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# Optimal singularities of initial functions for solvability of a semilinear parabolic system

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**Abstract.** Let $(u,v)$ be a nonnegative solution to the semilinear parabolic system

$$\text{(P)} \qquad \begin{cases} \partial_t u = D_1\Delta u + v^p, & x\in\mathbf{R}^N,\ \ t>0,\\ \partial_t v = D_2\Delta v + u^q, & x\in\mathbf{R}^N,\ \ t>0,\\ (u(\cdot,0),v(\cdot,0)) = (\mu,\nu), & x\in\mathbf{R}^N, \end{cases}$$

where $D_1$, $D_2>0$, $0<p\le q$ with $pq>1$ and $(\mu,\nu)$ is a pair of nonnegative Radon measures or nonnegative measurable functions in $\mathbf{R}^N$. In this paper we study sufficient conditions on the initial data for the solvability of problem (P) and clarify optimal singularities of the initial functions for the solvability.

## 1. Introduction

We are concerned with the Cauchy problem for a semilinear parabolic system

$$\text{(P)} \qquad \begin{cases} \partial_t u = D_1\Delta u + v^p & \text{in} \quad \mathbf{R}^N\times(0,T),\\ \partial_t v = D_2\Delta v + u^q & \text{in} \quad \mathbf{R}^N\times(0,T),\\ u,v\ge 0 & \text{in} \quad \mathbf{R}^N\times(0,T),\\ (u(\cdot,0),v(\cdot,0)) = (\mu,\nu) & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $N\ge 1$, $0<T\le\infty$, $D_1$, $D_2>0$, $0<p\le q$ with $pq>1$ and $(\mu,\nu)$ is a pair of (nonnegative) Radon measures or measurable functions in $\mathbf{R}^N$. Problem (P) is one of the simplest parabolic systems and it is an example of reaction-diffusion systems describing heat propagation in a two component combustible mixture. It has been studied extensively in many papers from various points of view, see e.g. [1, 3, 4, 5, 7, 14, 16, 19] and [17, Section 32]. The following properties have been already proved in [4], [14] and [16] for the case of $D_1=D_2$.

(1) Let $p\ge 1$ and $r_1,r_2\in(1,\infty)$. Assume

$$\max\{P(r_1,r_2),Q(r_1,r_2)\}\le 2,$$

where

$$P(r_1,r_2) := N\left(\frac{p}{r_2}-\frac{1}{r_1}\right), \qquad Q(r_1,r_2) := N\left(\frac{q}{r_1}-\frac{1}{r_2}\right).$$

Then, for any $(\mu,\nu)\in L^{r_1,\infty}(\mathbf{R}^N)\times L^{r_2,\infty}(\mathbf{R}^N)$, problem (P) possesses a solution for some $T>0$.

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(2) Assume that $\max\{P,Q\} > 2$. Then there exists $(\mu,\nu) \in L^{r_1}(\mathbf{R}^N) \times L^{r_2}(\mathbf{R}^N)$ such that problem (P) possess no solutions for any $T > 0$.

(3) If

$$\frac{q+1}{pq-1} < \frac{N}{2}, \tag{1.1}$$

then problem (P) possesses a global-in-time positive solution provided that $(\mu,\nu) \not\equiv (0,0)$ and both $\|\mu\|_{L^{r_1^*,\infty}(\mathbf{R}^N)}$ and $\|\nu\|_{L^{r_2^*,\infty}(\mathbf{R}^N)}$ are sufficiently small, where

$$r_1^* := \frac{N}{2}\frac{pq-1}{p+1}, \qquad r_2^* := \frac{N}{2}\frac{pq-1}{q+1}. \tag{1.2}$$

On the other hand, if $(p,q)$ does not satisfy (1.1), then problem (P) possesses no global-in-time positive solutions.

(See Definition 1.1 for the definition of the solution to problem (P).) Unfortunately, even for the case of $D_1 = D_2$, statements (1)–(3) are not available to the case of $(q+1)/(pq-1) \ge N/2$ and they are not enough to clarify optimal singularities of the initial functions for the solvability of problem (P). On the other hand, much less is known about the results on the solvability of problem (P) in the case of $D_1 \ne D_2$.

Recently, in [6] the authors of this paper studied qualitative property of the initial traces of the solutions to problem (P) and obtained necessary conditions on the initial data for the existence of solutions. More precisely, they divided problem (P) into the following six cases:

(A) $\displaystyle\frac{q+1}{pq-1} < \frac{N}{2}$;

(B) $\displaystyle\frac{q+1}{pq-1} = \frac{N}{2}$ and $p < q$; (C) $\displaystyle\frac{q+1}{pq-1} = \frac{N}{2}$ and $p = q$;

(D) $\displaystyle\frac{q+1}{pq-1} > \frac{N}{2}$ and $q > 1 + \frac{2}{N}$; (E) $\displaystyle\frac{q+1}{pq-1} > \frac{N}{2}$ and $q = 1 + \frac{2}{N}$;

(F) $\displaystyle\frac{q+1}{pq-1} > \frac{N}{2}$ and $q < 1 + \frac{2}{N}$

(see Figure 1) and proved the following theorem (see [6, Theorems 1.1 and 1.2]).

**Theorem 1.1.** *Let $N \ge 1$, $0 < p \le q$ with $pq > 1$ and $T > 0$. Let $(u,v)$ be a solution to problem (P) in $\mathbf{R}^N \times [0,T)$. Then the initial data $(\mu,\nu)$ satisfies the following:*

(a) *Consider case (A). Then there exists $\gamma_1 > 0$ such that*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma_1 \sigma^{N-\frac{2(p+1)}{pq-1}}, \qquad \sup_{x\in\mathbf{R}^N} \nu(B(x,\sigma)) \le \gamma_1 \sigma^{N-\frac{2(q+1)}{pq-1}},$$

*for $0 < \sigma \le T^{\frac{1}{2}}$;*

(b) *Consider case (B). Then there exists $\gamma_2 > 0$ such that*

$$\sup_{x\in\mathbf{R}^N} \int_0^\sigma \left[\frac{\mu(B(x,\tau))}{\tau^{N-\frac{2(p+1)}{pq-1}}}\right]^q \tau^{-1}\,d\tau + \sup_{x\in\mathbf{R}^N} \nu(B(x,\sigma)) \le \gamma_2 \left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-\frac{1}{pq-1}}$$

*for* $0 < \sigma \le T^{\frac{1}{2}}$;

(c) *Consider case* (C). *Then there exists* $\gamma_3 > 0$ *such that*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) + \sup_{x\in\mathbf{R}^N} \nu(B(x,\sigma)) \le \gamma_3 \left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-\frac{N}{2}}$$

*for* $0 < \sigma \le T^{\frac{1}{2}}$;

(d) *Consider case* (D). *Then there exists* $\gamma_4 > 0$ *such that*

$$\sup_{x\in\mathbf{R}^N} \int_0^{T^{\frac{1}{2}}} \left[\frac{\mu(B(x,\tau))}{\tau^{N-\frac{N+2}{q}}}\right]^q \tau^{-1}\, d\tau + \sup_{x\in\mathbf{R}^N} \nu(B(x,T^{\frac{1}{2}})) \le \gamma_4 T^{\frac{N}{2}-\frac{q+1}{pq-1}};$$

(e) *Consider case* (E). *Then there exists* $\gamma_5 > 0$ *such that*

$$\sup_{x\in\mathbf{R}^N} \int_0^{T^{\frac{1}{2}}} \mu(B(x,\tau))^q \tau^{-1}\, d\tau + \sup_{x\in\mathbf{R}^N} \nu(B(x,T^{\frac{1}{2}})) \le \gamma_5 T^{\frac{N}{2}-\frac{q+1}{pq-1}};$$

(f) *Consider case* (F). *Then there exists* $\gamma_6 > 0$ *such that*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,T^{\frac{1}{2}})) \le \gamma_6 T^{\frac{N}{2}-\frac{p+1}{pq-1}}, \quad \sup_{x\in\mathbf{R}^N} \nu(B(x,T^{\frac{1}{2}})) \le \gamma_6 T^{\frac{N}{2}-\frac{q+1}{pq-1}}.$$

*Here* $\gamma_1, \dots, \gamma_6$ *are positive constants depending only on* $N$, $p$, $q$, $D_1$ *and* $D_2$ *and* $B(x,r) := \{y \in \mathbf{R}^N : |x-y| < r\}$ *for* $x \in \mathbf{R}^N$ *and* $r > 0$.

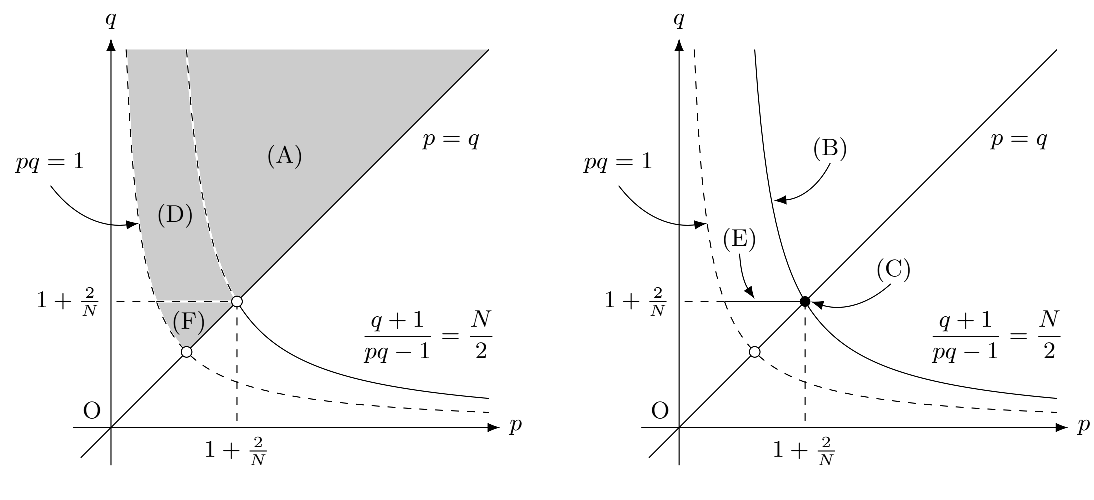


Figure 1

In this paper, as a continuation of [6], we obtain sufficient conditions on the existence of solutions to problem (P). Combining our main results with Theorem 1.1, we clarify optimal singularities of the initial functions for the existence of solutions to problem (P). Our results are independent of whether $D_1 = D_2$ or not.

Following [6], we formulate the definition of a solution to problem (P). Let $G = G(x,t)$ be the fundamental solution to the heat equation in $\mathbf{R}^N$, that is,

$$G(x,t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right). \tag{1.3}$$

For any Radon measure $\mu$ in $\mathbf{R}^N$, we set

$$[S(t)\mu](x) := \int_{\mathbf{R}^N} G(x-y,t)\,d\mu(y).$$

We also write

$$[S(t)\mu](x) = \int_{\mathbf{R}^N} G(x-y,t)\mu(y)\,dy$$

if $\mu$ is a nonnegative measurable function in $\mathbf{R}^N$.

**Definition 1.1.** *Let $\mu$ and $\nu$ be Radon measures in $\mathbf{R}^N$. Let $(u,v)$ be a pair of nonnegative measurable functions in $\mathbf{R}^N \times (0,T)$, where $0 < T \le \infty$. We say that $(u,v)$ is a solution to problem* (P) *in $\mathbf{R}^N \times [0,T)$ if $(u,v)$ satisfies $u(x,t) < \infty$, $v(x,t) < \infty$ and*

$$\begin{aligned} u(x,t) &= [S(D_1 t)\mu](x) + \int_0^t [S(D_1(t-s))v(s)^p](x)\,ds, \\ v(x,t) &= [S(D_2 t)\nu](x) + \int_0^t [S(D_2(t-s))u(s)^q](x)\,ds, \end{aligned} \tag{1.4}$$

*for almost all $x \in \mathbf{R}^N$ and $0 < t < T$. If $(u,v)$ satisfies* (1.4) *with $=$ replaced by $\ge$, then we say that $(u,v)$ is a supersolution to problem* (P).

**Remark 1.1.** (i) *It follows from* [14, Lemma 2.2]*, with a slight modification, that problem* (P) *possesses a solution in $\mathbf{R}^N \times [0,T)$ if and only if problem* (P) *possesses a supersolution in $\mathbf{R}^N \times [0,T)$.*
(ii) *Let $T > 0$ and let $(u,v)$ be a solution to problem* (P) *in $\mathbf{R}^N \times [0,T)$. Let $\alpha > 0$. Set*

$$\hat{u}(x,t) := T^{\frac{p+1}{pq-1}} u(\alpha T^{1/2}x, Tt), \qquad \hat{v}(x,t) := T^{\frac{q+1}{pq-1}} v(\alpha T^{1/2}x, Tt),$$

*for $x \in \mathbf{R}^N$ and $t \in (0,1)$. Then $(\hat{u},\hat{v})$ satisfies*

$$\begin{cases} \partial_t \hat{u} = D_1 \alpha^{-2} \Delta \hat{u} + \hat{v}^p & \text{in } \mathbf{R}^N \times (0,1), \\ \partial_t \hat{v} = D_2 \alpha^{-2} \Delta \hat{v} + \hat{u}^q & \text{in } \mathbf{R}^N \times (0,1), \\ \hat{u}, \hat{v} \ge 0 & \text{in } \mathbf{R}^N \times (0,T), \\ (\hat{u}(\cdot,0), \hat{v}(\cdot,0)) = (\hat{\mu}, \hat{\nu}) & \text{in } \mathbf{R}^N. \end{cases}$$

*Here $\hat{\mu}$ and $\hat{\nu}$ are Radon measures in $\mathbf{R}^N$ such that*

$$\hat{\mu}(K) = \alpha^{-N} T^{\frac{p+1}{pq-1} - \frac{N}{2}} \mu(\alpha T^{\frac{1}{2}} K), \quad \hat{\nu}(K) = \alpha^{-N} T^{\frac{q+1}{pq-1} - \frac{N}{2}} \nu(\alpha T^{\frac{1}{2}} K),$$

*for Borel sets $K$ in $\mathbf{R}^N$.*

We are ready to state one of the main results of this paper. By Theorem 1.2 we clarify optimal singularities of the initial functions for the solvability of problem (P).

**Theorem 1.2.** *Let $N \ge 1$ and $0 < p \le q$ with $pq > 1$.*

(a) *Consider case* (A). *Let*

$$\mu(x) = c_{a,1}|x|^{-\frac{2(p+1)}{pq-1}}\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$
$$\nu(x) = c_{a,2}|x|^{-\frac{2(q+1)}{pq-1}}\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$

*where $c_{a,1},\ c_{a,2} > 0$. Then problem* (P) *possesses no positive local-in-time solutions if either $c_{a,1}$ or $c_{a,2}$ is sufficiently large. On the other hand, problem* (P) *possesses a global-in-time solution if both of $c_{a,1}$ and $c_{a,2}$ are sufficiently small.*

(b) *Consider case* (B). *Let*

$$\mu(x) = c_{b,1}|x|^{-\frac{2(p+1)}{pq-1}}\left|\log\frac{|x|}{2}\right|^{-\frac{p}{pq-1}}\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$
$$\nu(x) = c_{b,2}|x|^{-N}\left|\log\frac{|x|}{2}\right|^{-\frac{1}{pq-1}-1}\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$

*where $c_{b,1},\ c_{b,2} > 0$. Then problem* (P) *possesses no positive local-in-time solutions if either $c_{b,1}$ or $c_{b,2}$ is sufficiently large. On the other hand, problem* (P) *possesses a local-in-time solution if both of $c_{b,1}$ and $c_{b,2}$ are sufficiently small.*

(c) *Consider case* (C). *Let*

$$\mu(x) = c_{c,1}|x|^{-N}\left|\log\frac{|x|}{2}\right|^{-\frac{N}{2}-1}\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$
$$\nu(x) = c_{c,2}|x|^{-N}\left|\log\frac{|x|}{2}\right|^{-\frac{N}{2}-1}\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$

*where $c_{c,1},\ c_{c,2} > 0$. Then problem* (P) *possesses no positive local-in-time solutions if either $c_{c,1}$ or $c_{c,2}$ is sufficiently large. On the other hand, problem* (P) *possesses a local-in-time solution if both of $c_{c,1}$ and $c_{c,2}$ are sufficiently small.*

(d) *Consider case* (D). *Let*

$$\mu(x) = |x|^{-\frac{N+2}{q}}h_1(|x|)\chi_{B(0,1)}(x) \quad in \quad \mathbf{R}^N,$$

*where $h_1$ is a positive increasing function in $(0,1]$ such that $h_1(1) < \infty$ and $r^{-\epsilon}h_1(r)$ is decreasing in $r$ for some $\epsilon > 0$. Let $\nu$ be a Radon measure in $\mathbf{R}^N$. Then problem* (P) *possesses no positive local-in-time solution if either*

$$\int_0^1 h_1(\tau)^q\tau^{-1}\,d\tau = \infty \quad or \quad \sup_{x\in\mathbf{R}^N}\nu(B(x,1)) = \infty.$$

*On the other hand, problem* (P) *possesses a local-in-time solution if*

$$\int_0^1 h_1(\tau)^q\tau^{-1}\,d\tau < \infty \quad and \quad \sup_{x\in\mathbf{R}^N}\nu(B(x,1)) < \infty.$$

(e) *Consider case* (E). *Let*

$$\mu(x) = |x|^{-N} h_2(|x|)\chi_{B(0,1)}(x) \quad \textit{in} \quad \mathbf{R}^N,$$

*where* $h_2$ *is a positive increasing function in* $(0,1]$ *satisfying* $h_2(1) < \infty$. *Let* $\nu$ *be a Radon measure in* $\mathbf{R}^N$. *Then problem* (P) *possesses no positive local-in-time solutions if either*

$$\int_0^1 \left[\int_0^r h_2(\tau)\tau^{-1}\,d\tau\right]^q r^{-1}\,dr = \infty \quad \textit{or} \quad \sup_{x\in\mathbf{R}^N} \nu(B(x,1)) = \infty.$$

*On the other hand, problem* (P) *possesses a local-in-time solution if*

$$\int_0^1 \left[\int_0^r h_2(\tau)\tau^{-1}\,d\tau\right]^q r^{-1}\,dr < \infty \quad \textit{and} \quad \sup_{x\in\mathbf{R}^N} \nu(B(x,1)) < \infty.$$

(f) *Consider case* (F). *Let* $\mu$ *and* $\nu$ *be Radon measures in* $\mathbf{R}^N$. *Then problem* (P) *possesses no positive local-in-time solutions if either*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,1)) = \infty \quad \textit{or} \quad \sup_{x\in\mathbf{R}^N} \nu(B(x,1)) = \infty.$$

*On ther other hand, problem* (P) *possesses a local-in-time solution if*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,1)) < \infty \quad \textit{and} \quad \sup_{x\in\mathbf{R}^N} \nu(B(x,1)) < \infty.$$

The statements of the nonexistence of local-in-time solutions in Theorem 1.2 follow from Theorem 1.1. (See also [6, Corollary 1.2].) The other statements in Theorem 1.2 follow from the results in Sections 3–5, which are the main ingredients of this paper and which are concerned with sufficient conditions on the solvability of problem (P) with singular initial functions.

Optimal singularities of the initial functions for the solvability of problem (P) depend on $(p,q)$ and so do our sufficient conditions on the existence of solutions. Problem (P) in cases (A), (C) and (F) can be regarded as generalizations of the Cauchy problem for

$$\partial_t u = \Delta u + u^p \quad \text{in} \quad \mathbf{R}^N \times (0,T)$$

in the cases $p > 1 + 2/N$, $p = 1 + 2/N$ and $p < 1 + 2/N$, respectively. We construct a supersolution in caces (A), (C) and (F) by applying similar arguments in [10] and [14]. Then, thanks to Remark 1.1 (i), we give sufficient conditions for the existence of solutions for problem (P) (see Theorems 3.1, 3.2 and 3.3 in Section 3). Cases (B), (D) and (E) are specific to the parabolic system and the construction of supersolutions requires delicate decay estimates of $S(t)\mu$ (see lemmas in Section 2) and more complicate arguments than those in cases (A), (C) and (F) (see Theorems 4.1 and 5.1).

The rest of this paper is organized as follows. In Section 2 we recall some properties on uniformly local weak $L^r$ spaces and obtain some decay estimates of $S(t)\varphi$. Section 3 is devoted to cases (A), (C) and (F). In Sections 4 and 5 we consider case (B) and cases (D) and (E), respectively. In Section 6 we apply the results in Sections 3, 4 and 5 to problem (P) and prove Theorem 1.2. This shows the validity of our sufficient conditions given in Sections 3, 4 and 5 for the existence of solutions to problem (P).

## 2. Preliminaries

In this section we introduce some notations and prove several lemmas on $S(t)\varphi$. In what follows, we denote by $C$ a generic constant depending only on $p$, $q$, $D_1$, $D_2$ and $N$, which may change line by line.

We introduce some notations. For any measurable set $\Omega$ in $\mathbf{R}^N$ and $1 \le r \le \infty$, $L^r(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^r(\Omega)}$. In the case of $\Omega = \mathbf{R}^N$, we write $L^r = L^r(\mathbf{R}^N)$ and $\|\cdot\|_{L^r} = \|\cdot\|_{L^r(\mathbf{R}^N)}$ for simplicity. Let $L^r_{\rm loc}$ be the local $L^r$ space in $\mathbf{R}^N$. We introduce the uniformly local Lebesuge space $L^r_{\rm uloc}$ by

$$L^r_{\rm uloc} := \{f \,:\, f \in L^r_{\rm loc} \text{ such that } |||f|||_r < \infty\},$$

where

$$|||f|||_r := \sup_{x\in\mathbf{R}^N} \|f\|_{L^r(B(x,1))}.$$

Note that $|||f|||_r < \infty$ if and only if

$$|||f|||_{r,\rho} := \sup_{x\in\mathbf{R}^N} \|f\|_{L^r(B(x,\rho))} < \infty, \quad \rho > 0.$$

We recall some properties of $S(t)\varphi$. Let $\varphi$ be a nonnegative measurable function in $\mathbf{R}^N$ and $\Phi$ an increasing convex function in $[0,\infty)$. It follows from the Jensen inequality that

$$[S(t)\varphi](x) \le \Phi^{-1}\left([S(t)\Phi(\varphi)](x)\right), \quad x \in \mathbf{R}^N,\ t > 0. \tag{2.1}$$

The following inequalities hold:

$$\|S(t)\varphi\|_{L^\ell} \le Ct^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{\ell}\right)}\|\varphi\|_{L^r}, \quad \varphi \in L^r, \tag{2.2}$$

$$|||S(t)\varphi|||_{\ell,\rho} \le C\left[t^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{\ell}\right)} + \rho^{-N\left(\frac{1}{r}-\frac{1}{\ell}\right)}\right]|||\varphi|||_{r,\rho}, \quad \varphi \in L^r_{\rm uloc}, \tag{2.3}$$

for $t > 0$, $\rho > 0$ and $1 \le r \le \ell \le \infty$ (see e.g., [15, (1.8)] for (2.3)). In particular,

$$||S(t)\varphi|||_\ell \le Ct^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{\ell}\right)}|||\varphi|||_r, \quad \varphi \in L^r_{\rm uloc} \tag{2.4}$$

for $0 < t \le 1$ and $1 \le r \le \ell \le \infty$. Furthermore, we have:

**Lemma 2.1.** *There exists $C > 0$ such that*

$$\|S(t)\mu\|_\infty \le Ct^{-\frac{N}{2}} \sup_{x\in\mathbf{R}^N} \mu(B(x,\sqrt{t})), \quad t > 0, \tag{2.5}$$

*for nonnegative Radon measures $\mu$ in $\mathbf{R}^N$.*

See also [10, Lemma 2.1].

We next obtain estimates of $S(t)\mu$ in an annular domain.

**Lemma 2.2.** *Let $\mu$ be a nonnegative measurable function in $\mathbf{R}^N$. Assume that there exist $a \in (0,N]$ and a nonnegative increasing function $f$ on $(0,1]$ such that $f(1) < \infty$ and*

$$\mu(x) \le |x|^{-a} f(|x|)\chi_{B(0,1)}(x) \quad \text{in } \mathbf{R}^N.$$

*Then, for any $r_* > N/a$, there exists $C > 0$ such that*

$$\|S(t)\mu\|_{L^{r_*}(B(0,1)\setminus B(0,\sqrt{t}))} \le Ct^{-\frac{N}{2}\left(\frac{a}{N}-\frac{1}{r_*}\right)}g(t) \tag{2.6}$$

*for sufficiently small $t > 0$, where $g$ is a function defined by*

$$g(t) = \begin{cases} f(t^{\frac{1}{6}}) + t^{\frac{ar_*-N}{4r_*}} & \text{if} \quad a < N, \\ f(t^{\frac{1}{6}}) + t^{\frac{Nr_*-N}{4r_*}} + \displaystyle\int_0^{\sqrt{t}} \tau^{-1} f(\tau)\, d\tau & \text{if} \quad a = N. \end{cases} \tag{2.7}$$

**Proof.** Let $t \in (0,1)$ and $|x| \ge \sqrt{t}$. By (1.3) we have

$$\begin{aligned}[S(t)\mu](x) &= (4\pi t)^{-\frac{N}{2}} \left[ \int_{\{|y-x|\le|x|/2\}} e^{-\frac{|x-y|^2}{4t}} \mu(y)\, dy + \int_{\{|y-x|\ge|x|/2\}} e^{-\frac{|x-y|^2}{4t}} \mu(y)\, dy \right] \\ &=: I(x,t) + J(x,t).\end{aligned}$$

Since $|y| \ge |x|/2$ and $|y| \le 3|x|/2$ if $|y-x| \le |x|/2$, we have

$$\begin{aligned} I(x,t) &\le C(4\pi t)^{-\frac{N}{2}} \int_{\{|y-x|\le|x|/2\}} e^{-\frac{|y-x|^2}{4t}}\, dy \cdot |x|^{-a} f\left(\frac{3|x|}{2}\right) \\ &\le C|x|^{-a} f\left(\frac{3|x|}{2}\right) \int_{\{|\eta|\le|x|/2\sqrt{t}\}} e^{-\frac{|\eta|^2}{4}}\, d\eta \le C|x|^{-a} f\left(\frac{3|x|}{2}\right). \end{aligned}$$

Then we obtain

$$\begin{aligned} \int_{\{\sqrt{t}\le|x|\le t^{\frac{1}{4}}\}} |I(x,t)|^{r_*}\, dx &\le C \int_{\{1\le|x|\le t^{-\frac{1}{4}}\}} |x|^{-ar_*} f\left(\frac{3\sqrt{t}|x|}{2}\right)^{r_*} dx \cdot t^{-\frac{ar_*}{2}+\frac{N}{2}} \\ &\le Ct^{-\frac{ar_*}{2}+\frac{N}{2}} f\left(\frac{3t^{\frac{1}{4}}}{2}\right)^{r_*} \int_{\{|x|\ge1\}} |x|^{-ar_*}\, dx \\ &\le Ct^{-\frac{ar_*}{2}+\frac{N}{2}} f(t^{\frac{1}{6}})^{r_*} \int_{\{|x|\ge1\}} |x|^{-ar_*}\, dx \end{aligned} \tag{2.8}$$

for sufficiently small $t > 0$. Note that the last integral converges due to $ar_* > N$. Similarly, we obtain

$$\begin{aligned} \int_{\{t^{\frac{1}{4}}\le|x|\le1\}} |I(x,t)|^{r_*}\, dx &\le Cf(1) \int_{\{1\le|x|\le t^{-\frac{1}{4}}\}} |x|^{-ar*}\, dx \cdot t^{-\frac{ar_*}{4}+\frac{N}{4}} \\ &\le Cf(1)t^{-\frac{ar_*}{2}+\frac{N}{2}} \cdot t^{\frac{ar_*}{4}-\frac{N}{4}} \int_{\{|x|\ge1\}} |x|^{-ar_*}\, dx. \end{aligned} \tag{2.9}$$

Therefore, by (2.8) and (2.9) we have

$$\left[ \int_{\{\sqrt{t}\le|x|\le1\}} |I(x,t)|^{r_*}\, dx \right]^{\frac{1}{r_*}} \le Ct^{-\frac{N}{2}\left(\frac{a}{N}-\frac{1}{r_*}\right)} \left[ f(t^{\frac{1}{6}}) + t^{\frac{ar_*-N}{4r_*}} \right] \tag{2.10}$$

for sufficiently small $t > 0$.

On the other hand, since $|y-x| \ge |x|/2$ if $|x| \ge \sqrt{t}$ and $|y| \le \sqrt{t}/2$, we see that

$$
\begin{aligned}
& J(x,t) = (4\pi t)^{-\frac{N}{2}} \left[ \int_{\{|y| \le \sqrt{t}/2\}} + \int_{\{|y| \ge \frac{\sqrt{t}}{2}\} \cap \{|y-x| \ge |x|/2\}} \right] e^{-\frac{|y-x|^2}{4t}} \mu(y)\, dy \\
& \le (4\pi t)^{-\frac{N}{2}} \int_{\{|y| \le \sqrt{t}/2\}} e^{-\frac{|x|^2}{16t}} \mu(y)\, dy + (4\pi t)^{-\frac{N}{2}} \int_{\{|y| \ge \sqrt{t}/2\} \cap \{|y-x| \ge |x|/2\}} e^{-\frac{|y-x|^2}{4t}} \mu(y)\, dy \\
& =: J_1(x,t) + J_2(x,t).
\end{aligned}
$$

We now divide the proof into two cases $a < N$ and $a = N$.
<u>Case $a < N$</u> : Since $|x| \ge 2|y|$ if $|x| \ge \sqrt{t}$ and $|y| \le \sqrt{t}/2$, we have

$$
\begin{aligned}
J_1(x,t) &\le (4\pi t)^{-\frac{N}{2}} \int_{\{|y| \le \sqrt{t}/2\}} |y|^{-a}\, dy \cdot e^{-\frac{|x|^2}{16t}} f\left(\frac{\sqrt{t}}{2}\right) \\
&\le C t^{-\frac{a}{2}} e^{-\frac{|x|^2}{16t}} f(\sqrt{t}) \int_{\{|y| \le 1/2\}} |y|^{-a}\, dy.
\end{aligned}
$$

The last integral converges due to $a < N$. Then we have

$$
\begin{aligned}
\int_{\{\sqrt{t} \le |x| \le 1\}} |J_1(x,t)|^{r_*}\, dx &\le C t^{-\frac{a r_*}{2}} f(\sqrt{t})^{r_*} \int_{\{|x| \le 1\}} e^{-\frac{r_* |x|^2}{16t}}\, dx \\
&\le C t^{-\frac{a r_*}{2} + \frac{N}{2}} f(\sqrt{t})^{r_*}.
\end{aligned}
\tag{2.11}
$$

On the other hand, since

$$
\mu(y) \le \begin{cases} C t^{-\frac{a}{2}} f(t^{\frac{1}{4}}) \text{ if } \frac{\sqrt{t}}{2} \le |y| \le t^{\frac{1}{4}}, \\ C t^{-\frac{a}{4}} f(1) \ \text{ if } |y| \ge t^{\frac{1}{4}}, \end{cases}
$$

we have

$$
\begin{aligned}
J_2(x,t) &\le C t^{-\frac{N}{2}} \int_{\{|y-x| \ge \frac{|x|}{2}\}} e^{-\frac{|y-x|^2}{4t}}\, dy \cdot \left[ t^{-\frac{a}{2}} f(t^{\frac{1}{4}}) + t^{-\frac{a}{4}} f(1) \right] \\
&\le C t^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|y-x|^2}{8t}}\, dy \cdot \left[ t^{-\frac{a}{2}} f(t^{\frac{1}{4}}) + t^{-\frac{a}{4}} f(1) \right] \cdot e^{-\frac{|x|^2}{32t}} \\
&\le C \left[ t^{-\frac{a}{2}} f(t^{\frac{1}{4}}) + t^{-\frac{a}{4}} f(1) \right] e^{-\frac{|x|^2}{32t}}.
\end{aligned}
$$

Then we see that

$$
\begin{aligned}
& \int_{\{\sqrt{t} \le |x| \le 1\}} |J_2(x,t)|^{r_*}\, dx \\
& \le C \left[ t^{-\frac{a r_*}{2}} f(t^{\frac{1}{4}})^{r_*} + t^{-\frac{a r_*}{4}} f(1)^{r_*} \right] \int_{\{\sqrt{t} \le |x| \le 1\}} e^{-\frac{r_* |x|^2}{32t}}\, dx \\
& \le C \left[ t^{-\frac{a r_*}{2} + \frac{N}{2}} f(t^{\frac{1}{4}})^{r_*} + t^{-\frac{a r_*}{4} + \frac{N}{2}} f(1)^{r_*} \right]
\end{aligned}
\tag{2.12}
$$

for sufficiently small $t > 0$. Therefore, by (2.11) and (2.12) we obtain

$$
\left[ \int_{\{\sqrt{t} \le |x| \le 1\}} |J(x,t)|^{r_*} \right]^{\frac{1}{r_*}} \le C t^{-\frac{N}{2}\left(\frac{a}{N} - \frac{1}{r_*}\right)} \left[ f(t^{\frac{1}{4}}) + t^{\frac{a}{4}} \right]
\tag{2.13}
$$

for sufficiently small $t > 0$.

Case $a = N$ : Since $|x| \geq 2|y|$ if $|x| \geq \sqrt{t}$ and $|y| \leq \sqrt{t}/2$, we have

$$\begin{aligned} J_1(x,t) &\leq (4\pi t)^{-\frac{N}{2}} \int_{\{|y|\leq\sqrt{t}/2\}} |y|^{-N} f(|y|)\,dy \cdot e^{-\frac{|x|^2}{16t}} \\ &\leq Ct^{-\frac{N}{2}} e^{-\frac{|x|^2}{16t}} \int_0^{\frac{\sqrt{t}}{2}} \tau^{-1} f(\tau)\,d\tau \leq Ct^{-\frac{N}{2}} e^{-\frac{|x|^2}{16t}} \int_0^{\sqrt{t}} \tau^{-1} f(\tau)\,d\tau. \end{aligned}$$

Setting

$$\tilde{f}(s) := \int_0^s \tau^{-1} f(\tau)\,d\tau, \qquad 0 < s \leq 1,$$

as in (2.11), we obtain

$$\int_{\{\sqrt{t}\leq|x|\leq1\}} |J_1(x,t)|^{r_*}\,dx \leq Ct^{-\frac{Nr_*}{2}+\frac{N}{2}} \tilde{f}(\sqrt{t})^{r_*}. \tag{2.14}$$

On the other hand, we can derive the same calculation for $J_2$ as in the case $a < N$, and by (2.12) and (2.14) we obtain

$$\left[\int_{\{\sqrt{t}\leq|x|\leq1\}} |J(x,t)|^{r_*}\,dx\right]^{\frac{1}{r_*}} \leq Ct^{-\frac{N}{2}\left(1-\frac{1}{r_*}\right)} \left[f(\sqrt{t}) + \tilde{f}(\sqrt{t}) + t^{\frac{a}{4}}\right] \tag{2.15}$$

for sufficiently small $t > 0$.

Thus, since $f$ is an increasing function in $(0,1]$ and $t^{\frac{1}{4}}, \sqrt{t} < t^{1/6}$ for $0 < t < 1$, by (2.10), (2.13) and (2.15) we obtain inequality (2.6). Thus Lemma 2.2 follows. □

At the end of this section we prove a lemma, which gives an upper bound of some integrals including logarithmic functions.

**Lemma 2.3.** *Let $a > -1$ and $b \in \mathbf{R}$. Then there exists $C > 0$ such that*

$$\int_0^t s^a \left|\log\frac{s}{2}\right|^b ds \leq Ct^{a+1} \left|\log\frac{t}{2}\right|^b, \quad 0 < t < 1.$$

**Proof.** Set

$$I(b,t) := \int_0^t s^a \left|\log\frac{s}{2}\right|^b ds$$

for $0 < t < 1$. If $b < 0$, then we have

$$\int_0^t s^a \left|\log\frac{s}{2}\right|^b ds \leq \left|\log\frac{t}{2}\right|^b \int_0^t s^a\,ds = \frac{1}{a+1} t^{a+1} \left|\log\frac{t}{2}\right|^b \tag{2.16}$$

for $0 < t < 1$. For $b \geq 0$, by integration by parts we have

$$I(b,t) = \int_0^t \left(\frac{1}{a+1}s^{a+1}\right)' \left|\log\frac{s}{2}\right|^b ds = \frac{t^{a+1}}{a+1} \left|\log\frac{t}{2}\right|^b + \frac{b}{a+1} I(b-1) \tag{2.17}$$

for $0 < t < 1$. Repeating the above argument, we see that

$$I(b,t) \le Ct^{a+1}\left|\log\frac{t}{2}\right|^b + CI(b-[b]-1,t)$$

for $0 < t < 1$, where $[b]$ denote the greatest integer less than or equal to $b$. Since $b-[b]-1<0$, by (2.16) and (2.17) we obtain the desired inequality. Thus Lemma 2.3 follows. □

## 3. Cases (A), (C) and (F)

In this section we focus on cases (A), (C) and (F) and obtain sufficient conditions on the existence of solutions to problem (P).

**Theorem 3.1.** *Let $N \ge 1$ and $0 < p \le q$ with $pq > 1$ be in case* (A). *Let*

$$1 < \alpha < \frac{pq+q}{q+1}. \tag{3.1}$$

*Then there exists $\gamma > 0$ such that, if $\mu$ and $\nu$ are nonnegative measurable functions in $\mathbf{R}^N$ and satisfy*

$$\left\|S(t)\mu^{\frac{\alpha(q+1)}{p+1}}\right\|_\infty + \left\|S(t)\nu^\alpha\right\|_\infty \le \gamma t^{-\frac{q+1}{pq-1}\alpha}, \quad 0<t<1, \tag{3.2}$$

*then problem* (P) *possesses a solution in $\mathbf{R}^N \times [0,1)$.*

**Theorem 3.2.** *Let $N \ge 1$ and $0 < p \le q$ with $pq > 1$ be in case* (C). *Let $\beta > 0$ and set $\Phi(\tau) := \tau[\log(e+\tau)]^\beta$ for $\tau \ge 0$. Then there exists $\gamma > 0$ such that, if $\mu$ and $\nu$ are nonnegative measurable functions in $\mathbf{R}^N$ and satisfy*

$$\|S(t)\Phi(\mu)\|_\infty + \|S(t)\Phi(\nu)\|_\infty \le \gamma t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{N}{2}+\beta}, \quad 0<t<1, \tag{3.3}$$

*then problem* (P) *possesses a solution in $\mathbf{R}^N \times [0,1)$.*

**Theorem 3.3.** *Let $N \ge 1$ and $0 < p \le q$ with $pq > 1$ be in case* (F). *Then there exists $\gamma > 0$ such that, if $\mu$ and $\nu$ are Radon measures in $\mathbf{R}^N$ and satisfy*

$$\|S(t)\mu\|_\infty + \|S(t)\nu\|_\infty \le \gamma t^{-\frac{N}{2}}, \quad 0<t<1, \tag{3.4}$$

*then problem* (P) *possesses a solution in $\mathbf{R}^N \times [0,1)$.*

Let $D := \min\{D_1, D_2\}$ and $D' := \max\{D_1, D_2\}$. Due to Remark 1.1 (ii), it suffices to consider the case where

$$T = 1, \qquad 0 < D \le D' = \max\{D_1, D_2\} = 1. \tag{3.5}$$

We construct supersolutions to problem (P) and prove Theorems 3.1, 3.2 and 3.3. It follows that

$$G(x, D_i t) = (4\pi D_i t)^{-\frac{N}{2}}\exp\left(-\frac{|x|^2}{4D_i t}\right) \le D^{-\frac{N}{2}}G(x, D't) = D^{-\frac{N}{2}}G(x,t) \tag{3.6}$$

for $x \in \mathbf{R}^N$ and $t > 0$, where $i \in \{1,2\}$. Let $(\tilde{u}, \tilde{v})$ be a solution to the Cauchy problem

$$\text{(P')} \qquad \begin{cases} \partial_t u = \Delta u + D^{-\frac{N}{2}} v^p & \text{in } \mathbf{R}^N \times (0,1), \\ \partial_t v = \Delta v + D^{-\frac{N}{2}} u^q & \text{in } \mathbf{R}^N \times (0,1), \\ u, v \geq 0 & \text{in } \mathbf{R}^N \times (0,1), \\ (u(0), v(0)) = (\mu_D, \nu_D) & \text{in } \mathbf{R}^N, \end{cases}$$

where $(\mu_D, \nu_D) := D^{-\frac{N}{2}}(\mu, \nu)$. By Definition 1.1, (3.5) and (3.6) we see that

$$\begin{aligned} \tilde{u}(x,t) &= D^{-\frac{N}{2}}[S(t)\mu](x) + D^{-\frac{N}{2}} \int_0^t [S(t-s)\tilde{v}(s)^p](x)\, ds \\ &\geq [S(D_1 t)\tilde{u}(0)](x) + \int_0^t [S(D_1(t-s))\tilde{v}(s)^p](x)\, ds \end{aligned}$$

for almost all $\mathbf{R}^N \times (0,1)$. Similarly, we have

$$\tilde{v}(x,t) \geq [S(D_2 t)\tilde{v}(0)](x) + \int_0^t S(D_2(t-s))\tilde{u}(s)^q](x)\, ds$$

for almost all $\mathbf{R}^N \times (0,1)$. This implies that $(\tilde{u}, \tilde{v})$ is a supersolution to problem (P). By Remark 1.1 (i) we see that problem (P) possesses a solution in $\mathbf{R}^N \times [0,1)$ if there exists a solution to problem (P') in $\mathbf{R}^N \times [0,1)$.

**Proof of Theorem 3.1.** It suffices to construct a supersolution to problem (P') in $\mathbf{R}^N \times [0,1)$. Set

$$\text{(3.7)} \qquad \begin{aligned} & w(x,t) := \left[ S(t)\mu_D^{\frac{\alpha(q+1)}{p+1}} \right](x) + [S(t)\nu_D^\alpha](x), \\ & \overline{u}(x,t) := 2w(x,t)^{\frac{p+1}{\alpha(q+1)}}, \quad \overline{v}(x,t) := 2w(x,t)^{\frac{1}{\alpha}}. \end{aligned}$$

Then

$$\text{(3.8)} \qquad 0 \leq \mu_D(x) \leq \overline{u}(x,0), \quad 0 \leq \nu_D(x) \leq \overline{v}(x,0), \qquad x \in \mathbf{R}^N.$$

Furthermore, it follows from (3.2) that

$$\text{(3.9)} \qquad \|w(t)\|_\infty \leq C\gamma t^{-\frac{q+1}{pq-1}\alpha}, \qquad 0 < t < 1.$$

By the Jensen inequality (see (2.1)), (3.1) and (3.7) we have

$$
\begin{aligned}
&[S(t)\nu_D](x) + D^{-\frac{N}{2}} \int_0^t [S(t-s)\overline{u}(s)^q](x)\,ds \\
&\le w(x,t)^{\frac{1}{\alpha}} + C\int_0^t \left[S(t-s)w(s)^{\frac{pq+q}{\alpha(q+1)}}\right](x)\,ds \\
&\le w(x,t)^{\frac{1}{\alpha}} + C\int_0^t \|w(s)\|_\infty^{\frac{pq+q}{\alpha(q+1)}-1}[S(t-s)w(s)](x)\,ds \qquad (3.10)\\
&\le w(x,t)^{\frac{1}{\alpha}} + Cw(x,t)\int_0^t \|w(s)\|_\infty^{\frac{pq+q}{\alpha(q+1)}-1}\,ds \\
&\le w(x,t)^{\frac{1}{\alpha}} + C\|w(t)\|_\infty^{1-\frac{1}{\alpha}} w(x,t)^{\frac{1}{\alpha}} \int_0^t \|w(s)\|_\infty^{\frac{pq+q}{\alpha(q+1)}-1}\,ds
\end{aligned}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. On the other hand, it follows from $\alpha > 1$ that

$$
-\frac{q+1}{pq-1}\alpha\left(\frac{pq+q}{\alpha(q+1)} - 1\right) = \frac{-pq-q+\alpha(q+1)}{pq-1} > -1.
$$

Then, by (3.9) and (3.10) we have

$$
\begin{aligned}
&[S(t)\nu_D](x) + D^{-\frac{N}{2}} \int_0^t [S(t-s)\overline{u}(s)^q](x)\,ds \\
&\le w(x,t)^{\frac{1}{\alpha}} + C\gamma^{\frac{pq+q}{\alpha(q+1)}-\frac{1}{\alpha}}[t^{-\frac{q+1}{pq-1}\alpha}]^{1-\frac{1}{\alpha}} w(x,t)^{\frac{1}{\alpha}} \int_0^t [s^{-\frac{q+1}{pq-1}\alpha}]^{\frac{pq+q}{\alpha(q+1)}-1}\,ds \\
&\le w(x,t)^{\frac{1}{\alpha}} + C\gamma^{\frac{pq-1}{\alpha(q+1)}} w(x,t)^{\frac{1}{\alpha}}
\end{aligned}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. Taking a sufficiently small $\gamma > 0$ if necessary, we see that

$$
[S(t)\nu_D](x) + D^{-\frac{N}{2}} \int_0^t [S(t-s)\overline{u}(s)^q](x)\,ds \le 2w(x,t)^{\frac{1}{\alpha}} = \overline{v}(x,t) \qquad (3.11)
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$.

Next, taking a sufficiently small $\gamma > 0$ if necessary, we show that

$$
[S(t)\mu_D](x) + D^{-\frac{N}{2}} \int_0^t [S(t-s)\overline{v}(s)^p](x)\,ds \le 2w(x,t)^{\frac{p+1}{\alpha(q+1)}} = \overline{u}(x,t) \qquad (3.12)
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. We consider the case of $p \ge \alpha$. It follows from $\alpha > 1$ that

$$
-\frac{q+1}{pq-1}\alpha\left(\frac{p}{\alpha} - 1\right) = \frac{(q+1)(\alpha-p)}{pq-1} > \frac{(q+1)(1-p)}{pq-1} = -1 + \frac{q-p}{pq-1} \ge -1.
$$

Then, by (2.1), (3.7) and (3.9) we have

$$
\begin{aligned}
&[S(t)\mu_D](x) + D^{-\frac{N}{2}}\int_0^t [S(t-s)\overline{v}(s)^p](x)\,ds \\
&\le w(x,t)^{\frac{p+1}{\alpha(q+1)}} + C\|w(t)\|_\infty^{1-\frac{p+1}{\alpha(q+1)}} w(x,t)^{\frac{p+1}{\alpha(q+1)}} \int_0^t \|w(s)\|_\infty^{\frac{p}{\alpha}-1}\,ds \\
&\le w(x,t)^{\frac{p+1}{\alpha(q+1)}} + C\gamma^{\frac{p}{\alpha}-\frac{p+1}{\alpha(q+1)}}[t^{-\frac{q+1}{pq-1}\alpha}]^{1-\frac{p+1}{\alpha(q+1)}} w(x,t)^{\frac{p+1}{\alpha(q+1)}} \int_0^t [s^{-\frac{q+1}{pq-1}\alpha}]^{\frac{p}{\alpha}-1}\,ds \\
&\le w(x,t)^{\frac{p+1}{\alpha(q+1)}} + C\gamma^{\frac{p}{\alpha}-\frac{p+1}{\alpha(q+1)}} w(x,t)^{\frac{p+1}{\alpha(q+1)}}
\end{aligned}
\tag{3.13}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. Since $p > (p+1)/(q+1)$, taking a sufficiently small $\gamma > 0$ if necessary, we see that

$$
[S(t)\mu_D](x) + D^{-\frac{N}{2}}\int_0^t [S(t-s)\overline{u}(s)^q](x)\,ds \le 2w(x,t)^{\frac{p+1}{\alpha(q+1)}} = \overline{u}(x,t)
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. Thus (3.12) holds in the case of $p \ge \alpha$.

We consider the case of $p < \alpha$. It follows from (2.1) that

$$
\begin{aligned}
&[S(t)\mu_D](x) + D^{-\frac{N}{2}}\int_0^t [S(t-s)\overline{v}(s)^p](x)\,ds \\
&\le w(x,t)^{\frac{p+1}{\alpha(q+1)}} + C\int_0^t S(t-s)w(s)^{\frac{p}{\alpha}}\,ds \\
&\le w(x,t)^{\frac{p+1}{\alpha(q+1)}} + C\int_0^t [S(t-s)w(s)]^{\frac{p}{\alpha}}\,ds \le w(x,t)^{\frac{p+1}{\alpha(q+1)}} + Ctw(x,t)^{\frac{p}{\alpha}}
\end{aligned}
\tag{3.14}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. Since $p > (p+1)/(q+1)$, by (3.9) we have

$$
\begin{aligned}
tw(x,t)^{\frac{p}{\alpha}} &\le t\|w(t)\|_\infty^{\frac{p}{\alpha}-\frac{p+1}{\alpha(q+1)}} w(x,t)^{\frac{p+1}{\alpha(q+1)}} \\
&\le C\gamma^{\frac{p}{\alpha}-\frac{p+1}{\alpha(q+1)}} t[t^{-\frac{q+1}{pq-1}\alpha}]^{\frac{p}{\alpha}-\frac{p+1}{\alpha(q+1)}} w(x,t)^{\frac{p+1}{\alpha(q+1)}} \\
&= C\gamma^{\frac{p}{\alpha}-\frac{p+1}{\alpha(q+1)}} w(x,t)^{\frac{p+1}{\alpha(q+1)}}
\end{aligned}
\tag{3.15}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. By (3.14) and (3.15), taking a sufficiently small $\gamma > 0$ if necessary, we obtain (3.12) in the case of $p < \alpha$. Thus (3.12) holds. Combining (3.8), (3.11) and (3.12), we deduce that $(\overline{u}, \overline{v})$ is a supersolution to problem (P') in $\mathbf{R}^N \times [0,1)$. Thus Theorem 3.1 follows. □

**Proof of Theorem 3.2.** It suffices to construct a supersolution to problem (P') in $\mathbf{R}^N \times [0,1)$. Let $L \ge e$ be such that $\Phi_L(s) := s[\log(L+s)]^\beta$ $(s \ge 0)$ satisfies the following properties:

- $\Phi_L$ is convex in $[0,\infty)$;
- $(0,1) \ni s \mapsto s^{\frac{p-1}{2}}[\log(L+s)]^{-p\beta}$ is monotone increasing.

Consider problem

$$
\begin{cases}
\partial_t w = \Delta w + D^{-\frac{N}{2}} w^{1+\frac{2}{N}}, & x \in \mathbf{R}^N,\ t > 0, \\
w(x,0) = \Phi_L^{-1}\left(\dfrac{1}{2}\Phi_L(2\mu_D) + \dfrac{1}{2}\Phi_L(2\nu_D)\right), & x \in \mathbf{R}^N.
\end{cases}
\tag{3.16}
$$

It follows from (3.3) that

$$\|S(t)\Phi_L(w(0))\|_\infty \le C\gamma t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{\beta-\frac{N}{2}}$$

for $0<t<1$. Taking a sufficiently small $\gamma>0$, by the same arguments as in the proof of [13, Theorem 5.3] (see also the proof of Proposition 4.1 and [10, Theorem 1.5]) we see that problem (3.16) possesses a solution $w$ in $\mathbf{R}^N\times[0,1)$. On the other hand, since $\Phi_L$ is convex, it follows that

$$\Phi_L(\mu_D(x)+\nu_D(x)) = \Phi_L\left(\frac{2\mu_D(x)+2\nu_D(x)}{2}\right) \le \frac{1}{2}(\Phi_L(2\mu_D(x))+\Phi_L(2\nu_D(x)))$$

for $x\in\mathbf{R}^N$. This implies that $(w,w)$ is a supersolution to problem (P') in $\mathbf{R}^N\times[0,1)$. Thus Theorem 3.2 follows. □

**Proof of Theorem 3.3.** Set

$$w(x,t) := [S(t)\mu_D](x)+[S(t)\nu_D](x).$$

It follows from (3.4) that $\|w(t)\|_\infty\le C\gamma t^{-\frac{N}{2}}$ for $0<t<1$. Since $q<1+2/N$, we have

$$\begin{aligned}
&[S(t)\nu_D](x)+D^{-\frac{N}{2}}\int_0^t[S(t-s)(2w(s))^q](x)\,ds\\
&\le w(x,t)+C\int_0^t\|w(s)\|_\infty^{q-1}[S(t-s)w(s)](x)\,ds \qquad (3.17)\\
&\le w(x,t)+C\gamma^{q-1}w(x,t)\int_0^t s^{-\frac{N}{2}(q-1)}\,ds \le w(x,t)+C\gamma^{q-1}w(x,t).
\end{aligned}$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Similarly, since $p<1+2/N$, we have

$$[S(t)\mu_D](x)+D^{-\frac{N}{2}}\int_0^t[S(t-s)(2w(s))^p](x)\,ds \le w(x,t)+C\gamma^{p-1}w(x,t). \qquad (3.18)$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. By (3.17) and (3.18), taking a sufficiently small $\gamma>0$, we see that $(2w,2w)$ is a supersolution to problem (P') in $\mathbf{R}^N\times[0,1)$. Thus Theorem 3.3 follows. □

## 4. Case (B)

In this section we obtain sufficient conditions on the existence of solutions to problem (P) in case (B). We prove the following theorem.

**Theorem 4.1.** *Let $N\ge1$ and $0<p\le q$ with $pq>1$ be in case* (B). *Let $\alpha>0$ and $0<\beta<1/(pq-1)$. Set*

$$\Psi(\tau) := \tau[\log(e+\tau)]^\alpha, \qquad \Phi(\tau) := \tau[\log(e+\tau)]^\beta \quad \textit{for} \quad \tau\ge0.$$

*Let*

$$\frac{q+1}{p+1}<r_*<q.$$

*Then there exists $\gamma>0$ such that, if $\mu$ and $\nu$ are nonnegative measurable functions in $\mathbf{R}^N$ and* *satisfy*

$$
\begin{aligned}
&|||S(t)\Psi(\mu)|||_{r_*} \le \gamma t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r_*}\right)}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}, \quad 0<t<1,\\
&\|S(t)\Phi(\nu)\|_\infty \le \gamma t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta}, \quad 0<t<1,
\end{aligned}
\tag{4.1}
$$

*then problem* (P) *possesses a solution in* $\mathbf{R}^N\times[0,1)$.

Similarly to Section 3, for the proof of Theorem 4.1, it suffices to consider the case where $T=1$ and $D'=1$. Let $(p,q)$ be in the case of (B). Then

$$
-\frac{N}{2}(pq-1)+q=-1. \tag{4.2}
$$

Let $K_1$ and $K_2$ be positive constants such that

$$
K_1a^p+K_1b^p\ge D^{-\frac{N}{2}}(a+b)^p, \qquad K_2a^q+K_2b^q\ge D^{-\frac{N}{2}}(a+b)^q, \tag{4.3}
$$

for $a$, $b\ge 0$. We obtain sufficient conditions on the existence of solutions to the Cauchy problem

$$
\text{(Q)}\qquad \begin{cases}
\partial_t u=\Delta u+K_1v^p & \text{in}\quad \mathbf{R}^N\times(0,1),\\
\partial_t v=\Delta v+K_2u^q & \text{in}\quad \mathbf{R}^N\times(0,1),\\
u,v\ge 0 & \text{in}\quad \mathbf{R}^N\times(0,1),\\
(u(\cdot,0),v(\cdot,0))=(\mu_D,\nu_D) & \text{in}\quad \mathbf{R}^N,
\end{cases}
$$

under the assumption either $\mu=0$ or $\nu=0$, and prove Theorem 4.1.

On the other hand, for any $L\ge e$ and $\lambda>0$, we set

$$
\Lambda(s):=s[\log(e+s)]^\lambda, \quad \Lambda_L(s):=s[\log(L+s)]^\lambda, \quad s\ge 0.
$$

Then

$$
\begin{aligned}
&C^{-1}\Lambda_L(s)\le\Lambda(s)\le C\Lambda_L(s),\\
&0\le\Lambda_L'(s)\le C[\log(L+s)]^\lambda, \quad 0\le\Lambda_L'(\Lambda_L^{-1}(s))\le C[\log(L+s)]^\lambda,\\
&C^{-1}s[\log(L+s)]^{-\lambda}\le\Lambda_L^{-1}(s)\le Cs[\log(L+s)]^{-\lambda},
\end{aligned}
\tag{4.4}
$$

for $s\ge 0$. Furthermore, for any $a>0$ and $b>0$, taking a sufficiently large $L$ if necessary, we have

(a) $\Lambda_L$ is convex in $[0,\infty)$;

(b) the function $(0,1)\ni s\mapsto s^a[\log(L+s)]^{-b}$ is monotone increasing.

We prove the following proposition on the existence of solutions to problem (Q) with $\mu_D=0$.

**Proposition 4.1.** *Assume the same conditions as in Theorem* 4.1 *and* $\mu=0$ *in* $\mathbf{R}^N$. *Then there exists* $\gamma>0$ *such that, if* $\nu$ *is a nonnegative measurable function in* $\mathbf{R}^N$ *satisfying*

$$
\|S(t)\Phi(\nu)\|_\infty\le\gamma t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta}, \quad 0<t<1, \tag{4.5}
$$

*then problem* (Q) *possesses a solution in* $\mathbf{R}^N\times[0,1)$.

**Proof.** Let $L \ge e$ and set

$$v_*(x,t) := \Phi_L^{-1}\left[S(t)\Phi_L(\nu_D)\right],$$

where $\Phi_L(s) := \Lambda_L(s)$ with $\lambda = \beta$. Let $0 < \gamma < 1$ and assume (4.5). It follows from (4.4) and (4.5) that

$$\|S(t)\Phi_L(\nu_D)\|_\infty \le C\gamma t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta} \equiv \gamma\xi(t), \quad 0<t<1. \tag{4.6}$$

Case $p > 1$ : Consider the case of $p > 1$. Taking a sufficiently large $L$ if necessary, we can assume that properties (a) and (b) hold with $a = (p-1)/2$ and $b = \beta p$. Set

$$a(t) := t^{-\frac{N}{2}(p-1)+1}\left|\log\frac{t}{2}\right|^{-\frac{p-1}{pq-1}-\beta}, \quad U(t) := a(t)S(t)\Phi_L(\nu_D), \quad V(t) := 2v_*(t).$$

We show that $(U,V)$ is a supersolution to problem (Q) in $\mathbf{R}^N \times [0,1)$. It follows from (4.2) and (4.6) that

$$\begin{aligned}
&a(s)^q\|S(s)\Phi_L(\nu_D)\|_\infty^{q-1}\\
&\le s^{-\frac{N}{2}(pq-q)+q}\left|\log\frac{s}{2}\right|^{-\frac{pq-q}{pq-1}-\beta q}\cdot(C\gamma)^{q-1}s^{-\frac{N}{2}(q-1)}\left|\log\frac{s}{2}\right|^{-\frac{q-1}{pq-1}+\beta(q-1)}\\
&\le C\gamma^{q-1}s^{-1}\left|\log\frac{s}{2}\right|^{-1-\beta}
\end{aligned} \tag{4.7}$$

for $0 < s < 1$. Furthermore, by (4.4) we have

$$\begin{aligned}
0 \le \frac{\Phi_L(v_*(x,t))}{v_*(x,t)} &= \frac{[S(t)\Phi_L(\nu_D)](x)}{\Phi_L^{-1}([S(t)\Phi_L(\nu_D)](x))} \le C[\log(L+[S(t)\Phi_L(\nu_D)](x))]^\beta\\
&\le C[\log(L+\gamma\xi(t))]^\beta \le C[\log(L+\xi(t))]^\beta \le C\left|\log\frac{t}{2}\right|^\beta
\end{aligned} \tag{4.8}$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. By (4.7) and (4.8) we see that

$$\begin{aligned}
&\int_0^t [S(t-s)U(s)^q](x)\,ds = \int_0^t [S(t-s)a(s)^q(S(s)\Phi_L(\nu_D))^q](x)\,ds\\
&\quad\le [S(t)\Phi_L(\nu_D)](x)\int_0^t a(s)^q\|S(s)\Phi_L(\nu_D)\|_\infty^{q-1}\,ds\\
&\quad\le \left\|\frac{\Phi_L(v_*(t))}{v_*(t)}\right\|_\infty v_*(x,t)\int_0^t a(s)^q\|S(s)\Phi_L(\nu_D)\|_\infty^{q-1}\,ds\\
&\quad\le C\left|\log\frac{t}{2}\right|^\beta v_*(x,t)\int_0^t a(s)^q\|S(s)\Phi_L(\nu_D)\|_\infty^{q-1}\,ds \le C\gamma^{q-1}v_*(x,t)
\end{aligned} \tag{4.9}$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. Taking a sufficiently small $\gamma > 0$ if necessary, by (2.1) and (4.9) we obtain

$$\begin{aligned}
&[S(t)\nu_D](x) + K_2\int_0^t [S(t-s)U(s)^q](x)\,ds\\
&\le v_*(x,t) + CK_2\gamma^{q-1}v_*(x,t) \le 2v_*(x,t) = V(x,t)
\end{aligned} \tag{4.10}$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$.

On the other hand, by property (b) (with $a=(p-1)/2$ and $b=\beta p$), (4.4) and (4.6) we have

$$
\begin{aligned}
0 &\le \frac{v_*(x,t)^p}{[S(t)\Phi_L(\nu_D)](x)} \le C[S(t)\Phi_L(\nu_D)](x)^{p-1}[\log(L+[S(t)\Phi_L(\nu_D)](x))]^{-p\beta}\\
&= C[S(t)\Phi_L(\nu_D)](x)^{\frac{p-1}{2}}[S(t)\Phi_L(\nu_D)](x)^{\frac{p-1}{2}}[\log(L+[S(t)\Phi_L(\nu_D)](x))]^{-p\beta}\\
&\le C[\gamma\xi(t)]^{\frac{p-1}{2}}[\gamma\xi(t)]^{\frac{p-1}{2}}[\log(L+\gamma\xi(t))]^{-p\beta}\\
&\le C[\gamma\xi(t)]^{\frac{p-1}{2}}\xi(t)^{\frac{p-1}{2}}[\log(L+\xi(t))]^{-p\beta}\\
&\le C\gamma^{\frac{p-1}{2}}t^{-\frac{N}{2}(p-1)}\left|\log\frac{t}{2}\right|^{-\frac{p-1}{pq-1}-\beta}
\end{aligned}
\tag{4.11}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Recalling that $p<1+2/N$ and taking a sufficiently small $\gamma$ if necessary, by Lemma 2.3 and (4.11) we have

$$
\begin{aligned}
K_1\int_0^t[S(t-s)V(s)^p](x)\,ds &= 2^pK_1\int_0^t[S(t-s)v_*(s)^p](x)\,ds\\
&\le 2^pK_1\int_0^t\left\|\frac{v_*(s)^p}{S(s)\Phi_L(\nu_D)}\right\|_\infty[S(t-s)S(s)\Phi_L(\nu_D)](x)\,ds\\
&\le C\gamma^{\frac{p-1}{2}}[S(t)\Phi(\nu_D)](x)\int_0^t s^{-\frac{N}{2}(p-1)}\left|\log\frac{s}{2}\right|^{-\frac{p-1}{pq-1}-\beta}ds\\
&\le C\gamma^{\frac{p-1}{2}}t^{-\frac{N}{2}(p-1)+1}\left|\log\frac{t}{2}\right|^{-\frac{p-1}{pq-1}-\beta}[S(t)\Phi_L(\nu_D)](x)\\
&= C\gamma^{\frac{p-1}{2}}a(t)[S(t)\Phi_L(\nu_D)](x)\le a(t)[S(t)\Phi_L(\nu_D)](x)=U(x,t)
\end{aligned}
\tag{4.12}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Therefore, by (4.10) and (4.12) we see that $(U,V)$ is a supersolution to problem (Q) in $\mathbf{R}^N\times[0,1)$ in the case of $\mu=0$. Thus Proposition 4.1 follows in the case of $p>1$.

<u>Case $0<p\le 1$</u> : Consider the case of $0<p\le1$. Let $\delta\in(0,1)$ be such that

$$
\frac{N}{2}p(1-\delta)<1 \quad\text{and}\quad \delta pq>1. \tag{4.13}
$$

Taking a sufficiently large $L$ if necessary, we can assume that properties (a) and (b) hold with $a=(1-\delta)/2$ and $b=\beta$. Set

$$
\begin{aligned}
&\tilde a(t) := t^{-\frac{N}{2}p(1-\delta)+1}\left|\log\frac{t}{2}\right|^{-p\frac{1-\delta}{pq-1}-\beta\delta p},\\
&\tilde U(x,t) := \tilde a(t)[S(t)\Phi_L(\nu)](x)^{\delta p},\quad V(x,t):=2v_*(x,t),
\end{aligned}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. We show that $(\tilde U,V)$ is a supersolution to problem (Q) in $\mathbf{R}^N\times[0,1)$. It follows from (4.4), (4.6) and property (b) (with $a=(1-\delta)/2$ and $b=\beta$) that

$$
\begin{aligned}
\frac{v_*(x,t)}{[S(t)\Phi(\nu_D)]^\delta} &\le C[S(t)\Phi_L(\nu_D)](x)^{1-\delta}[\log(L+[S(t)\Phi_L(\nu_D)](x))]^{-\beta}\\
&= C[S(t)\Phi_L(\nu_D)](x)^{\frac{1-\delta}{2}}[S(t)\Phi_L(\nu_D)](x)^{\frac{1-\delta}{2}}[\log(L+[S(t)\Phi_L(\nu_D)](x))]^{-\beta}\\
&\le C(\gamma\xi(t))^{\frac{1-\delta}{2}}(\gamma\xi(t))^{\frac{1-\delta}{2}}[\log(L+\gamma\xi(t))]^{-\beta}\\
&\le C\gamma^{\frac{1-\delta}{2}}\xi(t)^{1-\delta}[\log(L+\xi(t))]^{-\beta}\\
&\le C\gamma^{\frac{1-\delta}{2}}t^{-\frac{N}{2}(1-\delta)}\left|\log\frac{t}{2}\right|^{-\frac{1-\delta}{pq-1}-\beta\delta}
\end{aligned}
\tag{4.14}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Furthermore, by (4.4) and (4.6) we have

$$
\begin{aligned}
\frac{[S(t)\Phi_L(\nu_D)](x)}{v_*(x,t)} &\le C[\log(L+[S(t)\Phi_L(\nu_D)](x))]^\beta \le C[\log(L+C\gamma\xi(t))]^\beta\\
&\le C[\log(L+C\xi(t))]^\beta \le C\left|\log\frac{t}{2}\right|^\beta
\end{aligned}
\tag{4.15}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Taking a sufficiently small $\gamma>0$ if necessary, by Lemma 2.3, (2.1), (4.13) and (4.15) we see that

$$
\begin{aligned}
K_1\int_0^t[S(t-s)V(s)^p](x)\,ds &= 2^pK_1\int_0^t[S(t-s)v_*(s)^p](x)\,ds\\
&\le 2^pK_1\int_0^t\left\|\frac{v_*(s)}{[S(s)\Phi_L(\nu_D)]^\delta}\right\|_\infty^p[S(t-s)[S(s)\Phi_L(\nu_D)]^{\delta p}](x)\,ds\\
&\le 2^pK_1\int_0^t\left\|\frac{v_*(s)}{[S(s)\Phi_L(\nu_D)]^\delta}\right\|_\infty^p[[S(t-s)S(s)\Phi_L(\nu_D)](x)]^{\delta p}\,ds\\
&\le C\gamma^{\frac{p(1-\delta)}{2}}[S(t)\Phi_L(\nu_D)](x)^{\delta p}\int_0^t s^{-\frac{N}{2}p(1-\delta)}\left|\log\frac{s}{2}\right|^{-p\frac{1-\delta}{pq-1}-\beta\delta p}\,ds\\
&\le C\gamma^{\frac{p(1-\delta)}{2}}t^{-\frac{N}{2}p(1-\delta)+1}\left|\log\frac{t}{2}\right|^{-p\frac{1-\delta}{pq-1}-\beta\delta p}[S(t)\Phi_L(\nu_D)](x)^{\delta p}\\
&= C\gamma^{\frac{p(1-\delta)}{2}}\tilde a(t)[S(t)\Phi_L(\nu_D)](x)^{\delta p}\le\tilde a(t)[S(t)\Phi_L(\nu_D)](x)^{\delta p}=\tilde U(x,t)
\end{aligned}
\tag{4.16}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. On the other hand, taking a sufficiently small $\gamma>0$ if necessary, by (4.2), (4.6), (4.13) and (4.15) we have

$$
\begin{aligned}
K_2\int_0^t[S(t-s)\tilde U(s)^q](x)\,ds &= K_2\int_0^t[S(t-s)\tilde a(s)^q[S(s)\Phi_L(\nu_D)]^{\delta pq}](x)\,ds\\
&\le K_2[S(t)\Phi_L(\nu)](x)\int_0^t\tilde a(s)^q\,\|S(s)\Phi_L(\nu_D)\|_\infty^{\delta pq-1}\,ds\\
&\le C\left|\log\frac{t}{2}\right|^\beta v_*(x,t)\int_0^t s^{-\frac{N}{2}pq(1-\delta)+q}\left|\log\frac{s}{2}\right|^{-pq\frac{1-\delta}{pq-1}-\beta\delta pq}\\
&\qquad\qquad\times\gamma^{\delta pq-1}s^{-\frac{N}{2}(\delta pq-1)}\left|\log\frac{s}{2}\right|^{-\frac{\delta pq-1}{pq-1}+\beta(\delta pq-1)}\,ds
\end{aligned}
$$

$$\leq C\gamma^{\delta pq-1}\left|\log\frac{t}{2}\right|^{\beta} v_*(x,t)\int_0^t s^{-1}\left|\log\frac{s}{2}\right|^{-1-\beta}ds \leq C\gamma^{\delta pq-1}v_*(x,t)\leq v_*(x,t)$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Therefore, by (2.1) we see that

$$[S(t)\nu_D](x)+K_2\int_0^t[S(t-s)\tilde{U}(s)^q](x)\,ds \leq v_*(x,t)+v_*(x,t)=2v_*(x,t)=V(x,t) \tag{4.17}$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$.

By (4.16) and (4.17) we see that $(\tilde{U},V)$ is a supersolution to problem (Q) in $\mathbf{R}^N\times[0,1)$ in the case of $\mu=0$. Thus Proposition 4.1 follows in the case of $0<p\leq 1$. Therefore the proof of Proposition 4.1 is complete. □

Next we consider problem (Q) in the case of $\nu=0$.

**Proposition 4.2.** *Assume the same conditions as in Theorem* 4.1 *and* $\nu=0$ *in* $\mathbf{R}^N$. *Then there exists* $\gamma>0$ *such that, if* $\mu$ *is a nonnegative measurable function in* $\mathbf{R}^N$ *satisfying*

$$|||S(t)\Psi(\mu)|||_{r_*}\leq\gamma t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r_*}\right)}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha},\quad 0<t<1, \tag{4.18}$$

*for some* $r_*\in((q+1)/(p+1),q)$, *then problem* (Q) *possesses a solution in* $\mathbf{R}^N\times[0,1)$.

**Proof.** Let $\alpha>0$, $0<\beta<1/(pq-1)$, $\delta\in(0,1)$ and $L\geq e$. By (4.4) and (4.18) we have

$$|||S(t)\Psi_L(\mu)|||_{r_*}\leq C\gamma t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r_*}\right)}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha},\quad 0<t<1, \tag{4.19}$$

where $\Psi_L(s):=\Lambda(s)$ with $s=\alpha$. Since $S(t)\Psi_L(\mu)=S(t/2)S(t/2)\Psi_L(\mu)$, taking a sufficiently small $\gamma>0$ if necessary, by (2.4) and (4.19) we have

$$\begin{aligned}|||S(t)\Psi_L(\mu)|||_r&\leq Ct^{-\frac{N}{2}\left(\frac{1}{r_*}-\frac{1}{r}\right)}\left|\left|\left|S\left(\frac{t}{2}\right)\Psi_L(\mu)\right|\right|\right|_{r_*}\\&\leq\delta t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}\end{aligned} \tag{4.20}$$

for $0<t<1$ and $r_*\leq r\leq\infty$. Let $r'>0$ be such that

$$p^{-1}<r'<\frac{q+1}{p+1}. \tag{4.21}$$

Let $\epsilon>0$ be such that $\epsilon<q$ and $p(q-\epsilon)>1$. Let $a>0$ be such that

$$\begin{aligned}&0<a<\min\left\{q-r_*,p-1,\epsilon,\frac{p(q-\epsilon)-1}{q-\epsilon}\right\}&&\text{if}\quad p>1,\\&0<a<\min\left\{q-r_*,pr'-1,\epsilon,r'\cdot\frac{p(q-\epsilon)-1}{q-\epsilon}\right\}&&\text{if}\quad 0<p\leq 1,\end{aligned} \tag{4.22}$$

and $b=\beta q$. Then

$$a<\epsilon<q,\qquad (q-\epsilon)(p-a)>1. \tag{4.23}$$

Taking a sufficiently large $L \geq e$ if necessary, we can assume that $\Psi_L$ and $\Phi_L$ have property (a) and property (b) holds.

Let $(U,V)$ be a solution to the Cauchy problem

$$\begin{cases} U_t = \Delta U + K_1 \Psi_L'(\Psi_L^{-1}(U))[\Phi_L^{-1}(V)]^p, & x \in \mathbf{R}^N,\ t > 0, \\ V_t = \Delta V + K_2 \Phi_L'(\Phi_L^{-1}(V))[\Psi_L^{-1}(U)]^q, & x \in \mathbf{R}^N,\ t > 0, \\ U(x,0) = \Psi_L(\mu),\ V(x,0) = 0, & x \in \mathbf{R}^N. \end{cases} \tag{4.24}$$

Then it follows from property (a) that $(\tilde{u},\tilde{v}) := (\Psi_L^{-1}(U), \Phi_L^{-1}(V))$ is a supersolution to problem (P) with $\nu = 0$. Therefore it suffices to prove the existence of a solution $(U,V)$ to Cauchy problem (4.24) in $\mathbf{R}^N \times [0,1)$. Set $(U_0,V_0) := (S(t)\Psi_L(\mu), 0)$. Define $\{(U_n,V_n)\}_{n=1}^\infty$ inductively by

$$\begin{aligned} U_{n+1}(x,t) &:= S(t)\Psi(\mu) + K_1 \int_0^t S(t-s)\Psi_L'(\Psi_L^{-1}(U_n(s)))[\Phi_L^{-1}(V_n(s))]^p\, ds, \\ V_{n+1}(x,t) &:= K_2 \int_0^t S(t-s)\Phi_L'(\Phi_L^{-1}(V_n(s)))[\Psi_L^{-1}(U_n(s))]^q\, ds, \end{aligned} \tag{4.25}$$

for $x \in \mathbf{R}^N$ and $t > 0$, where $n = 0,1,2,\dots$. Then

$$\begin{aligned} 0 \leq U_0(x,t) \leq U_1(x,t) \leq \cdots \leq U_n(x,t) \leq \cdots, \\ 0 \leq V_0(x,t) \leq V_1(x,t) \leq \cdots \leq V_n(x,t) \leq \cdots, \end{aligned} \tag{4.26}$$

for $x \in \mathbf{R}^N$ and $t > 0$.

Let $\delta \in (0,1)$ be sufficiently small. We show that

$$|||U_n(t)|||_r \leq 2\delta t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log \frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} =: \delta \xi_r(t), \tag{4.27}$$

$$|||V_n(t)|||_\ell \leq \delta^{q-\epsilon} t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \left|\log \frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta} =: \delta^{q-\epsilon} \eta_\ell(t), \tag{4.28}$$

for $0 < t < 1$, $r_* \leq r \leq \infty$, $1 \leq \ell \leq \infty$ and $n = 0,1,2,\dots$. Since $V_0 \equiv 0$, by (4.20) we see that (4.27) and (4.28) hold for $n = 0$.

Assume that (4.27) and (4.28) hold for $n = k \in \{0,1,2,\dots\}$. Recall that

$$\frac{q+1}{pq-1} = \frac{N}{2}. \tag{4.29}$$

By property (b) (with $a = q - r_* - a$ and $b = q\alpha$), (4.4), (4.22), (4.27) and (4.28) with $n = k$ we have

$$
\begin{aligned}
0 &\le \frac{\Phi_L'(\Phi_L^{-1}(V_k(x,t)))[\Psi_L^{-1}(U_k(x,t))]^q}{U_k(x,t)^{r_*}}\\
&\le C[\log(L+V_k(x,t))]^\beta U_k(x,t)^{q-r_*-a}U_k(x,t)^a[\log(L+U_k(x,t))]^{-q\alpha}\\
&\le C[\log(L+\delta^{q-\epsilon}\eta_\infty(t))]^\beta(\delta\xi_\infty(t))^{q-r_*-a}(\delta\xi_\infty(t))^a[\log(L+\delta\xi_\infty(t))]^{-q\alpha}\\
&\le C\delta^{q-r_*-a}[\log(L+\eta_\infty(t))]^\beta\xi_\infty(t)^{q-r_*}[\log(L+\xi_\infty(t))]^{-q\alpha}\\
&\le C\delta^{q-r_*-a}\left|\log\frac{t}{2}\right|^\beta t^{-\frac{N}{2}\cdot\frac{p+1}{q+1}(q-r_*)}\left[\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}\right]^{q-r_*}\left|\log\frac{t}{2}\right|^{-q\alpha}\\
&= C\delta^{q-r_*-a}t^{-\frac{N}{2}\cdot\frac{p+1}{q+1}(q-r_*)}\left|\log\frac{t}{2}\right|^{\beta-\alpha r_*-\frac{p}{pq-1}(q-r_*)}
\end{aligned}
\tag{4.30}
$$

for $(x,t)\in\mathbf{R}^N\times(0,1)$. Then

$$
\begin{aligned}
&\left|\left|\left|\int_{t/2}^t S(t-s)\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q\,ds\right|\right|\right|_\ell\\
&\le C\int_{t/2}^t \left|\left|\left|\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q\right|\right|\right|_\ell\,ds\\
&\le C\int_{t/2}^t\left\|\frac{\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q}{U_k(\cdot,s)^{r_*}}\right\|_\infty |||U_k(\cdot,s)^{r_*}|||_\ell\,ds\\
&\le C\delta^{q-r_*-a}\int_{t/2}^t s^{-\frac{N}{2}\cdot\frac{p+1}{q+1}(q-r_*)}\left|\log\frac{s}{2}\right|^{\beta-\alpha r_*-\frac{p}{pq-1}(q-r_*)}|||U_k(s)|||_{\ell r_*}^{r_*}\,ds
\end{aligned}
\tag{4.31}
$$

for $0<t<1$. It follows from (4.27) that

$$
|||U_k(s)|||_{\ell r_*}\le 2\delta s^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{\ell r_*}\right)}\left|\log\frac{s}{2}\right|^{-\frac{p}{pq-1}+\alpha}
$$

for $0<s<1$. This together with (4.29) and (4.31) implies that

$$
\begin{aligned}
&\left|\left|\left|\int_{t/2}^t S(t-s)\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q\,ds\right|\right|\right|_\ell\\
&\le C\delta^{q-a}\int_{t/2}^t s^{-\frac{N}{2}\left(\frac{p+1}{q+1}q-\frac{1}{\ell}\right)}\left|\log\frac{s}{2}\right|^{\beta-\alpha r_*-\frac{p}{pq-1}(q-r_*)-\frac{p}{pq-1}r_*+\alpha r_*}\,ds\\
&\le C\delta^{q-a}\int_{t/2}^t s^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)-1}\left|\log\frac{s}{2}\right|^{\beta-1-\frac{1}{pq-1}}\,ds\\
&\le C\delta^{q-a}t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)}\int_{t/2}^t s^{-1}\left|\log\frac{s}{2}\right|^{\beta-1-\frac{1}{pq-1}}\,ds
\end{aligned}
\tag{4.32}
$$

for $0 < t < 1$. Similarly, by (2.4) we have

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q\,ds \right|\right|\right|_\ell \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} |||\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q|||_1\,ds \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \left\| \frac{\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q}{U_k(\cdot,s)^{r_*}} \right\|_\infty |||U_k(s)|||_{r_*}^{r_*}\,ds \qquad (4.33)\\
&\le C\delta^{q-a} t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \int_0^{t/2} s^{-1} \left|\log\frac{s}{2}\right|^{\beta-\alpha r_* - \frac{p}{pq-1}(q-r_*) - \frac{p}{pq-1}r_* + \alpha r_*} ds \\
&\le C\delta^{q-a} t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \int_0^{t/2} s^{-1} \left|\log\frac{s}{2}\right|^{\beta-1-\frac{1}{pq-1}} ds
\end{aligned}
$$

for $0 < t < 1$. Since $\beta < 1/(pq-1)$, applying (4.32) and (4.33) to (4.25), we obtain

$$
\begin{aligned}
|||V_{k+1}(t)|||_\ell &\le K_2 \left|\left|\left| \int_0^{t} S(t-s)\Phi_L'(\Phi_L^{-1}(V_k(\cdot,s)))[\Psi_L^{-1}(U_k(\cdot,s))]^q\,ds \right|\right|\right|_\ell \\
&\le C\delta^{q-a} t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \int_0^{t} s^{-1} \left|\log\frac{s}{2}\right|^{\beta-1-\frac{1}{pq-1}} ds \\
&\le C\delta^{q-a} t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta}
\end{aligned}
$$

for $0 < t < 1$. Therefore, taking a sufficiently small $\delta > 0$ if necessary, by (4.23) we obtain (4.28) with $n = k+1$.

We prove (4.27) with $n = k+1$. Let us consider the case of $p > 1$. Then, by (4.4), (4.22), (4.27), (4.28) and property (b) we have

$$
\begin{aligned}
0 &\le \Psi_L'(\Psi_L^{-1}(U_k(x,t)))\frac{[\Phi_L^{-1}(V_k(x,t))]^p}{V_k(x,t)} \\
&\le C[\log(L+U_k(x,t))]^\alpha V_k(x,t)^{p-1-a} V_k(x,t)^a [\log(L+V_k(x,t))]^{-\beta p} \\
&\le C[\log(L+\delta\xi_\infty(t))]^\alpha (\delta^{q-\epsilon}\eta_\infty(t))^{p-1-a} (\delta^{q-\epsilon}\eta_\infty(t))^a [\log(L+\delta^{q-\epsilon}\eta_\infty(t))]^{-\beta p} \qquad (4.34)\\
&\le C(\delta^{q-\epsilon})^{p-1-a} [\log(L+\xi_\infty(t))]^\alpha \eta_\infty(t)^{p-1} [\log(L+\eta_\infty(t))]^{-\beta p} \\
&\le C(\delta^{q-\epsilon})^{p-1-a} \left[ t^{-\frac{N}{2}} \left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta} \right]^{p-1} \left|\log\frac{t}{2}\right|^{\alpha-\beta p}
\end{aligned}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$. This together with (4.28) implies that

$$
\begin{aligned}
&\left|\left|\left| \int_{t/2}^t S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p\, ds \right|\right|\right|_r \\
&\le C\int_{t/2}^t |||\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p|||_r\, ds \\
&\le C(\delta^{q-\epsilon})^{p-1-a}\int_{t/2}^t \left[s^{-\frac{N}{2}}\left|\log\frac{s}{2}\right|^{-\frac{1}{pq-1}+\beta}\right]^{p-1}\left|\log\frac{s}{2}\right|^{\alpha-\beta p} ||V_k(s)||_r\, ds \\
&\le C(\delta^{q-\epsilon})^{p-a} t\left[t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta}\right]^{p-1}\left|\log\frac{t}{2}\right|^{\alpha-\beta p} t^{-\frac{N}{2}\left(1-\frac{1}{r}\right)}\left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta} \\
&\le C(\delta^{q-\epsilon})^{p-a} t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} \\
&= C(\delta^{q-\epsilon})^{p-a} t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}
\end{aligned} \tag{4.35}
$$

for $0<t<1$. Here we used the relation

$$
-\frac{N}{2}p+1 = -\frac{N}{2}p+\frac{N}{2}\cdot\frac{pq-1}{q+1} = -\frac{N}{2}\cdot\frac{p+1}{q+1}, \tag{4.36}
$$

which follows from (4.29). On the other hand, since $p>1$, by (2.4), (4.28) and (4.34) we have

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p\, ds \right|\right|\right|_r \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(1-\frac{1}{r}\right)} |||\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p|||_1\, ds \\
&\le C(\delta^{q-\epsilon})^{p-1-a} t^{-\frac{N}{2}\left(1-\frac{1}{r}\right)}\int_0^{t/2} s^{-\frac{N}{2}(p-1)}\left|\log\frac{s}{2}\right|^{-\frac{p-1}{pq-1}+\beta(p-1)+\alpha-\beta p} |||V_k(s)|||_1\, ds \\
&\le C(\delta^{q-\epsilon})^{p-a} t^{-\frac{N}{2}\left(1-\frac{1}{r}\right)}\int_0^{t/2} s^{-\frac{N}{2}(p-1)}\left|\log\frac{s}{2}\right|^{-\frac{p-1}{pq-1}-\beta+\alpha}\left|\log\frac{s}{2}\right|^{-\frac{1}{pq-1}+\beta} ds
\end{aligned} \tag{4.37}
$$

for $0<t<1$. Recalling $p<1+2/N$ and (4.36) and combining Lemma 2.3 and (4.37), we obtain

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p\, ds \right|\right|\right|_r \\
&\le C(\delta^{q-\epsilon})^{p-a} t^{-\frac{N}{2}\left(1-\frac{1}{r}\right)} t^{-\frac{N}{2}(p-1)+1}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} \\
&= C(\delta^{q-\epsilon})^{p-a} t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)}\left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}
\end{aligned} \tag{4.38}
$$

for $0<t<1$. Taking a sufficiently small $\delta$ if necessary, by (4.20), (4.23), (4.35) and (4.38) we obtain

$$
|||U_{k+1}(t)|||_r
$$

$$
\begin{aligned}
&\le |||S(t)\Psi_L(\mu)|||_r + K_1 \left|\left|\left| \int_0^t S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(\cdot,s)))[\Phi_L^{-1}(V_k(\cdot,s))]^p\,ds \right|\right|\right|_r \\
&\le \left[\delta + C(\delta^{q-\epsilon})^{p-a}\right] t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} \\
&\le 2\delta t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}
\end{aligned}
$$

for $0 < t < 1$. This implies that (4.27) holds for $n = k+1$ in the case of $p > 1$.

We consider the case of $0 < p \le 1$. Similarly to (4.34), by (4.4), (4.22), (4.27), (4.28) with $n = k$ and property (b) we have

$$
\begin{aligned}
0 &\le \frac{\left[\Psi_L'(\Psi_L^{-1}(U_k(x,t)))[\Phi_L^{-1}(V_k(x,t))]^p\right]^{r'}}{V_k(x,t)} \\
&\le C[\log(L+U_k(x,t))]^{\alpha r'} V_k(x,t)^{pr'-1-a} V_k(x,t)^a [\log(L+V_k(x,s))]^{-\beta p r'} \\
&\le C[\log(L+\delta\xi_\infty(t))]^{\alpha r'} (\delta^{q-\epsilon}\eta_\infty(t))^{pr'-1-a} \\
&\qquad \times (\delta^{q-\epsilon}\eta_\infty(t))^a [\log(L+\delta^{q-\epsilon}\eta_\infty(t))]^{-\beta p r'} \\
&\le C(\delta^{q-\epsilon})^{pr'-1-a} [\log(L+\xi_\infty(t))]^{\alpha r'} (\eta_\infty(t))^{pr'-1} [\log(L+\eta_\infty(t))]^{-\beta p r'} \\
&\le C(\delta^{q-\epsilon})^{pr'-1-a} \left|\log\frac{t}{2}\right|^{\alpha r'} \left[ t^{-\frac{N}{2}} \left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta} \right]^{pr'-1} \left|\log\frac{t}{2}\right|^{-\beta p r'} \\
&\le C(\delta^{q-\epsilon})^{pr'-1-a} t^{-\frac{N}{2}(pr'-1)} \left|\log\frac{t}{2}\right|^{-\frac{pr'-1}{pq-1}+\alpha r'-\beta}
\end{aligned}
\tag{4.39}
$$

for $(x,t) \in \mathbf{R}^N \times (0,1)$, where $r'$ is as in (4.21). By (2.4), (4.28), (4.36) and (4.39) we have

$$
\begin{aligned}
&\left|\left|\left| \int_{t/2}^t S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p\,ds \right|\right|\right|_r \\
&\le C\int_{t/2}^t \left\| \frac{\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p}{V_k(s)^{1/r'}} \right\|_\infty |||V_k(s)^{1/r'}|||_r\,ds \\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-1-a}{r'}} t^{-\frac{N}{2}\left(p-\frac{1}{r'}\right)} \left|\log\frac{t}{2}\right|^{-\frac{pr'-1}{r'(pq-1)}+\alpha-\frac{\beta}{r'}} \int_{t/2}^t |||V_k(s)|||_{r/r'}^{1/r'}\,ds \\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-1-a}{r'}} t^{-\frac{N}{2}\left(p-\frac{1}{r'}\right)} \left|\log\frac{t}{2}\right|^{-\frac{pr'-1}{r'(pq-1)}+\alpha-\frac{\beta}{r'}} \\
&\qquad\qquad \times (\delta^{q-\epsilon})^{\frac{1}{r'}} t^{-\frac{N}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)+1} \left|\log\frac{t}{2}\right|^{-\frac{1}{r'(pq-1)}+\frac{\beta}{r'}} \\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-a}{r'}} t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} \\
&= C(\delta^{q-\epsilon})^{\frac{pr'-a}{r'}} t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}
\end{aligned}
\tag{4.40}
$$

for $0 < t < 1$. Furthermore, by (2.4), Lemma 2.3, (4.28), (4.36) and (4.39) we obtain

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p\, ds \right|\right|\right|_r \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} |||\Psi_L'(\Psi_L^{-1}(U_k(s)))[\Phi_L^{-1}(V_k(s))]^p|||_{r'}\, ds \\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-1-a}{r'}} t^{-\frac{N}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \int_0^{t/2} s^{-\frac{N}{2}\left(p-\frac{1}{r'}\right)} \left|\log\frac{s}{2}\right|^{-\frac{pr'-1}{r'(pq-1)}+\alpha-\frac{\beta}{r'}} |||V_k(s)|||_1^{1/r'}\, ds \\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-1-a}{r'}} t^{-\frac{N}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \int_0^{t/2} s^{-\frac{N}{2}\left(p-\frac{1}{r'}\right)} \left|\log\frac{s}{2}\right|^{-\frac{pr'-1}{r'(pq-1)}+\alpha-\frac{\beta}{r'}} \\
&\qquad\qquad \times (\delta^{q-\epsilon})^{\frac{1}{r'}} \left|\log\frac{s}{2}\right|^{-\frac{1}{r'(pq-1)}+\frac{\beta}{r'}}\, ds \qquad (4.41)\\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-a}{r'}} t^{-\frac{N}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \int_0^{t/2} s^{-\frac{N}{2}\left(p-\frac{1}{r'}\right)} \left|\log\frac{s}{2}\right|^{-\frac{p}{pq-1}+\alpha}\, ds \\
&\le C(\delta^{q-\epsilon})^{\frac{pr'-a}{r'}} t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} \\
&= C(\delta^{q-\epsilon})^{\frac{pr'-a}{r'}} t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}
\end{aligned}
$$

for $0 < t < 1$. Here we used the relation

$$\frac{N}{2}\left(p-\frac{1}{r'}\right) < \frac{N}{2}\left(p-\frac{p+1}{q+1}\right) = \frac{N}{2}\frac{pq-1}{q+1} = 1,$$

which follows from (4.21). Therefore, applying (4.20), (4.40) and (4.41) to (4.25), we obtain

$$
\begin{aligned}
&|||U_{k+1}(t)|||_r \\
&\le |||S(t)\Psi_L(\mu)|||_r + K_1 \left|\left|\left| \int_0^t S(t-s)\Psi_L'(\Psi_L^{-1}(U_k(\cdot,s)))[\Phi_L^{-1}(V_k(\cdot,s))]^p\, ds \right|\right|\right|_r \\
&\le \left[\delta + C(\delta^{q-\epsilon})^{\frac{pr'-a}{r'}}\right] t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha} \\
&= \delta\left[1 + C\delta^{(q-\epsilon)(p-\frac{a}{r'})-1}\right] t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r}\right)} \left|\log\frac{t}{2}\right|^{-\frac{p}{pq-1}+\alpha}
\end{aligned}
$$

for $0 < t < 1$. Since $(q-\epsilon)(p-a/r') > (q-\epsilon)(p-a)$, taking a sufficiently small $\delta > 0$ if necessary, by (4.23) we obtain (4.27) with $n = k+1$. Therefore we see that (4.27) and (4.28) hold for $n \in \{0, 1, 2, \dots\}$. Then, by (4.26), (4.27) and (4.28) we see that the limit functions

$$U(x,t) := \lim_{n\to\infty} U_n(x,t), \qquad V(x,t) := \lim_{n\to\infty} V_n(x,t),$$

can be defined for $x \in \mathbf{R}^N$ and $t \in (0,1)$ and $(U,V)$ is a solution to problem (4.24) in $\mathbf{R}^N \times [0,1)$. This implies that problem (P) possesses a supersolution in $\mathbf{R}^N \times [0,1)$. Thus Proposition 4.2 follows. $\Box$

Now we are ready to complete the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Under assumptions of Theorem 4.1, if $\gamma$ is sufficiently small, then problem (Q) in the case of $\mu = 0$ posses a solution $(u_1, v_1)$ in $\mathbf{R}^N \times [0,1)$. Similarly, problem (Q) in the case of $\nu = 0$ possesses a solution $(u_2, v_2)$ in $\mathbf{R}^N \times [0,1)$. Set $u_* = u_1 + u_2$ and $v_* = v_1 + v_2$. Then, by (4.3) we have

$$\begin{cases} \partial_t u_* - \Delta u_* = K_1 v_1^p + K_1 v_2^p \ge D^{-\frac{N}{2}}(v_1+v_2)^p = D^{-\frac{N}{2}} v_*^p \text{ in } & \mathbf{R}^N \times (0,1), \\ \partial_t v_* - \Delta v_* = K_2 u_1^q + K_2 u_2^q \ge D^{-\frac{N}{2}}(u_1+u_2)^q = D^{-\frac{N}{2}} u_*^q \text{ in } & \mathbf{R}^N \times (0,1), \\ (u_*(\cdot,0), v_*(\cdot,0)) = (\mu_D, \nu_D) \qquad\qquad\qquad\qquad\quad \text{ in } & \mathbf{R}^N. \end{cases}$$

This means that $(u_*, v_*)$ is a supersolution to problem (P'). Thus problem (P) possesses a solution in $\mathbf{R}^N \times [0,1)$. The proof of Theorem 4.1 is complete. $\Box$

## 5. Cases (D) and (E)

In this section we consider cases (D) and (E), and prove the following theorem.

**Theorem 5.1.** *Let $N \ge 1$ and $0 < p \le q$ with $pq > 1$ be in case* (D) *or* (E). *Let $f$ be a positive continuous function in $(0,1)$ such that*

$$\int_0^1 \tau^{-1} f(\tau)\, d\tau < \infty. \tag{5.1}$$

*Let*

$$\frac{Nq}{N+2} < r_* < q. \tag{5.2}$$

*Then there exists $\gamma > 0$ such that, if $\mu$ and $\nu$ are Radon measures in $\mathbf{R}^N$ and satisfy*

$$|||S(t)\mu|||_{r_*} \le \gamma t^{-\frac{N}{2}\left(\frac{N+2}{Nq} - \frac{1}{r_*}\right)} f(\sqrt{t})^{\frac{1}{q}}, \qquad \sup_{x \in \mathbf{R}^N} \nu(B(x,1)) \le \gamma^q, \tag{5.3}$$

*for $0 < t < 1$, then problem* (P) *possesses a solution in $\mathbf{R}^N \times [0,1)$.*

We modify the argument in the proof of Theorem 4.1 to prove Theorem 5.1. As in Section 3, it suffices to consider the case of $T = 1$ and $D' = 1$.

**Proof of Theorem 5.1.** Let $\delta \in (0,1)$ be a sufficiently small constant. Similarly to the proof of Theorem 4.1, by (2.4) and (5.3), taking a sufficiently small $\gamma > 0$ if necessary, we have

$$\begin{aligned} &|||S(t)\mu|||_r \le C\left(\frac{t}{2}\right)^{-\frac{N}{2}\left(\frac{1}{r_*} - \frac{1}{r}\right)} \left|\left|\left| S\left(\frac{t}{2}\right)\mu \right|\right|\right|_{r_*} \le \delta t^{\frac{N}{2r} - \frac{N+2}{2q}} \tilde{f}(\sqrt{t})^{\frac{1}{q}}, \\ &|||S(t)\nu|||_\ell \le \delta^q t^{-\frac{N}{2}\left(1 - \frac{1}{\ell}\right)}, \end{aligned} \tag{5.4}$$

for $0 < t < 1$, where $\tilde{f}(t) := f(t/2)$ and

$$\max\left\{r_*, \frac{1}{p}\right\} \le r \le \infty, \qquad 1 \le \ell \le \infty. \tag{5.5}$$

Set $(u_0, v_0) := (S(t)\mu, S(t)\nu)$. Define $\{(u_n, v_n)\}_{n=1}^{\infty}$ inductively by

$$
\begin{aligned}
u_{n+1}(x,t) &:= S(t)\mu + D^{-\frac{N}{2}} \int_0^t S(t-s) v_n(s)^p \, ds, \\
v_{n+1}(x,t) &:= S(t)\nu + D^{-\frac{N}{2}} \int_0^t S(t-s) u_n(s)^q \, ds,
\end{aligned}
\tag{5.6}
$$

for $x \in \mathbf{R}^N$ and $t > 0$, where $n = 0, 1, 2, \dots$. Then

$$
\begin{aligned}
&0 \le u_0(x,t) \le u_1(x,t) \le \cdots \le u_n(x,t) \le \cdots, \\
&0 \le v_0(x,t) \le v_1(x,t) \le \cdots \le v_n(x,t) \le \cdots,
\end{aligned}
\tag{5.7}
$$

for $x \in \mathbf{R}^N$ and $t > 0$.

Let $\epsilon > 0$ satisfy

$$
\epsilon < q, \qquad p(q-\epsilon) > 1. \tag{5.8}
$$

Taking a sufficiently small $\delta \in (0,1)$ if necessary, we show that

$$
|||u_n(t)|||_r \le \delta t^{\frac{N}{2r} - \frac{N+2}{2q}} \tilde{f}(\sqrt{t})^{\frac{1}{q}} + \delta t^{-\frac{N}{2}(p - \frac{1}{r}) + 1}, \tag{5.9}
$$

$$
|||v_n(t)|||_\ell \le \delta^{q-\epsilon} t^{-\frac{N}{2}(1 - \frac{1}{\ell})}, \tag{5.10}
$$

for $0 < t < 1$, $n = 0, 1, 2, \dots$, where $r$ and $\ell$ satisfy (5.5). By (5.4) we see that (5.9) and (5.10) hold for $n = 0$.

Assume that (5.9) and (5.10) hold for $n = k \in \{0, 1, 2, \dots\}$. It follows from (2.4) that

$$
\left|\left|\left| \int_{t/2}^t S(t-s) u_k(s)^q \, ds \right|\right|\right|_\ell \le C \int_{t/2}^t |||u_k(s)^q|||_\ell \, ds = C \int_{t/2}^t |||u_k(s)|||_{q\ell}^q \, ds \tag{5.11}
$$

for $0 < t < 1$. In cases (D) and (E), we have

$$
\frac{N}{2} - \frac{N}{2} pq + q + 1 = q + 1 - \frac{N}{2}(pq - 1) = (pq-1)\left( \frac{q+1}{pq-1} - \frac{N}{2} \right) > 0. \tag{5.12}
$$

Since

$$
\int_0^1 s^{-1} f(\sqrt{s}) \, ds = 2 \int_0^1 \tau^{-1} f(\tau) \, d\tau < \infty, \tag{5.13}
$$

by (5.1), (5.9), (5.11) and (5.12) we have

$$
\begin{aligned}
&\left|\left|\left| \int_{t/2}^t S(t-s) u_k(s)^q \, ds \right|\right|\right|_\ell \\
&\le C\delta^q \int_{t/2}^t \left\{ \left[ s^{\frac{N}{2}\frac{1}{q\ell} - \frac{N+2}{2q}} \tilde{f}(\sqrt{s})^{\frac{1}{q}} \right]^q + s^{-\frac{N}{2}(pq - \frac{1}{\ell}) + q} \right\} ds \\
&\le C\delta^q \int_{t/2}^t s^{\frac{N}{2\ell} - \frac{N+2}{2}} \tilde{f}(\sqrt{s}) \, ds + C\delta^q t^{\frac{N}{2\ell} - \frac{N}{2} pq + q + 1} \\
&\le C\delta^q t^{-\frac{N}{2}(1 - \frac{1}{\ell})} \left[ \int_{t/2}^t s^{-1} \tilde{f}(\sqrt{s}) \, ds + t^{\frac{N}{2} - \frac{N}{2} pq + q + 1} \right] \le C\delta^q t^{-\frac{N}{2}(1 - \frac{1}{\ell})}
\end{aligned}
\tag{5.14}
$$

for $0 < t < 1$. Similarly to (5.14), by (2.4) we have

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)u_k(s)^q\,ds \right|\right|\right|_\ell \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} |||u_k(s)^q|||_1\,ds \le Ct^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \int_0^{t/2} |||u_k(s)|||_q^q\,ds \\
&\le C\delta^q t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \int_0^{t/2} \left\{ \left[ s^{\frac{N}{2q}-\frac{N+2}{2q}} \tilde{f}(\sqrt{s})^{\frac{1}{q}} \right]^q + s^{-\frac{N}{2}(pq-1)+q} \right\} ds \\
&\le C\delta^q t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \left[ \int_0^{t/2} s^{-1}\tilde{f}(\sqrt{s})\,ds + t^{(pq-1)\left(\frac{q+1}{pq-1}-\frac{N}{2}\right)} \right] \le C\delta^q t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)}
\end{aligned}
\tag{5.15}
$$

for $0 < t < 1$. By (5.4), (5.8), (5.14) and (5.15), taking a sufficiently small $\delta > 0$ if necessary, we have

$$
\begin{aligned}
|||v_{k+1}(t)|||_\ell &\le |||S(t)\nu|||_\ell + D^{-\frac{N}{2}} \left|\left|\left| \int_0^t S(t-s)u_k(s)^q\,ds \right|\right|\right|_\ell \\
&\le \delta^q t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} + C\delta^q t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)} \le \delta^{q-\epsilon} t^{-\frac{N}{2}\left(1-\frac{1}{\ell}\right)}
\end{aligned}
\tag{5.16}
$$

for $0 < t < 1$. Then (5.10) holds for $n = k+1$.

On the other hand, by (2.4) and (5.10) we have

$$
\begin{aligned}
\left|\left|\left| \int_{t/2}^t S(t-s)v_k(s)^p\,ds \right|\right|\right|_r &\le C\int_{t/2}^t |||v_k(s)^p|||_r\,ds = C\int_{t/2}^t |||v_k(s)|||_{pr}^p\,ds \\
&\le C(\delta^{q-\epsilon})^p \int_{t/2}^t s^{-\frac{N}{2}\left(p-\frac{1}{r}\right)}\,ds \le C\delta^{p(q-\epsilon)} t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1}
\end{aligned}
\tag{5.17}
$$

for $0 < t < 1$. Since $p < 1 + 2/N$, if $p \ge 1$, then by (2.4) we see that

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)v_k(s)^p\,ds \right|\right|\right|_r \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(1-\frac{1}{r}\right)} |||v_k(s)^p|||_1\,ds \le Ct^{-\frac{N}{2}\left(1-\frac{1}{r}\right)} \int_0^{t/2} |||v_k(s)|||_p^p\,ds \\
&\le C\delta^{p(q-\epsilon)} t^{-\frac{N}{2}\left(1-\frac{1}{r}\right)} \int_0^{t/2} s^{-\frac{N}{2}(p-1)}\,ds \le C\delta^{p(q-\epsilon)} t^{-\frac{N}{2}\left(1-\frac{1}{r}\right)-\frac{N(p-1)}{2}+1} \\
&\le C\delta^{p(q-\epsilon)} t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1}
\end{aligned}
\tag{5.18}
$$

for $0 < t < 1$. If $0 < p < 1$, then by (2.4) we observe that

$$
\begin{aligned}
&\left|\left|\left| \int_0^{t/2} S(t-s)v_k(s)^p\,ds \right|\right|\right|_r \\
&\le C\int_0^{t/2} (t-s)^{-\frac{N}{2}\left(p-\frac{1}{r}\right)} |||v_k(s)^p|||_{p^{-1}}\,ds \le Ct^{-\frac{N}{2}\left(p-\frac{1}{r}\right)} \int_0^{t/2} |||v_k(s)|||_1^p\,ds \\
&\le C\delta^{p(q-\epsilon)} t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1}
\end{aligned}
\tag{5.19}
$$

for $0 < t < 1$. By (5.4), (5.17), (5.18) and (5.19) we have

$$\begin{aligned}|||u_{k+1}(t)|||_r &\le |||S(t)\mu|||_r + D^{-\frac{N}{2}}\left|\left|\left|\int_0^t S(t-s)v_k(s)^p\,ds\right|\right|\right|_r \\ &\le \delta t^{\frac{N}{2r}-\frac{N+2}{2q}}\tilde{f}(\sqrt{t})^{\frac{1}{q}} + C\delta^{p(q-\epsilon)}t^{-\frac{N}{2}\left(p-\frac{1}{r}\right)+1}\end{aligned}$$

for $0 < t < 1$. Therefore, taking a sufficiently small $\delta > 0$ if necessary, by (5.8), we obtain (5.9) with $n = k+1$. Thus (5.9) and (5.10) hold for $n \in \{0, 1, 2, \dots\}$. Then, by (5.7), (5.9) and (5.10) we see that the limit functions

$$u(x,t) := \lim_{n\to\infty} u_n(x,t), \qquad v(x,t) := \lim_{n\to\infty} v_n(x,t),$$

can be defined for for $x \in \mathbf{R}^N$ and $t \in (0,1)$ and $(u,v)$ is a solution to problem (P') in $\mathbf{R}^N \times [0,1)$. Thus Theorem 5.1 follows. □

## 6. Proof of Theorem 1.2

As application of Theorems 3.1, 3.2, 3.3, 4.1 and 5.1, we prove Theorem 1.2.

**Proof of Theorem 1.2.** All of the statements on the nonexistence of solutions has already been proved in [6] as a corollary of Theorem 1.1. See [6, Corollary 1.5]. It suffices to prove the statements on the existence of solutions.

For any $T > 0$, set

$$\mu_T(K) = T^{\frac{p+1}{pq-1}}\mu(T^{\frac{1}{2}}K), \quad \nu_T(K) = T^{\frac{q+1}{pq-1}}\nu(T^{\frac{1}{2}}K), \tag{6.1}$$

for Borel sets $K$ in $\mathbf{R}^N$. Then problem (P) possesses a solution in $\mathbf{R}^N \times [0,T)$ if problem (P) possesses a solution in $\mathbf{R}^N \times [0,1)$ with the initial data $(\mu_T, \nu_T)$. See Remark 1.1 (ii).

Case (A) : Let $(p,q)$ be in case (A) and $T > 0$. It follows that

$$0 \le \mu_T(x) \le c_{a,1}|x|^{-\frac{2(p+1)}{pq-1}}, \quad 0 \le \nu_T(x) \le c_{a,2}|x|^{-\frac{2(q+1)}{pq-1}}, \quad x \in \mathbf{R}^N.$$

Let $\alpha > 1$ satisfy

$$\frac{2(q+1)}{pq-1}\alpha < N.$$

Then

$$\begin{aligned}&\sup_{x\in\mathbf{R}^N}\int_{B(x,\sigma)}\mu_T(y)^{\frac{\alpha(q+1)}{p+1}}\,dy + \sup_{x\in\mathbf{R}^N}\int_{B(x,\sigma)}\nu_T(y)^\alpha\,dy \\ &\le c_{a,1}^{\frac{\alpha(q+1)}{p+1}}\int_{B(0,\sigma)}|y|^{-\frac{2(p+1)}{pq-1}\frac{\alpha(q+1)}{p+1}}\,dy + c_{a,2}^\alpha\int_{B(0,\sigma)}|y|^{-\frac{2\alpha(q+1)}{pq-1}}\,dy \\ &\le C\left(c_{a,1}^{\frac{\alpha(q+1)}{p+1}} + c_{a,2}^\alpha\right)\sigma^{N-\frac{2\alpha(q+1)}{pq-1}}\end{aligned}$$

for $0 < \sigma \le 1$. This together with Lemma 2.1 implies that

$$\left\|S(t)\mu_T^{\frac{\alpha(q+1)}{p+1}}\right\|_\infty + \left\|S(t)\nu_T^\alpha\right\|_\infty \le C\left(c_{a,1}^{\frac{\alpha(q+1)}{p+1}} + c_{a,2}^\alpha\right)t^{-\frac{q+1}{pq-1}\alpha}$$

for $0 < t \le 1$. By Theorem 3.1 we see that if both of $c_{a,1}$ and $c_{a,2}$ are sufficiently small, then there exists a solution in $\mathbf{R}^N \times [0,1)$ to problem (P) with the initial data $(\mu_T, \nu_T)$. Since $T$ is arbitrary, we obtain the desired conclusion for case (A).

Case (B) : Let $(p,q)$ be in case (B) and $r_*$ be such that

$$\frac{q+1}{p+1} < r_* < q.$$

Let $0 < \beta < 1/(pq-1)$ and set

$$\Psi(\tau) := \tau[\log(e+\tau)]^{\frac{p}{pq-1}}, \qquad \Phi(\tau) := \tau[\log(e+\tau)]^{\beta}, \tag{6.2}$$

for $\tau \ge 0$. Let $0 < c_{b,1} \le 1$. Then

$$\begin{aligned}
&0 \le \Psi(\mu(x))\\
&\le c_{b,1}|x|^{-\frac{2(p+1)}{pq-1}}\left|\log\frac{|x|}{2}\right|^{-\frac{p}{pq-1}}\left[\log\left(e + c_{b,1}|x|^{-\frac{2(p+1)}{pq-1}}\left|\log\frac{|x|}{2}\right|^{-\frac{p}{p-1}}\right)\right]^{\frac{p}{pq-1}}\chi_{B(0,1)}\\
&\le c_{b,1}|x|^{-\frac{2(p+1)}{pq-1}}\left|\log\frac{|x|}{2}\right|^{-\frac{p}{pq-1}}\left[\log\left(e + |x|^{-\frac{2(p+1)}{pq-1}}\left|\log\frac{|x|}{2}\right|^{-\frac{p}{p-1}}\right)\right]^{\frac{p}{pq-1}}\chi_{B(0,1)}\\
&\le Cc_{b,1}|x|^{-\frac{2(p+1)}{pq-1}}\chi_{B(0,1)}
\end{aligned} \tag{6.3}$$

for $x \in \mathbf{R}^N$. Since $(p,q)$ is in case (B), we see that $p < q$. Furthermore, it follows from (4.2) that

$$\frac{2(p+1)}{pq-1} = N\frac{p+1}{q+1}.$$

These imply that $\Psi(\mu)$ belongs to the Lorentz space $L^{\frac{q+1}{p+1},\infty}(\mathbf{R}^N)$ (see e.g. [9] for the definition of Lorentz spaces). Then we apply the Young-O'Neil inequality (see e.g. [9, Chapter 1]) to obtain

$$\begin{aligned}
|||S(t)\Psi(\mu)|||_{r_*} &\le \|S(t)\Psi(\mu)\|_{L^{r_*}}\\
&\le C\|G(t)\|_{L^{\bar r}}\|\Psi(\mu)\|_{L^{r_0,\infty}} \le Cc_{b,1}t^{-\frac{N}{2}\left(1-\frac{1}{\bar r}\right)} \le Cc_{b,1}t^{-\frac{N}{2}\left(\frac{p+1}{q+1}-\frac{1}{r_*}\right)}
\end{aligned} \tag{6.4}$$

for $0 < t < 1$, where

$$r_0 := \frac{q+1}{p+1} > 1, \qquad 1 + \frac{1}{r_*} = \frac{1}{\bar r} + \frac{1}{r_0}.$$

On the other hand, similarly to (6.3), we have

$$0 \le \Phi(\nu(x)) \le Cc_{b,2}|x|^{-N}\left|\log\frac{|x|}{2}\right|^{-\frac{1}{pq-1}-1+\beta}\chi_{B(0,1)}(x)$$

for $x \in \mathbf{R}^N$. Then, by Lemma 2.1 we have

$$\begin{aligned}
&\|S(t)\Phi(\nu)\|_\infty \le Ct^{-\frac{N}{2}}\sup_{x\in\mathbf{R}^N}\int_{B(x,\sqrt{t})}\Phi(\nu)\,dy\\
&\le Cc_{b,2}t^{-\frac{N}{2}}\int_{B(0,\sqrt{t})}|y|^{-N}\left|\log\frac{|y|}{2}\right|^{-\frac{1}{pq-1}-1+\beta}dy \le Cc_{b,2}t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{1}{pq-1}+\beta}
\end{aligned} \tag{6.5}$$

for $0<t<1$. By (6.4) and (6.5) we apply Theorem 4.1 with (6.2) to see that problem (P) possesses a solution in $\mathbf{R}^N\times[0,1)$ if $c_{b,1}$ and $c_{b,2}$ are sufficiently small. Thus Theorem 1.2 follows in case (B).

Case (C) : Let $(p,q)$ be in case (C). For $\beta>0$, set $\Phi(\tau):=\tau[\log(e+\tau)]^\beta$ for $\tau\ge 0$. As in case (B), by Lemma 2.1 we see that

$$\|S(t)\Phi(\mu)\|_\infty+\|S(t)\Phi(\nu)\|_\infty\le C(c_{c,1}+c_{c,2})t^{-\frac{N}{2}}\left|\log\frac{t}{2}\right|^{-\frac{N}{2}+\beta}$$

for $0<t<1$. Then we apply Theorem 3.2 to see that problem (P) possesses a solution if $c_{c,1}$ and $c_{c,2}$ are sufficiently small. Thus Theorem 1.2 follows in case (C).

Case (D) : Let $0<T<1$ and

$$\frac{Nq}{N+2}<r_*<q.$$

Since $h_1$ is a increasing function on $(0,1]$, it follows that

$$\begin{aligned}0\le\mu_T(x)&=T^{\frac{p+1}{pq-1}}\mu(T^{\frac12}x)=T^{\frac{p+1}{pq-1}-\frac{N+2}{2q}}|x|^{-\frac{N+2}{q}}h_1(T^{\frac12}|x|)\chi_{B(0,1)}(T^{\frac12}x)\\&\le T^{\frac{p+1}{pq-1}-\frac{N+2}{2q}}|x|^{-\frac{N+2}{q}}h_1(|x|)\end{aligned}\tag{6.6}$$

for $x\in B(0,1)$. By (2.3) and (6.6) we have

$$\begin{aligned}\sup_{x\in\mathbf{R}^N}\|S(t)\mu_T\|_{L^{r_*}(B(x,\sqrt t))}&\le Ct^{-\frac N2\left(1-\frac{1}{r_*}\right)}\sup_{x\in\mathbf{R}^N}\|\mu_T\|_{L^1(B(x,\sqrt t))}\\&\le Ct^{-\frac N2\left(1-\frac{1}{r_*}\right)}\int_{B(0,\sqrt t)}\mu_T(y)\,dy\\&\le Ct^{-\frac N2\left(1-\frac{1}{r_*}\right)}T^{\frac{p+1}{pq-1}-\frac N2}\int_{B(0,\sqrt t)}|z|^{-\frac{N+2}{q}}h_1(|z|)\,dz\\&\le Ct^{-\frac N2\left(1-\frac{1}{r_*}\right)}T^{\frac{p+1}{pq-1}-\frac N2}t^{\frac N2-\frac{N+2}{2q}}h_1(\sqrt t)=CT^{\frac{p+1}{pq-1}-\frac N2}t^{-\frac N2\left(\frac{N+2}{Nq}-\frac{1}{r_*}\right)}h_1(\sqrt t)\end{aligned}\tag{6.7}$$

for $0<t<1$. On the other hand, since

$$\frac{N+2}{q}r_*>\frac{N+2}{q}\frac{Nq}{N+2}=N,\tag{6.8}$$

by (6.6) we apply Lemma 2.2 with $a=(N+2)/q$ to obtain

$$\|S(t)\mu_T\|_{L^{r_*}(B(0,1)\setminus B(0,\sqrt t))}\le CT^{\frac{p+1}{pq-1}-\frac{N+2}{2q}}t^{-\frac N2\left(\frac{N+2}{Nq}-\frac{1}{r_*}\right)}\left[h_1(t^{\frac16})+t^{\frac{(N+2)r_*-Nq}{4qr_*}}\right]$$

for $0<t<1$. This together with (6.7) implies that

$$\begin{aligned}|||S(t)\mu_T|||_{r_*}&=\|S(t)\mu_T\|_{L^{r_*}(B(0,1))}\\&\le\|S(t)\mu_T\|_{L^{r_*}(B(0,\sqrt t))}+\|S(t)\mu_T\|_{L^{r_*}(B(0,1)\setminus B(0,\sqrt t))}\\&\le CT^{\frac{p+1}{pq-1}-\frac{N+2}{2q}}t^{-\frac N2\left(\frac{N+2}{Nq}-\frac{1}{r_*}\right)}\left[\left(h_1(\sqrt t)+h_1(t^{\frac16})+t^{\frac{(N+2)r_*-Nq}{4qr_*}}\right)^q\right]^{\frac1q}\\&\le CT^{\frac{p+1}{pq-1}-\frac{N+2}{2q}}t^{-\frac N2\left(\frac{N+2}{Nq}-\frac{1}{r_*}\right)}\left[h_1(\sqrt t)^q+h_1(t^{\frac16})^q+t^{\frac{(N+2)r_*-Nq}{4r_*}}\right]^{\frac1q}\end{aligned}\tag{6.9}$$

for $0 < t < 1$. Since

$$\int_0^1 h_1(\tau)^q \tau^{-1}\,d\tau < \infty,$$

for any $k > 0$, it follows that

$$\int_0^1 h_1(\tau^k)^q \tau^{-1}\,d\tau = \frac{1}{k}\int_0^1 h_1(\tau)^q \tau^{-1}\,d\tau < \infty.$$

This together with (6.8) implies that

$$\int_0^1 \left[h_1(\sqrt{\tau})^q + h_1(\tau^{\frac{1}{6}})^q + \tau^{\frac{(N+2)r_*-Nq}{4r_*}}\right]\tau^{-1}\,d\tau < \infty. \tag{6.10}$$

On the other hand, it follows from (6.1) that

$$\sup_{x\in\mathbf{R}^N} \nu_T(B(x,1)) = T^{\frac{q+1}{pq-1}-\frac{N}{2}} \sup_{x\in\mathbf{R}^N} \nu(B(x,T^{\frac{1}{2}})) \le T^{\frac{q+1}{pq-1}-\frac{N}{2}} \sup_{x\in\mathbf{R}^N} \nu(B(x,1)). \tag{6.11}$$

Since

$$\frac{p+1}{pq-1} - \frac{N+2}{2q} = \frac{2(q+1)-N(pq-1)}{2q(pq-1)} = \frac{1}{q}\left(\frac{q+1}{pq-1} - \frac{N}{2}\right) > 0,$$

taking a sufficiently small $T > 0$ if necessary, by (6.9), (6.10) and (6.11) we apply Theorem 5.1 to see that problem (P) possesses a solution in $\mathbf{R}^N \times [0,1)$ with the initial data $(\mu_T, \nu_T)$. This means that problem (P) possesses a local-in-time solution. Thus Theorem 1.2 follows in case (D).

Case (E) : Let $0 < T < 1$ and $r_* \in (1,q)$. It follows that

$$\int_0^1 f(r) r^{-1}\,dr < \infty, \quad \text{where} \quad f(r) := \left[\int_0^r h_2(\tau)\tau^{-1}\,d\tau\right]^q. \tag{6.12}$$

Since $h_2$ is a positive function in $(0,1)$, similarly to (6.6), we see that

$$\begin{aligned}\sup_{x\in\mathbf{R}^N}\int_{B(x,\sqrt{t})} \mu_T(y,0)\,dy &\le CT^{\frac{p+1}{pq-1}-\frac{N}{2}}\int_{B(0,\sqrt{t})} |y|^{-N} h_2(|y|)\,dy\\ &\le CT^{\frac{p+1}{pq-1}-\frac{N}{2}} f(\sqrt{t})^{\frac{1}{q}}\end{aligned}$$

for $0 < t < 1$. Then, by (2.3) we have

$$\sup_{x\in\mathbf{R}^N} \|S(t)\mu_T\|_{L^{r_*}(B(x,\sqrt{t}))} \le CT^{\frac{p+1}{pq-1}-\frac{N}{2}} t^{-\frac{N}{2}\left(1-\frac{1}{r_*}\right)} f(\sqrt{t})^{\frac{1}{q}} \tag{6.13}$$

for $0 < t < 1$. On the other hand, we apply Lemma 2.2 with $a = N$ to obtain

$$\|S(t)\mu_T\|_{L^{r_*}(B(0,1)\setminus B(0,\sqrt{t}))} \le CT^{\frac{p+1}{pq-1}-\frac{N}{2}} t^{-\frac{N}{2}\left(1-\frac{1}{r_*}\right)} \left[h_2(t^{\frac{1}{6}}) + t^{\frac{N}{4}\left(1-\frac{1}{r_*}\right)} + f(\sqrt{t})^{\frac{1}{q}}\right]$$

for $0 < t < 1$. This together with (6.13) implies that

$$
\begin{aligned}
|||S(t)\mu_T|||_{r_*} &= \|S(t)\mu_T\|_{L^{r_*}(B(0,1))} \\
&\le \|S(t)\mu_T\|_{L^{r_*}(B(0,\sqrt{t}))} + \|S(t)\mu_T\|_{L^{r_*}(B(0,1)\setminus B(0,\sqrt{t}))} \\
&\le CT^{\frac{p+1}{pq-1}-\frac{N}{2}} t^{-\frac{N}{2}\left(1-\frac{1}{r_*}\right)} \left[h_2(t^{\frac{1}{6}})^q + t^{\frac{Nq}{4}\left(1-\frac{1}{r_*}\right)} + f(\sqrt{t})\right]^{\frac{1}{q}}
\end{aligned} \tag{6.14}
$$

for $0 < t < 1$. Since $q > 1$, $f(1) < \infty$ and $h_2$ is increasing in $(0, 1)$, we have

$$
\int_0^1 h_2(\tau^{\frac{1}{6}})^q \tau^{-1}\, d\tau = 6 \int_0^1 h_2(\tau)^q \tau^{-1}\, d\tau \le 6 h_2(1)^{q-1} f(1)^{\frac{1}{q}} < \infty.
$$

Then, since

$$
\int_0^1 \tau^{-1} f(\sqrt{\tau})\, d\tau = 2 \int_0^1 \tau^{-1} f(\tau)\, d\tau,
$$

by (6.12) we obtain

$$
\int_0^1 \left[h_2(\tau^{\frac{1}{6}})^q + \tau^{\frac{Nq}{4}\left(1-\frac{1}{r_*}\right)} + f(\sqrt{\tau})\right] \tau^{-1} d\tau < \infty. \tag{6.15}
$$

On the other hand, similarly to (6.11), we have

$$
\sup_{x\in\mathbf{R}^N} \nu_T(B(x,1)) \le T^{\frac{q+1}{pq-1}-\frac{N}{2}} \sup_{x\in\mathbf{R}^N} \nu(B(x,1)). \tag{6.16}
$$

In case (E), since $q = 1 + 2/N > p$, it follows that

$$
\frac{q+1}{pq-1} - \frac{N}{2} > \frac{p+1}{pq-1} - \frac{N}{2} = \frac{p+1}{pq-1} - \frac{1}{q-1} = \frac{q-p}{(pq-1)(q+1)} > 0.
$$

Therefore, taking a sufficiently small $T > 0$, by (6.14), (6.15) and (6.16) we apply Theorem 5.1 to see that problem (P) possesses a solution in $\mathbf{R}^N \times [0, 1)$ with the initial data $(\mu_T, \nu_T)$. This means that problem (P) possesses a local-in-time solution. Thus Theorem 1.2 follows in case (E).

Case (F) : By Lemma 2.1 and (6.1) we have

$$
\|S(t)\mu_T\|_\infty + \|S(t)\nu_T\|_\infty \le C\left(T^{\frac{p+1}{pq-1}-\frac{N}{2}} + T^{\frac{q+1}{pq-1}-\frac{N}{2}}\right) t^{-\frac{N}{2}}
$$

for $0 < t < 1$. Since it follows from $p \le q$ and $q < 1 + \frac{2}{N}$ that

$$
\frac{q+1}{pq-1} - \frac{N}{2} \ge \frac{p+1}{pq-1} - \frac{N}{2} > \frac{p+1}{pq-1} - \frac{1}{q-1} = \frac{q-p}{(pq-1)(q-1)} \ge 0,
$$

taking a sufficiently small $T > 0$, we apply Theorem 3.3 to see that problem (P) possesses a solution. Thus Theorem 1.2 follows in case (F). □

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## References

[1] D. Andreucci, M. A. Herrero and J. J. L. Velázquez, Liouville theorems and blow up behaviour in semilinear reaction diffusion systems, Ann. Inst. H. Poincaré Anal. Non Linéaire, **14** (1997), 1–53.

[2] P. Baras and M. Pierre, Critère d'existence de solutions positives pour des équations semi-linéaires non monotones, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), 185–212.

[3] M. Escobedo and M. A. Herrero, A uniqueness result for a semilinear reaction-diffusion system, Proc. Amer. Math. Soc. **112** (1991), 175–185.

[4] M. Escobedo and M. A. Herrero, Boundedness and blow up for a semilinear reaction-diffusion system, J. Differential Equations **89** (1991), 176–202.

[5] Y. Fujishima and K. Ishige, Blowing up solutions for nonlinear parabolic systems with unequall elliptic operators, J. Dynam. Differential Equations **32** (2020), 1219–1231.

[6] Y. Fujishima and K. Ishige, Initial traces and solvability of Cauchy problem to a semilinear parabolic system, to appear in J. Math. Soc. Japan.

[7] Y. Fujishima, K. Ishige and H. Maekawa, Blow-up set of type I blowing up solutions for nonlinear parabolic systems, Math. Ann. **369** (2017), 1491–1525.

[8] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 109–124.

[9] L. Grafakos, *Classical Fourier Analysis*, Springer-Verlag, 2008.

[10] K. Hisa and K. Ishige, Existence of solutions for a fractional semilinear parabolic equation with singular initial data, Nonlinear Anal. **175** (2018), 108–132.

[11] K. Hisa and K. Ishige, Solvability of the heat equation with a nonlinear boundary condition, SIAM J. Math. Anal. **51** (2019), 565–594.

[12] N. Ioku, K. Ishige and E. Yanagida, Sharp decay estimates in Lorentz spaces for nonnegative Schrödinger heat semigroups, J. Math. Pures Appl. **103** (2015), 900–923.

[13] K. Ishige, T. Kawakami and S. Okabe, Existence of solutions for a higher-order semilinear parabolic equation with singular initial data, Ann. Inst. H. Poincaré Anal. Non Linéaire **37** (2020), 1185–1209.

[14] K. Ishige, T. Kawakami and M. Sierżęga, Supersolutions for a class of nonlinear parabolic systems, J. Differential Equations **260** (2016), 6084–6107.

[15] Y. Maekawa and Y. Terasawa, The Navier-Stokes equations with initial data in uniformly local $L^p$ spaces, Differential Integral Equations **19** (2006), 369–400.

[16] P. Quittner and P. Souplet, Admissible $L_p$ norms for local existence and for continuation in semilinear parabolic systems are not the same, Proc. Roy. Soc. Edinburgh **131** (2001), 1435–1456.

[17] P. Quittner and P. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2007.

[18] J. C. Robinson and M. Sierżęga, Supersolutions for a class of semilinear heat equations, Rev. Mat. Compltu. **26** (2013), 341–360.

[19] J. Takahashi, Solvability of a semilinear parabolic equation with measures as initial data, Proceedings of the 4th Italian-Japanese workshop on Geometric Properties for Parabolic and Elliptic PDE's, 257–276, Springer Proceedings in Mathematics & Statistics **176**, 2016.

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