = m_{0,\delta}(O)$. Then $F_{\delta,\omega,\lambda}$ has the kernel functions $\mathcal{K}_\pm^{\delta,\omega,\lambda}(\cdot) = \chi_{\pm\delta}(u(\cdot))a_Q^{\pm\sqrt{-1}\lambda+\rho_Q}(\cdot)\mathcal{K}_\pm^\omega(m_{Q,\delta}(\cdot))$ by Lemma 4.5 with $(\sigma,W_\sigma) = (\chi_\delta\otimes\omega,\mathbb{C}_\delta\otimes V_\omega)$ and Lemma 4.8 with $\phi(um_{Q,\delta}(\cdot)) = \chi_\delta(u(\cdot))\phi_\omega(m_{Q,\delta}(\cdot))$. We note that $F_{\delta,\omega,\lambda}(x^{-1}y) = \int_H \mathcal{K}_+^{\delta,\omega,\lambda}(hy)\mathcal{K}_-^{\delta,\omega,\lambda}(hx)dh$ holds for $x,y\in O$.

If a compactly supported continuous function f is supported on O, we set

$$\widetilde{f}(b,\delta,\omega,\lambda) = \int_G f(x)\mathcal{K}_-^{\delta,\omega,\lambda}(b^{-1}x)dx$$

for $b\in B^\delta = H/M^\delta$, where $M^\delta = Z_{H_\delta}(\mathfrak{a}^\delta)$ as in (4.1). We obtain the following.

THEOREM 5.1. *We let $f = \phi*\phi^*$ with ϕ a compactly supported smooth function on G. Suppose that f is supported on O. Then we have for $g\in O$,*

$$f(g) = \int_{\sqrt{-1}\mathfrak{u}^*/L} \sum_{(Q,A_Q)\succsim(P_0,A_0)} C(A_Q) \sum_{\omega\in\mathcal{E}_2^{M_{Q_\delta}\cap H}(M_{Q_\delta})} d(\omega) \tag{5.6}$$
$$\times \int_{\mathfrak{a}_Q^*}\int_{B^\delta} \widetilde{f}(b,\delta,\omega,\lambda)\mathcal{K}_+^{\delta,\omega,\lambda}(b^{-1}g)db\,\mu(\omega,\sqrt{-1}\lambda)d\lambda d\bar{\nu}_{\mathfrak{u}^*}(\bar{\delta}).$$

Proof. Let $g \in O$. We put $F(\cdot) = \int_H f(gh\cdot^{-1})dh$. Then for any $(Q, A_Q) \succsim (P_0, A_0)$, $\omega \in \mathcal{E}_2^{M_{Q_\delta}\cap H}(M_{Q_\delta})$ and $\lambda \in \mathfrak{a}_Q^*$ we have

$$\begin{aligned}\widehat{F}(\delta,\omega,\lambda) &= \int_G F(x)F_{\delta,\omega,\lambda}(x)dx \\ &= \int_G \int_H f(ghx^{-1})dhF_{\delta,\omega,\lambda}(x)dx. \qquad (5.7)\end{aligned}$$

Since elementary H-spherical functions are H-biinvariant, we have

$$(5.7) = \int_G f(gx^{-1})F_{\delta,\omega,\lambda}(x)dx = \int_G f(x^{-1})F_{\delta,\omega,\lambda}(xg)dx = \int_G f(x)F_{\delta,\omega,\lambda}(x^{-1}g)dx.$$

By Corollary 4.20, we find

$$\widehat{F}(\delta,\omega,\lambda) = \int_{B^\delta} \widetilde{f}(b,\delta,\omega,\lambda)\mathcal{K}_+^{\delta,\omega,\lambda}(b^{-1}g)db. \qquad (5.8)$$

Noting that $f(g) = F(e)$ we obtain (5.6) from (5.5) combined with (5.8). ☐

Example 5.2. *We let* $(G, H) = (SO_e(1, 4m), U(2m))$. *By* [Ko94, Corollary 5.6(a)] *and* [Ko98, Remark 6.3(2)] $\mathcal{E}_2^H(G)$ *is non-empty and (5.6) involves discrete spectrum.*

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