Symmetric and asymmetric nodal solutions for the Moore-Nehari differential equation

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Abstract

We consider the Moore-Nehari equation, $u'' + h(x,\lambda)|u|^{p-1}u = 0$ in $(-1,1)$ with $u(-1) = u(1) = 0$, where $p > 1$, $h(x,\lambda) = 0$ for $|x| < \lambda$, $h(x,\lambda) = 1$ for $\lambda \le |x| \le 1$ and $\lambda \in (0,1)$ is a parameter. We prove the existence of a solution which has exactly m zeros in the interval $(-1,0)$ and exactly n zeros in $(0,1)$ for given nonnegative integers m and n.

1 Introduction

We study the existence of nodal solutions for the Moore-Nehari differential equation

$$u'' + h(x,\lambda)|u|^{p-1}u = 0 \quad \text{in } (-1,1), \qquad u(-1) = u(1) = 0. \tag{1.1}$$

where $p > 1$, $h(x,\lambda) = 0$ for $|x| < \lambda$ and $h(x,\lambda) = 1$ for $\lambda \le |x| \le 1$ and $\lambda \in (0,1)$ is a parameter. Since $h(x,\lambda)$ is not continuous, a solution of (1.1) is not of $C^2(-1,1)$ class. We give a definition of solutions.

Definition 1.1. We call $u(x)$ a *solution* of (1.1) if $u \in C^2((-1,1) \setminus \{\pm\lambda\})$, $u \in C^1[-1,1]$, $u(-1) = u(1) = 0$ and $u(x)$ satisfies (1.1) pointwisely in $(-1,1) \setminus \{\pm\lambda\}$.

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Moore and Nehari [9] proved the result below.

Theorem A (Moore and Nehari [9, pp.31-32]). *There exists a* $\lambda \in (0,1)$ *such that* (1.1) *has at least three positive solutions: an even positive solution, a noneven positive solution* $u(x)$ *and its reflection* $u(-x)$.

We explain why Moore and Nehari [9] introduced (1.1) and elucidate the significance of the discontinuous function $h(x,\lambda)$. Instead of (1.1), we consider the initial value problem

$$u'' + |u|^{p-1}u = 0, \tag{1.2}$$

$$u(0) = 0, \quad u'(0) = 1.$$

Denote the unique solution of the problem above by $U(x)$. For $\mu > 0$, we define

$$U(x,\mu) := \mu^{2/(p-1)}U(\mu x).$$

Then it is a solution of (1.2) with $U(0,\mu) = 0$ and $U'(0,\mu) = \mu^{(p+1)/(p-1)}$. Therefore $U(x,\mu)$ with $\mu > 0$ represents all the solutions of (1.2) satisfying $U(0) = 0$ and $U'(0) > 0$. It is known (see Lemma 2.9) that $U(x)$ is a periodic solution having zeros. Denote the first zero of U in $(0,\infty)$ by z, that is, $U(x) > 0$ in $(0,z)$ and $U(z) = 0$. Then $z(\mu) := z/\mu$ is the first zero of $U(x,\mu)$ in $(0,\infty)$. Clearly, $z(\mu)$ is strictly decreasing with respect to $\mu > 0$, $\lim_{\mu\to+0} z(\mu) = \infty$ and $\lim_{\mu\to\infty} z(\mu) = 0$. These facts conclude that (1.2) with the boundary condition $u(0) = u(a) = 0$ has a unique positive solution for any $a > 0$. The existence and uniqueness of positive solutions for the boundary value problem follows from the monotonicity of the first zero of solutions. If the monotonicity of the first zero would be broken, the boundary value problem would have two or more positive solutions. For the autonomous equation (1.2), the monotonicity holds. Therefore Moore and Nehari introduced the nonautonomous equation (1.1). Then $h(x,\lambda)$ is discontinuous, but its definition is very simple. For this weight function $h(x,\lambda)$, the first zero is not monotone and the equation (1.1) has at least three positive solutions. This is reason why Moore and Nehari introduced (1.1). The weight function $h(x,\lambda)$ is significant for the existence of the multiple positive solutions.

Kajikiya, Sim and Tanaka [7] investigated the bifurcation of positive solutions for (1.1). Their result is as follows. For any $\lambda \in (0,1)$, there exists a unique even positive solution $u(x,\lambda)$ of (1.1). From the curve $(\lambda, u(x,\lambda))$, a noneven positive solution bifurcates at a unique point $\lambda = \lambda_*$. For λ close to 1, (1.1) has an even positive solution $u(x,\lambda)$, a noneven positive solution

$v(x, \lambda)$ and its reflection $v(-x, \lambda)$. Gritsans and Sadyrbaev [3] proved that (1.1) with $p = 3$ has infinitely many sign-changing solutions.

The purpose of the present paper is to prove the existence of sign-changing symmetric solutions and sign-changing asymmetric solutions. Another purpose is to prove the existence of a solution with prescribed numbers of zeros in the left interval $(-1, 0)$ and in the right interval $(0, 1)$. In fact, we shall obtain a solution which has exactly m zeros in $(-1, 0)$ and exactly n zeros in $(0, 1)$ for given nonnegative integers m, n.

Definition 1.2. For an integer $n \geq 0$, a solution u is called an *n-nodal solution* if it has exactly n zeros in $(-1, 1)$.

We shall show the existence of an n-nodal solution in Theorem 1.4 later on.

Definition 1.3. A solution u is called *symmetric* if it is even or odd. A solution u is called an *n-nodal symmetric solution* if it is both an n-nodal solution and a symmetric solution.

We note that if u is an n-nodal solution, so is $-u$. To fix an n-nodal solution, we impose the condition $u'(-1) > 0$. Then we have the next result.

Theorem 1.4. *Let $\lambda \in (0, 1)$. For any integer $n \geq 0$, (1.1) has a unique n-nodal symmetric solution satisfying $u'(-1) > 0$. Denote it by $u_n(x)$. Then the set of all nontrivial symmetric solutions consists only of $\pm u_n(x)$ with $n = 0, 1, 2, \ldots$.*

Theorem 1.5. *For each integer $n \geq 0$, there exists a constant $\varepsilon_n \in (0, 1)$ such that when $\lambda \in (0, \varepsilon_n)$, an n-nodal solution satisfying $u'(-1) > 0$ is unique and hence it must be symmetric.*

Theorem 1.4 means that an n-nodal symmetric solution is unique for any λ, that is, a solution of (1.1) is unique in a set of n-nodal symmetric solutions. However, Theorem 1.5 says that a solution is unique in a set of n-nodal solutions when $\lambda > 0$ is sufficiently small. If $\lambda \in (0, 1)$ is close to 1, we shall find asymmetric solutions in Theorem 1.7 later on.

Definition 1.6. For integers $m, n \geq 0$, we say that u is an *(m, n)-solution* if it is a solution of (1.1) which has exactly m zeros in $(-1, 0)$ and exactly n zeros in $(0, 1)$.

For different integers $m, n \geq 0$, we shall find an (m, n)-solution in Theorem 1.7 later on. Such a solution must be asymmetric (neither an even solution nor an odd solution). On the other hand, let us choose a $2n$-nodal even solution or a $(2n+1)$-nodal odd solution. Then it has exactly n zeros in $(-1,0)$ and in $(0,1)$. Therefore it becomes an (n,n)-solution. We have a question whether any (n,n)-solution becomes symmetric or whether an asymmetric (n,n)-solution exists. For $(0,0)$, the problem has been solved in Theorem A and in [7]. Indeed, as stated after Theorem A, when λ is close to 1, (1.1) has a noneven positive solution, i.e., an asymmetric $(0,0)$-solution. For more general (n,n), we consider this problem. Our answer is as follows.

Theorem 1.7. *For each nonnegative integer m, there exists a $\lambda_m \in (0,1)$ close to* 1 *satisfying the condition below. If $m \geq 1$, $\lambda \in (\lambda_m, 1)$ and $0 \leq n \leq m-1$, then there exists an (m,n)-solution of* (1.1). *If $m \geq 0$ and $\lambda \in (\lambda_m, 1)$, then there exists an asymmetric (m,m)-solution of* (1.1).

We shall explain the theorem above. The number λ_m is chosen sufficiently close to 1. Then we obtain an (m,n)-solution with $0 \leq n \leq m-1$ and an (m,m)-asymmetric solution also. Theorem 1.5 says that for given n, when λ is close to 0, an n-nodal solution must be symmetric. However, Theorem 1.7 ensures that when λ is close to 1, an n-nodal solution is not necessary symmetric. Indeed, there exists an (m,n)-solution for $0 \leq n \leq m-1$, which is an asymmetric $(m+n)$-nodal solution.

If $u(x)$ is an (m,n)-solution, then $u(-x)$ is an (n,m)-solution and hence we have (m,n)-solutions for all pairs (m,n). Many authors (see [10], [11] and the references cited therein) proved the existence of solutions having prescribed numbers of zeros in $(-1,1)$, for instance, for each integer $n \geq 0$, they obtained a solution having exactly n zeros in $(-1,1)$. We stress that an (m,n)-asymmetric solution is an $(m+n)$-nodal solution, but an (m,n)-solution gives us more detailed information than that of an $(m+n)$-nodal solution. To our knowledge, it seems that no authors study an (m,n)-solution except for our paper [6], where we studied the sublinear case $0 < p < 1$.

Observe that if u is a solution of (1.1), then the reflection of u with respect to the x-axis or y-axis is also a solution, that is, $u(-x)$, $-u(x)$ and $-u(-x)$ are solutions. We identify these solutions and introduce the next definition.

Definition 1.8. We say that solutions u and v of (1.1) are *equivalent* if they satisfy one of the following relations: $u(x) = v(x)$, $u(x) = v(-x)$, $u(x) = -v(x)$ or $u(x) = -v(-x)$.

We consider (m,n)-solutions. If u is an (m,n)-solution and v is an (n,m)-solution, then u may be possibly equivalent to v. However, an (m,n)-solution must not be equivalent to any (m',n')-solution if $(m,n) \neq (m',n')$ and $(m,n) \neq (n',m')$. Therefore Theorem 1.7 shows readily the next corollary.

Corollary 1.9. *Let n be a positive integer. Choose $\lambda \in (0,1)$ close enough to 1. Then the set of $2n$-nodal solutions includes $n+2$ nonequivalent solutions:*

$$(2n-i,i) \text{ with } 0 \le i \le n-1, \quad \text{asymmetric } (n,n), \quad \text{even } (n,n),$$

where $(2n-i,i)$ denotes a $(2n-i,i)$-solution, asymmetric (n,n) means an asymmetric (n,n)-solution and even (n,n) stands for a $2n$-nodal even solution. The first $n+1$ solutions are asymmetric and the last one is symmetric.

The set of $(2n+1)$-nodal solutions includes $n+2$ nonequivalent solutions:

$$(2n+1-i,i) \text{ with } 0 \le i \le n, \quad \text{odd } (n,n),$$

where odd (n,n) means an (n,n)-solution with oddness, i.e., a $(2n+1)$-nodal odd solution.

This paper is structured as bellow. In Section 2, we study the Emden-Fowler equation and give several energy estimates for later use. In Section 3, we study the Moore-Nehari equation, prove the continuous dependence of solutions on the initial data and give some a priori estimates of solutions. In Section 4, we prove all the theorems.

2 Emden-Fowler equation

Note that (1.1) is piecewise autonomous. Indeed, a solution of (1.1) satisfies the Emden-Fowler equation

$$u'' + |u|^{p-1}u = 0, \tag{2.1}$$

in the interval $[-1,-\lambda]$ and in $[\lambda,1]$. Moreover, $u''=0$ in $(-\lambda,\lambda)$. Hence the graph of $u(x)$ in $(-\lambda,\lambda)$ is a line segment. In this section, we study (2.1) and give some technical lemmas. For a solution u of (2.1), the energy function $E(u)(x)$ is defined by

$$E(u)(x) := \frac{1}{2}u'(x)^2 + \frac{1}{p+1}|u(x)|^{p+1}. \tag{2.2}$$

Then we have the the next lemma (for the proof, see [2] or [5, Lemma 3.1]).

Lemma 2.1. *For a nontrivial solution $u(x)$ of* (2.1)*, the following assertions hold.*

(i) *$E(u)(x)$ is a constant independent of $x \in \mathbb{R}$.*

(ii) *u is a periodic solution which has zeros.*

(iii) *If* $u(x_0)=0$*, then* $u(x_0+x)=-u(x_0-x)$ *for all* $x\in\mathbb{R}$*. If* $u'(x_1)=0$*, then* $u(x_1+x)=u(x_1-x)$ *for all* x*.*

(iv) *The functions* $\mu^{2/(p-1)}u(\mu x-x_0)$ *with* $\mu>0$ *and* $x_0\in\mathbb{R}$ *represent all nontrivial solutions of* (2.1)*.*

We consider the boundary value problem

$$\phi''+|\phi|^{p-1}\phi=0,\quad \phi(x)>0\quad\text{in }(0,1),\quad \phi(0)=\phi(1)=0. \tag{2.3}$$

Lemma 2.2. *There exists a unique solution* $\phi(x)$ *of* (2.3)*. Moreover,* $\phi'(x)>0$ *in* $[0,1/2)$*,* $\phi'(x)<0$ *in* $(1/2,1]$ *and* $\phi'(1/2)=0$*.*

Proof. Let $u(x)$ be a unique solution of the initial value problem

$$u''+|u|^{p-1}u=0,\quad u(0)=0,\quad u'(0)=1. \tag{2.4}$$

By Lemma 2.1, u is is a periodic solution which has zeros. Denote the first zero of $u(x)$ in $(0,\infty)$ by z. We define $\phi(x):=z^{2/(p-1)}u(zx)$. Then it is a solution of (2.3). Conversely, let $\phi(x)$ be any solution of (2.3). We shall show that $\phi(x)=z^{2/(p-1)}u(zx)$. If this claim would be proved, then the uniqueness of solutions for (2.3) holds. We first note that $\phi'(0)\ge 0$. If $\phi'(0)=0$, then $\phi(0)=\phi'(0)=0$ and hence $\phi(x)$ identically vanishes, because of the uniqueness of solutions for the initial value problem. Therefore $\phi'(0)>0$. Put $\mu:=\phi'(0)^{(p-1)/(p+1)}$ and $v(x):=\mu^{2/(p-1)}u(\mu x)$. Then $v(0)=\phi(0)=0$ and $v'(0)=\mu^{(p+1)/(p-1)}=\phi'(0)$, and hence $v(x)\equiv\phi(x)$. Thus $\phi(x)=\mu^{2/(p-1)}u(\mu x)$. Since $\phi(0)=\phi(1)=0$ and $\phi(x)>0$ in $(0,1)$, $u(0)=u(\mu)=0$ and $u(\mu x)>0$ in $(0,1)$. Accordingly, μ is the first zero of u and $\mu=z$. Therefore we have $\phi(x)=z^{2/(p-1)}u(zx)$.

Let $\phi(x)$ be the unique solution of (2.3). ϕ is concave in $(0,1)$ because $\phi''=-\phi^p<0$. Therefore $\phi(x)$ has a unique critical point in $(0,1)$. Denote it by t, i.e., $\phi'(t)=0$. By (iii) of Lemma 2.1, $\phi(t-x)=\phi(t+x)$ for any x. This fact with $\phi(0)=\phi(1)=0$ proves $t=1/2$, that is, $\phi'(1/2)=0$ and $\phi'(x)>0$ in $[0,1/2)$, $\phi'(x)<0$ in $(1/2,1]$. The proof is complete. □

From now on, we denote the unique solution of (2.3) by $\phi(x)$ and extend it as a solution in $\mathbb{R}$. Then it is a periodic solution with period 2 by Lemma 2.1. Since $E(u)(x)$ given by (2.2) is independent of x, we sometimes denote it by $E(u)$. By Lemma 2.1, we see that all zeros of a nontrivial solution for (2.1) lie in equal intervals. We shall investigate the relation between this interval and the energy. The next proposition can be easily proved, however it is crucial for the later lemmas.

Proposition 2.3. *Let $u \not\equiv 0$ be any solution of* (2.1) *and denote by l the distance between two adjacent zeros of u. Then it holds that*

$$E(u) = c_1 l^{-2(p+1)/(p-1)}, \quad \|u\|_\infty = c_2 l^{-2/(p-1)}, \tag{2.5}$$

where $c_1 := \phi'(0)^2/2$, $c_2 := \|\phi\|_\infty = \phi(1/2)$, $\phi(x)$ is defined by (2.3) *and $\|\phi\|_\infty$ denotes the L^∞-norm of ϕ.*

Proof. Let u and l satisfy the assumption of the lemma. After translation, we may suppose that $u(0) = u(l) = 0$ and $u(x) > 0$ in $(0, l)$. Let ϕ be a unique solution of (2.3). Then u is represented as $u(x) = l^{-2/(p-1)}\phi(x/l)$. Since $u(0) = 0$ and $u'(0) = l^{-(p+1)/(p-1)}\phi'(0)$, we have (2.5). The proof is complete. □

The next lemma means that the order relation of positive solutions implies that of the energies.

Lemma 2.4. *Let (a, b) be any finite interval. Let u and v be solutions of* (2.1) *such that $u(x) = v(x)$ at $x = a, b$ and $v(x) > u(x) > 0$ in (a, b). Then $E(v) > E(u)$.*

Proof. By assumption, $u'(a) \le v'(a)$. If $u'(a) = v'(a)$, then $u \equiv v$, a contradiction. Therefore $u'(a) < v'(a)$. Similarly, $u'(b) > v'(b)$. If $u'(a) \ge 0$, then we have

$$E(u) = \frac{1}{2}u'(a)^2 + \frac{1}{p+1}|u(a)|^{p+1} < \frac{1}{2}v'(a)^2 + \frac{1}{p+1}|v(a)|^{p+1} = E(v).$$

Thus the conclusion holds. Let $u'(a) < 0$. Since u is concave in (a, b), we have $u'(b) < 0$. Therefore $v'(b) < u'(b) < 0$, and it follows that $E(v) > E(u)$. The proof is complete. □

By Proposition 2.3, we find that the distance between two consecutive zeros of a solution is smaller as the energy is larger. Indeed, Proposition 2.3 readily ensures the next lemma.

Lemma 2.5. *Let u and v be nontrivial solutions of* (2.1) *satisfying $E(u) < E(v)$. Then the distance between two adjacent zeros of v is shorter than that of u. In particular, if $u(x)$ vanishes at a pair of points $x = x_1, x_2$ $(x_1 < x_2)$, then $v(x)$ has at least one zero in (x_1, x_2).*

The next lemma shows that a solution with a large energy must have many zeros.

Lemma 2.6. *Let* (a, b) *be any finite interval,* m *be a positive integer and* c_1 *be given by Proposition 2.3. If* $u(x)$ *is a solution of* (2.1) *satisfying*

$$E(u) > c_1((b-a)/m)^{-2(p+1)/(p-1)}, \tag{2.6}$$

then it has at least m *zeros in the interval* (a, b)*.*

Proof. Let u be a solution satisfying the assumption of the lemma and let l be the distance between consecutive zeros of u. Combining Proposition 2.3 with (2.6), we have $l < (b-a)/m$. Therefore $u(x)$ has at least m zeros in (a, b). □

For a finite interval (a, b), we consider the problem

$$u'' + |u|^{p-1}u = 0, \quad u(x) > 0 \quad \text{in } (a, b), \quad u(a) = u(b) = 0. \tag{2.7}$$

In the same method as in the proof of Lemma 2.2, we can prove the following lemma.

Lemma 2.7. (2.7) *has a unique solution* $u(x)$*. Furthermore,* $u'(x) > 0$ *in* $[a, (a+b)/2)]$*,* $u'((a+b)/2) = 0$ *and* $u'(x) < 0$ *in* $((a+b)/2, b]$*.*

We consider an n-nodal solution of the problem

$$u'' + |u|^{p-1}u = 0, \quad u(-1) = u(1) = 0. \tag{2.8}$$

Proposition 2.8. *For any nonnegative integer* n*, there exists a unique* n*-nodal solution of* (2.8) *satisfying* $u'(-1) > 0$*. Denote it by* $\phi_n(x)$*. Then the set of all nontrivial solutions consists only of* $\pm\phi_n(x)$ *with* $n = 0, 1, 2, \ldots$*.*

Proof. Put $l := 2/(n+1)$. Then l is a distance between two adjacent zeros of an n-nodal solution of (2.8). Observe that $-1 + jl$ with $1 \le j \le n$ are all interior zeros of an n-nodal solution.

Let $u(x)$ be an n-nodal solution of (2.8) satisfying $u'(-1) > 0$. Then u satisfies (2.7) with $a = -1$ and $b = -1 + l$. Conversely, if u satisfies (2.7) with (a, b) replaced by $(-1, -1 + l)$, it can be extended in $\mathbb{R}$ as a solution of the Emden-Fowler equation. Then it becomes an n-nodal solution of (2.8). Consequently, the existence and the uniqueness of n-nodal solutions for (2.8) with $u'(-1) > 0$ is equivalent to those of (2.7) with (a, b) replaced by $(-1, -1 + l)$. From Lemma 2.7, the conclusion follows. The proof is complete. □

For $a > 0$, we consider the problem

$$\psi'' + |\psi|^{p-1}\psi = 0, \quad \psi > 0 \quad \text{in } (0, a), \quad \psi'(0) = \psi(a) = 0. \tag{2.9}$$

We need the next technical lemma to prove Theorem 1.7.

Lemma 2.9. *For $a > 0$, (2.9) has a unique solution. Denote it by $\psi(x, a)$. Then the following assertions hold.*

(i) *$\psi'(x, a)/\psi(x, a)$ is strictly decreasing with respect to $x \in (0, a)$, where $\psi'(x, a)$ denotes the derivative with respect to x.*

(ii) *For any $M > 0$, the equation $\psi'(\mu, a)/\psi(\mu, a) = -M$ has a unique solution $\mu \in (0, a)$.*

(iii) *Denote μ defined as above by $\mu(M, a)$, let ϕ be given by Lemma 2.2 and put*

$$c_3 := -\frac{\phi'(3/4)}{2\phi(3/4)} \ (> 0).$$

If $a < c_3/M$, then $\mu(M, a) < a/2$.

(iv) *Define $\mu(M, a)$ by (iii) and put $c_2 := \|\phi\|_\infty$. If M satisfies $\|\psi(\cdot, a)\|_\infty > c_2(2c_3/M)^{-2/(p-1)}$, then $\mu(M, a) < a/2$.*

Proof. Observe that (2.9) is equivalent to the next equation

$$\psi'' + |\psi|^{p-1}\psi = 0, \quad \psi > 0 \quad \text{in } (-a, a), \quad \psi(-a) = \psi(a) = 0. \tag{2.10}$$

Indeed, we extend a solution ψ of (2.9) in $\mathbb{R}$ as a solution of the Emden-Fowler equation. Then it must be even because of Lemma 2.1 (iii) with $\psi'(0) = 0$. Hence it satisfies (2.10). Conversely, if ψ is a solution of (2.10), then it satisfies $\psi'(0) = 0$ because of Lemma 2.7. Thus (2.9) is equivalent to (2.10). The existence and the uniqueness of solutions for (2.10) (or (2.9)) follows from Lemma 2.7.

We shall show (i). Denote $\psi(x, a)$ by $\psi(x)$. Then

$$\left(\frac{\psi'(x)}{\psi(x)}\right)' = \frac{\psi''\psi - (\psi')^2}{\psi^2} = \frac{-|\psi|^{p+1} - (\psi')^2}{\psi^2} < 0.$$

Hence (i) holds .

Since $\psi'(x)/\psi(x)$ is strictly decreasing and

$$\lim_{x\to+0} \psi'(x)/\psi(x) = 0, \quad \lim_{x\to a-0} \psi'(x)/\psi(x) = -\infty,$$

the equation $\psi'(\mu, a)/\psi(\mu, a) = -M$ has a unique solution μ.

We shall show (iii). The unique solution $\psi(x, a)$ of (2.9) is explicitly written as

$$\psi(x, a) = (2a)^{-2/(p-1)}\phi((2a)^{-1}x + 2^{-1}),$$

where ϕ is defined by Lemma 2.2. Put $\mu := \mu(M, a)$. Assume that $0 < a < c_3/M$. Suppose to the contrary that $\mu \geq a/2$. Since $\psi'(x)/\psi(x)$ is strictly decreasing, we have

$$-M = \psi'(\mu, a)/\psi(\mu, a) \leq \psi'(a/2, a)/\psi(a/2, a).$$

The right hand side is computed as

$$\psi'(a/2, a)/\psi(a/2, a) = (2a)^{-1}\phi'(3/4)/\phi(3/4) = -a^{-1}c_3.$$

Hence $-M \leq -a^{-1}c_3$ or equivalently $c_3/M \leq a$. A contradiction occurs. Accordingly, (iii) holds.

We shall prove (iv). Suppose that $\|\psi\|_\infty > c_2(2c_3/M)^{-2/(p-1)}$. By (2.10), the distance between consecutive zeros of ψ is $2a$. By Proposition 2.3, $\|\psi\|_\infty = c_2(2a)^{-2/(p-1)}$. Therefore $a < c_3/M$. By (iii), we have $\mu < a/2$. The proof is complete. □

We call a nontrivial solution u of (2.8) *nondegenerate* if the linearized operator $(d^2/dx^2) + p|u|^{p-1}$ does not have a zero eigenvalue, i.e., the problem

$$v'' + p|u|^{p-1}v = 0, \quad v(-1) = v(1) = 0,$$

has no solution except for the zero solution $v(x) \equiv 0$.

Proposition 2.10. *All nontrivial solutions of* (2.8) *are nondegenerate.*

The nondegeneracy of a positive solution is known (for example, see [8, p.555, Lemma 2.6]). However it seems unclear whether any nodal solution is nondegenerate. The nondegeneracy will be used for proving Theorem 1.5. To prove the the proposition above, let n be a nonnegative integer and u be an n-nodal solution of (2.8). Extend u as a solution of (2.1) into $\mathbb{R}$. Let v be a solution of the equation

$$v'' + p|u|^{p-1}v = 0, \quad v(-1) = 0, \quad v \not\equiv 0. \tag{2.11}$$

Lemma 2.11. *Let n be a nonnegative integer and $u(x)$ be an n-nodal solution of* (2.8)*. Extend u as a solution of* (2.1) *into $\mathbb{R}$. Let $v(x)$ be a solution of* (2.11)*. Then v is defined in $\mathbb{R}$ and has infinitely many zeros in $(-\infty, -1)$ and in $(-1, \infty)$. Let x_i be the i-th zero of v in $(-1, \infty)$. Then $x_{n+1} < 1 < x_{n+2}$. In particular, $v(1) \neq 0$.*

Lemma 2.11 directly proves Proposition 2.10.

Proof of Lemma 2.11. Let n be any nonnegative integer and let u be an n-nodal solution of (2.8). Then it is extended into $\mathbb{R}$ as a solution. We may suppose that $u'(-1) > 0$. Otherwise, we replace u by $-u$. Denote the i-th critical point and the i-th zero of $u(x)$ in $(-1, \infty)$ by t_i and z_i, respectively. Put $z_0 := -1$. Then $u'(t_i) = 0$, $u(z_i) = 0$ and

$$-1 = z_0 < t_1 < z_1 < \cdots < t_{n+1} < z_{n+1} = 1 < t_{n+2} < \cdots .$$

Let v be a solution of (2.11). Since u is bounded in $\mathbb{R}$, v is defined in the whole $\mathbb{R}$. Compare the first equation of (2.8) with (2.11). Since $|u|^{p-1} < p|u|^{p-1}$, the Sturm comparison theorem (see [4, p.334, Theorem 3.1]) ensures that $v(x)$ has at least one zero in an interval between two consecutive zeros of u. Therefore v has infinitely many zeros in $(-\infty, -1)$ and in $(-1, \infty)$. Denote the i-th zero of v in $(-1, \infty)$ by x_i. By the Sturm comparison theorem, we have

$$x_i < z_i \quad \text{for all } i \geq 1. \tag{2.12}$$

We define $U(x) := u'(x)$. Then it fulfills

$$U'' + p|u|^{p-1}U = 0.$$

Thus U and v are solutions of the same equation. Since $U(-1) = u'(-1) \neq 0$ and $v(-1) = 0$, U and v are linearly independent. By the Sturm separation theorem (see [4, p.335, Corollary 3.1]), the zeros of v and U separate each other, i.e., v has a unique zero in (t_i, t_{i+1}) and U has a unique zero in (x_j, x_{j+1}).

We shall show that $t_1 < x_1 < z_1$. If v has at least one zero in $(-1, t_1]$, then $x_1 \leq t_1$ and $v(-1) = v(x_1) = 0$. By the Sturm separation theorem, $U = u'$ has a zero in $(-1, x_1)$. This is impossible. Therefore v has no zeros in $(-1, t_1]$. Since $\{x_i\}$ and $\{t_i\}$ separate each other, x_1 lies in (t_1, t_2). Since $x_1 < z_1$ by (2.12), we find that $t_1 < x_1 < z_1$. By induction, we shall prove that $t_i < x_i$ for all i. Assume that $t_{i-1} < x_{i-1}$. We shall show $t_i < x_i$. Suppose to the contrary that $x_i \leq t_i$. Then $t_{i-1} < x_{i-1} < x_i \leq t_i$. Thus U has no zeros in (x_{i-1}, x_i). This is a contradiction. Accordingly, $t_i < x_i$. By (2.12), $x_i < z_i$. Consequently, $t_i < x_i < z_i$ for all i. In particular, $t_{n+1} < x_{n+1} < z_{n+1} = 1 < t_{n+2} < x_{n+2}$. The proof is complete. □

3 Moore-Nehari equation

We study the Moore-Nehari differential equation and give some lemmas for later use. In Definition 1.1, we have already defined solutions for (1.1). Since $h \in L^\infty(-1, 1)$, a solution can be considered in a Sobolev space $W^{2,\infty}(-1, 1)$

also. This space is the set of all functions $u \in L^\infty(-1,1)$ such that $u', u'' \in L^\infty(-1,1)$. It is well known that $W^{1,\infty}(-1,1) = \mathrm{Lip}[-1,1]$ (see [1, p.207, Proposition 8.4]), where $\mathrm{Lip}[-1,1]$ denotes the set of all Lipschitz continuous functions on $[-1,1]$. Therefore we have

$$W^{2,\infty}(-1,1) = \{u \in C^1[-1,1] : u' \in \mathrm{Lip}[-1,1]\}.$$

A solution in $W^{2,\infty}(-1,1)$ is equivalent to that of a class $C^2((-1,1)\setminus\{\pm\lambda\}) \cap C^1[-1,1]$. We shall show this fact in the next lemma.

Lemma 3.1. *The following are equivalent.*

(i) $u \in C^2((-1,1)\setminus\{\pm\lambda\}) \cap C^1[-1,1]$, $u(-1) = u(1) = 0$ *and* $u(x)$ *satisfies* (1.1) *pointwisely in* $(-1,1) \setminus \{\pm\lambda\}$.

(ii) $u \in W^{2,\infty}(-1,1)$, $u(-1) = u(1) = 0$ *and* $u(x)$ *satisfies* (1.1) *in the distribution sense.*

We adopted (i) as a definition of solutions in Definition 1.1.

Proof. Let $u(x)$ satisfy (ii). Since $W^{2,\infty}(-1,1)$ is embedded in $C^1[-1,1]$, $u(x)$ belongs to $C^1[-1,1]$. Since u satisfies (1.1) in the distribution sense, we have

$$-\int_{-1}^{1} u'(x)\phi'(x)dx + \int_{-1}^{1} h(x,\lambda)|u|^{p-1}u\phi dx = 0 \quad \text{for } \phi \in C_0^\infty(-1,1).$$

Let $\phi \in C_0^\infty(-1,-\lambda)$. Since $h(x,\lambda) = 1$ in $(-1,-\lambda)$, we have

$$-\int_{-1}^{-\lambda} u'(x)\phi'(x)dx + \int_{-1}^{-\lambda} |u|^{p-1}u\phi dx = 0,$$

and hence $u(x)$ satisfies

$$u'' + |u|^{p-1}u = 0 \quad \text{in } (-1,-\lambda),$$

in the distribution sense. Thus $u \in C^2(-1,-\lambda)$. In the same way, u belongs to $C^2(-\lambda,\lambda)$ and to $C^2(\lambda,1)$. Therefore u satisfies (i).

Conversely, suppose that u satisfies (i). Since $u \in C^1[-1,1]$, it belongs to $W^{1,\infty}(-1,1)$. We shall show that u satisfies (1.1) in the distribution sense. Let $\phi \in C_0^\infty(-1,1)$. We compute

$$\int_{-1}^{1} u'(x)\phi'(x)dx = \int_{-1}^{-\lambda} u'(x)\phi'(x)dx + \int_{-\lambda}^{\lambda} u'(x)\phi'(x)dx + \int_{\lambda}^{1} u'(x)\phi'(x)dx. \tag{3.1}$$

Note that $\phi(-1) = \phi(1) = 0$ and use the integration by parts. Then each term is computed as

$$\int_{-1}^{-\lambda} u'(x)\phi'(x)dx = u'(-\lambda)\phi(-\lambda) - \int_{-1}^{-\lambda} u''(x)\phi(x)dx,$$

$$\int_{-\lambda}^{\lambda} u'(x)\phi'(x)dx = u'(\lambda)\phi(\lambda) - u'(-\lambda)\phi(-\lambda) - \int_{-\lambda}^{-\lambda} u''(x)\phi(x)dx,$$

$$\int_{\lambda}^{1} u'(x)\phi'(x)dx = -u'(\lambda)\phi(\lambda) - \int_{\lambda}^{1} u''(x)\phi(x)dx.$$

Substituting the three equations above in (3.1), we obtain

$$\int_{-1}^{1} u'(x)\phi'(x)dx = -\int_{-1}^{1} u''(x)\phi(x)dx.$$

Using the inequality above, we find that

$$-\int_{-1}^{1} u'(x)\phi'(x)dx + \int_{-1}^{1} h|u|^{p-1}u\phi dx = \int_{-1}^{1} \left(u'' + h|u|^{p-1}u\right)\phi dx = 0,$$

because u satisfies (1.1) pointwisely in $(-1,1)\backslash\{\pm\lambda\}$. Accordingly, u satisfies (1.1) in the distribution sense, and hence the distributional derivative u'' belongs to L^∞, because

$$u'' = -h|u|^{p-1}u \in L^\infty(-1,1).$$

Therefore u belongs to $W^{2,\infty}(-1,1)$. The proof is complete. □

Hereafter we extend the definition domain of $h(x,\lambda)$ to $(-\infty,\infty)$ by putting $h(x,\lambda) = 1$ for $|x| \geq 1$. Hence $h(x,\lambda) = 0$ for $|x| < \lambda$ and $h(x,\lambda) = 1$ for $|x| \geq \lambda$. We consider the initial value problem

$$u'' + h(x,\lambda)|u|^{p-1}u = 0, \quad u(x_0) = \alpha, \quad u'(x_0) = \beta. \tag{3.2}$$

Lemma 3.2. *For any $\alpha, \beta, x_0 \in \mathbb{R}$, there exists a unique solution $u(x,\alpha,\beta)$ of* (3.2)*, which is defined for all $x \in (-\infty,\infty)$.*

Proof. It is clear that (3.2) is equivalent to the integral equation

$$u(x) = \alpha + \beta(x - x_0) - \int_{x_0}^{x}\int_{x_0}^{s} h(t,\lambda)|u(t)|^{p-1}u(t)dtds. \tag{3.3}$$

Although $h(x,\lambda)$ is not continuous, the local Lipschitz continuity of $|u|^{p-1}u$ ensures the local existence and the uniqueness of solutions for (3.3). Note that the graph of u is linear in $[-\lambda,\lambda]$ and u has the energy $E(u)$ in $(-\infty,-\lambda]$ and another energy $E(u)$ in $[\lambda,\infty)$. These facts give an a priori bound of u. Therefore $u(x)$ can be extended in $(-\infty,\infty)$. The proof is complete. □

We denote the $L^\infty(-\infty,\infty)$ norm of u by $\|u\|_\infty$. Then $W^{2,\infty}(-\infty,\infty)$ norm is defined by

$$\|u\|_{W^{2,\infty}} := \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty.$$

Let $u(x,\alpha,\beta)$ be a solution given by Lemma 3.2. We show the continuous dependence of solutions $u(x,\alpha,\beta)$ on initial data.

Lemma 3.3. *$u(x,\alpha,\beta)$ is continuous with respect to α and β, more precisely we have*

$$\|u(\cdot,\alpha,\beta) - u(\cdot,\alpha_0,\beta_0)\|_{W^{2,\infty}(-X,X)} \to 0 \quad \text{as } (\alpha,\beta) \to (\alpha_0,\beta_0), \tag{3.4}$$

for any $X > 0$.

Proof. Let $x_0 \in \mathbb{R}$. We shall prove that for each $R > 0$, there exists a $C > 0$ such that

$$\|u(\cdot,\alpha,\beta)\|_{W^{2,\infty}(-\infty,\infty)} \leq C \quad \text{when } |\alpha| + |\beta| \leq R. \tag{3.5}$$

Hereafter C denotes a positive constant independent of u, which may vary from line to line. We define the C^1 norm by

$$\|u\|_{C^1} := \|u\|_\infty + \|u'\|_\infty.$$

Recall that the graph of u is linear in $[-\lambda,\lambda]$ and u is a periodic solution with a constant energy $E(u)$ in the interval $(-\infty,-\lambda]$ and another energy in $[\lambda,\infty)$. Using these facts, we can easily prove that for any $R > 0$, there exists a $C > 0$ such that

$$\|u(\cdot,\alpha,\beta)\|_{C^1} \leq C \quad \text{when } |\alpha| + |\beta| \leq R.$$

By (3.2), we find that $\|u''\|_\infty = \|h(\cdot,\lambda)|u|^p\|_\infty \leq C^p$. This inequality proves (3.5).

We shall show (3.4). Let $\alpha_0, \beta_0 \in \mathbb{R}$ and $X > 0$. By (3.5), we have a constant $C > 0$ such that

$$\|u(\cdot,\alpha,\beta)\|_{W^{2,\infty}(-\infty,\infty)} \leq C \quad \text{when } |\alpha-\alpha_0| + |\beta-\beta_0| \leq 1.$$

Let (α_n,β_n) be a sequence which converges to (α_0,β_0) as $n \to \infty$. Set $u_n(x) := u(x,\alpha_n,\beta_n)$. Since u_n is bounded in $W^{2,\infty}(-X,X)$, the compact embedding ensures that a subsequence (denoted again by u_n) of u_n converges to a limit u in $C^1[-X,X]$. Since u_n is a solution of (3.2), we integrate it to obtain

$$u_n'(x) = \beta_n - \int_{x_0}^{x} h(t,\lambda)|u_n(t)|^{p-1}u_n(t)dt.$$

Letting $n \to \infty$, we obtain

$$u'(x) = \beta_0 - \int_{x_0}^{x} h(t,\lambda)|u(t)|^{p-1}u(t)dt.$$

Therefore u satisfies the first equation of (3.2). Letting $n \to \infty$ in $u_n(x_0) = \alpha_n$ and $u_n'(x_0) = \beta_n$, we have $u(x_0) = \alpha_0$ and $u'(x_0) = \beta_0$. Therefore u satisfies (3.2) with α and β replaced by α_0 and β_0, respectively. Hence $u(\cdot, \alpha_n, \beta_n)$ converges to $u(\cdot, \alpha_0, \beta_0)$ in $C^1[-X, X]$. Since the limit $u(\cdot, \alpha_0, \beta_0)$ is unique, $u(\cdot, \alpha_n, \beta_n)$ itself (without extracting a subsequence) converges to $u(\cdot, \alpha_0, \beta_0)$. Since

$$\|u_n'' - u''\|_{L^\infty(-X,X)} = \|h(\cdot,\lambda)(|u_n|^{p-1}u_n - |u|^{p-1}u)\|_{L^\infty(-X,X)} \to 0,$$

we obtain (3.4). The proof is complete. □

In the same way as in the proof of Lemma 2.1 (iii), we can readily show the lemma below.

Lemma 3.4. *Let* $u(\not\equiv 0)$ *be a solution of* (1.1). *Then it is odd if and only if* $u(0) = 0$.

The next technical lemma will be used in the later section.

Lemma 3.5. *Let* (a, b) *be a finite interval and let* u *and* v *be different solutions for the first equation of* (3.2) *in* (a, b) *such that* $v(x) > 0$ *in* (a, b) *and* $v(a) = v(b) = 0$. *Then* $u(x_0) < v(x_0)$ *at some* $x_0 \in (a, b)$.

Proof. Let u and v satisfy the assumption of the lemma. Since $v(x) > 0$ in (a, b) and $v(a) = v(b) = 0$, $h(x, \lambda)$ does not identically vanish in (a, b). Suppose that the conclusion is false, i.e., $u(x) \geq v(x)$ in (a, b). If $u(x_1) = v(x_1)$ at some $x_1 \in (a, b)$, then $u'(x_1) = v'(x_1)$ and hence $u \equiv v$. A contradiction occurs. Therefore $u(x) > v(x)$ in (a, b). Since $h(x,\lambda)|v|^{p-1} \leq h(x,\lambda)|u|^{p-1}$ and the strict inequality holds somewhere, the Sturm comparison theorem says that u has a zero in (a, b). This is impossible. Therefore $u(x_0) < v(x_0)$ at some $x_0 \in (a, b)$. □

To prove the main theorems, we use the shooting method. Let us consider the initial value problem

$$u'' + h(x,\lambda)|u|^{p-1}u = 0, \quad u(-1) = 0, \quad u'(-1) = \alpha > 0. \tag{3.6}$$

Denote a unique solution of (3.6) by $u(x, \alpha)$. It has infinitely many zeros in $(-\infty, -1]$ and in $[1, \infty)$. Each zero of u is a continuous function of the initial value α because of Lemma 3.3.

Lemma 3.6. *If $\alpha > 0$ is small enough, then $u(x, \alpha) > 0$ in $(-1, 1]$.*

Proof. Choose $\delta > 0$ satisfying $2\sqrt{\delta} < \pi$. By Lemma 3.3 with the fact that $u(x, 0) \equiv 0$, $\|u(\cdot, \alpha)\|_{L^\infty(-1,1)}$ is small enough when $\alpha > 0$ is small. In particular, we choose $\alpha_0 > 0$ such that $|u(x, \alpha)|^{p-1} < \delta$ for $x \in [-1, 1]$ when $\alpha \in (0, \alpha_0)$. Therefore $0 \le h(x, \lambda)|u|^{p-1} < \delta$. Fix $\alpha \in (0, \alpha_0)$ and put $u(x) := u(x, \alpha)$. We shall show that $u(x)$ has no zeros in $(-1, 1]$. Suppose to the contrary that it has a zero $x_0 \in (-1, 1]$. We compare (3.6) with the equation

$$v'' + \delta v = 0.$$

Put $v(x) := \sin \sqrt{\delta}(x + 1)$. Since $h(x, \lambda)|u|^{p-1} < \delta$ and $u(-1) = u(x_0) = 0$, the Sturm comparison theorem ensures that v has a zero in $(-1, x_0)$. This contradict the assumption that $2\sqrt{\delta} < \pi$. Therefore u has no zeros and $u(x) > 0$ in $(-1, 1]$. The proof is complete. □

Observe Lemma 3.4. To obtain an n-nodal odd solution, we consider the initial value problem

$$v'' + h(x, \lambda)|v|^{p-1}v = 0, \quad v(0) = 0, \quad v'(0) = \mu > 0. \tag{3.7}$$

Denote a unique solution of (3.7) by $v(x, \mu)$. It has infinitely many zeros in $[1, \infty)$. In the same way as in the proof of Lemma 3.6, we can prove the next lemma.

Lemma 3.7. *When $\mu > 0$ is small enough, $v(x, \mu) > 0$ for $x \in (0, 1]$.*

Definition 3.8. For a nontrivial solution u of (1.1), $E(u)(x)$ is defined by (2.2). Then $E(u)(x)$ is a constant $E_-(u)$ in $(-\infty, -\lambda]$ and another constant $E_+(u)$ in $[\lambda, \infty)$, i.e.,

$$E_-(u) := E(u)(x)|_{(-\infty,-\lambda]}, \quad E_+(u) := E(u)(x)|_{[\lambda,\infty)}.$$

Lemma 3.9. *For any positive integer n, there exists a constant $\mu_n > 0$ such that when $\mu > \mu_n$, the solution $v(x, \mu)$ of (3.7) has at least n zeros in $(0, 1)$.*

Proof. Put $v(x) := v(x, \mu)$. Since $v'(x) \equiv v'(0) = \mu$ in $[0, \lambda]$, its energy $E_+(v)$ is estimated as

$$E_+(v)(\lambda) \ge \frac{1}{2}v'(\lambda)^2 = \frac{1}{2}\mu^2.$$

If $\mu > 0$ is large enough, so is $E_+(v)$. Note that v satisfies (2.1) in $(\lambda, 1)$. Using Lemma 2.6 with $(a, b) = (\lambda, 1)$, we obtain the conclusion. □

To prove Theorem 1.5, we need the next lemma, which provides an a priori estimate of nodal solutions by means of the number of zeros.

Lemma 3.10. *For each nonnegative integer n, there exists a positive constant C_n independent of λ such that if u is an n-nodal solution of* (1.1) *with $\lambda \in (0, 1/2)$, then $\|u\|_{W^{2,\infty}(-1,1)} \le C_n$.*

Proof. We use Lemma 2.6 with $(a, b) = (\lambda, 1)$, $\lambda \in (0, 1/2)$ and $m := n + 1$. Put $C := c_1(2m)^{2(p+1)/(p-1)}$. Let u be an n-nodal solution. If $E_+(u) > C$, then we have

$$E_+(u) > c_1(2m)^{2(p+1)/(p-1)} > c_1((1-\lambda)/m)^{-2(p+1)/(p-1)},$$

because $\lambda \in (0, 1/2)$. Thus (2.6) holds. By Lemma 2.6, u has at least $n + 1$ zeros in $(\lambda, 1)$. This is a contradiction. Therefore, if u is an n-nodal solution of (1.1), then $E_+(u) \le C$. Similarly, putting $(a, b) = (-1, -\lambda)$, we obtain $E_-(u) \le C$. Therefore the C^1-norms of u in the intervals $[-1, -\lambda]$ and $[\lambda, 1]$ are bounded from above by a certain constant $C_0 > 0$ independent of λ. Then $|u(\pm\lambda)|, |u'(\pm\lambda)| \le C_0$, and so the $C^1[-\lambda, \lambda]$-norm of u is estimated by C_0. Therefore u has an a priori estimate for the $C^1[-1, 1]$-norm. Then (1.1) proves that

$$\|u''\|_{L^\infty(-1,1)} = \|h(x, \lambda)|u|^p\|_{L^\infty(-1,1)} \le C_0^p.$$

The proof is complete. □

We give an a priori lower estimate of nontrivial solutions.

Lemma 3.11. *There exists a constant $c > 0$ independent of $\lambda \in (0, 1)$ such that if $u(x)$ is a nontrivial solution of* (1.1)*, then $\|u\|_{C^1[-1,1]} \ge c$.*

Proof. We denote the $L^q(-1, 1)$-norm of u by $\|u\|_q$. Multiplying (1.1) by u and integrating it over $(-1, 1)$, we have

$$\int_{-1}^{1} u'(x)^2 dx = \int_{-1}^{1} h|u|^{p+1} dx \le \|u\|_{p+1}^{p+1} \le C\|u'\|_2^{p+1},$$

with some $C > 0$ because of the Sobolev embedding with the Poincaré inequality. Dividing both sides by $\|u'\|_2^2$, we obtain

$$1 \le C\|u'\|_2^{p-1} \le C'\|u\|_{C^1[-1,1]}^{p-1},$$

with some $C' > 0$. The proof is complete. □

4 Proofs of the main theorems

We shall prove the main theorems.

Definition 4.1. For a function $u(x)$ defined in an interval (a,b), we denote by $N[u,(a,b)]$ the number of zeros of $u(x)$ in (a,b).

To obtain an n-nodal odd solution, we study the boundary value problem

$$v'' + h(x,\lambda)|v|^{p-1}v = 0 \quad \text{in } (0,1), \quad v(0) = 0, \quad v(1) = 0. \tag{4.1}$$

Proposition 4.2. *For any nonnegative integer n, (4.1) has a solution v such that $v'(0) > 0$ and $v(x)$ has exactly n zeros in $(0,1)$.*

Proof. By using the shooting method, we shall prove the proposition. Let $N[v,(a,b)]$ be given by Definition 4.1. Denote the unique solution of (3.7) by $v(x,\mu)$. Each zero of $v(x,\mu)$ is continuous with respect to μ because of Lemma 3.3. By Lemma 3.7, $N[v(\cdot,\mu),(0,1)] = 0$ for $\mu > 0$ small. By Lemma 3.9, $N[v(\cdot,\mu),(0,1)] \geq 1$ for $\mu > 0$ large. Define

$$\mu_0 := \sup\{\mu > 0 : N[v(\cdot,\mu),(0,1)] = 0\}.$$

We shall prove that $v(x,\mu_0)$ becomes a positive solution of (4.1). If $v(x,\mu_0)$ has a zero x_0 in $(0,1)$, then $v(x,\mu)$ has also a zero in a neighborhood of x_0 when μ is close to μ_0. This fact conflicts with the definition of μ_0. Hence $v(x,\mu_0) > 0$ in $(0,1)$. We shall prove that $v(1,\mu_0) = 0$. To the contrary, suppose that $v(1,\mu_0) > 0$. Then $N[v(\cdot,\mu),(0,1)] = 0$ for μ slightly larger than μ_0. This is impossible. Therefore $v(1,\mu_0) = 0$ and $v(x,\mu_0)$ is a positive solution of (4.1).

For $\mu \in (\mu_0, \mu_0 + \varepsilon)$ with a small $\varepsilon > 0$, $v(x,\mu)$ has exactly one zero in $(0,1)$ and $v(1,\mu) < 0$. Put

$$\mu_1 := \sup\{\mu > 0 : N[v(\cdot,\mu),(0,1)] = 1\}.$$

Since $\mu_1 > \mu_0$, $v(x,\mu_1)$ has at least one zero in $(0,1)$. In the same argument as in μ_0, we can prove that $v(x,\mu_1)$ is a solution of (4.1) having exactly one zero in $(0,1)$. Repeating the discussion above, we can define

$$\mu_n := \sup\{\mu > 0 : N[v(\cdot,\mu),(0,1)] = n\}.$$

Then $v(x,\mu_n)$ is a solution having exactly n zeros in $(0,1)$. The proof is complete. □

We shall show the uniqueness of n-nodal solutions for (4.1).

Proposition 4.3. *Let n be a nonnegative integer. Then (4.1) possesses a unique solution $v(x)$ which has exactly n zeros in $(0,1)$ and satisfies $v'(0) > 0$.*

Proof. It is enough to prove the uniqueness of n-nodal solutions because the existence has been shown in Proposition 4.2. Let $n = 0$. Then v is a positive solution of (4.1). We shall show the uniqueness of positive solutions. Suppose that our claim is false. Then (4.1) has different positive solutions u and v. Suppose that $u'(0) > v'(0) > 0$ because the case $u'(0) < v'(0)$ also can be treated in the same way as below. Since u and v are linear in $[0, \lambda]$, it readily follows that $u(x) > v(x) > 0$ in $(0, \lambda]$ and $u'(\lambda) > v'(\lambda) > 0$. Computing the energy at λ, we have

$$E_+(u) > E_+(v). \tag{4.2}$$

We divide the proof into three cases.

(i) The graphs of u and v do not intersect in $(\lambda, 1)$.

(ii) The graphs of u and v intersect exactly once in $(\lambda, 1)$.

(iii) The graphs of u and v intersect at least twice in $(\lambda, 1)$.

We shall show that each case leads to a contradiction.

Suppose (i). Then $u(x) > v(x) > 0$ in $(0, 1)$ and $u(x) = v(x) = 0$ at $x = 0, 1$. This fact contradicts Lemma 3.5.

Suppose (ii). Let z_1 be a unique intersection point. Then $u(x) < v(x)$ in $(z_1, 1)$. Since $u(1) = v(1) = 0$, we have $v'(1) < u'(1) < 0$. Hence $E(v)(1) > E(u)(1)$. This contradicts (4.2).

Suppose (iii). Denote the first and the second intersection points by z_1 and z_2, respectively. Then $v(x) > u(x)$ in (z_1, z_2). Since $\lambda < z_1 < z_2 < 1$, u and v satisfy (2.1) in $[z_1, z_2]$. By Lemma 2.4, $E_+(v) > E_+(u)$. This contradicts (4.2). Consequently, (i), (ii) and (iii) yield a contradiction. Therefore the conclusion for $n = 0$ holds.

Let $n \geq 1$. Suppose that u and v are different solutions of (4.1) such that $N[u, (0, 1)] = N[v, (0, 1)] = n$ and $u'(0) > v'(0) > 0$. Denote the i-th zero of u and v in $(0, 1]$ by x_i and y_i, respectively. Then $x_{n+1} = y_{n+1} = 1$. Since $u'(0) > v'(0)$, it holds that $u(\lambda) > v(\lambda)$ and $u'(\lambda) > v'(\lambda) > 0$. Hence we have (4.2). We note that all zeros x_i and y_i are larger than λ because $u(x) > v(x) > 0$ in $(0, \lambda]$. Since u and v satisfy the Emden-Fowler equation in $(\lambda, 1)$, we use Lemma 2.5 in this interval with (4.2). Then the distance between two adjacent zeros of u is shorter than that of v, i.e., $x_{i+1} - x_i < y_{i+1} - y_i$ for $1 \leq i \leq n$. Since $x_{n+1} = y_{n+1} = 1$, we conclude that $y_1 < x_1$. Then

$$u > 0 \quad \text{in } (0, x_1), \quad u(0) = u(x_1) = 0, \tag{4.3}$$

$$v > 0 \quad \text{in } (0, y_1), \quad v(0) = v(y_1) = 0. \tag{4.4}$$

Consider the intersection of the graphs of $u(x)$ and $v(x)$ in the interval $(0, y_1)$. Divide the proof in three cases (i)–(iii) with $(\lambda, 1)$ replaced by $(0, y_1)$. Observing (4.3), (4.4), noting that $y_1 < x_1$ and drawing the graphs of u and v, we find that case (ii) does not occur. In case (i), $u(x) > v(x)$ in $(0, y_1)$. This contradicts Lemma 3.5. In case (iii), we note that the first two intersection points z_1 and z_2 satisfy $\lambda < z_1 < z_2 < y_1$ because $v(x) < u(x)$ in $(0, \lambda]$. Therefore $u(x)$ and $v(x)$ satisfy (2.1) in $[z_1, z_2]$. Since $u(x) = v(x)$ at $x = z_1, z_2$ and $v > u > 0$ in (z_1, z_2), we have $E_+(v) > E_+(u)$ by Lemma 2.4. This contradicts (4.2). Consequently, an n-nodal solution $v(x)$ of (4.1) satisfying $v'(0) > 0$ is unique. The proof is complete. □

Let v be a solution obtained by Proposition 4.3. Extend it as an odd function in $[-1, 1]$ and denote it by u. Then it is an $(2n + 1)$-nodal odd solution. Put $m := (2n + 1)$. If $u'(-1) < 0$ (i.e., n is even), we replace u by $-u$. Then $u'(-1) > 0$. By Propositions 4.3 with the argument above, we have the following theorem.

Theorem 4.4. *Let $\lambda \in (0, 1)$. For any odd integer $m \geq 1$, there exists a unique m-nodal odd solution u of* (1.1) *satisfying $u'(-1) > 0$.*

We investigate n-nodal even solutions in the theorem below. Such solutions can be easily constructed.

Theorem 4.5. *Let $\lambda \in (0, 1)$. For any nonnegative even integer n, there exists a unique n-nodal even solution $u(x)$ of* (1.1) *satisfying $u'(-1) > 0$.*

Proof. Let $n = 0$. A 0-nodal even solution satisfying $u'(-1) > 0$ is a positive even solution. If u is a positive even solution, it satisfies $u'(-\lambda) = 0$ because $u'(x) \equiv u'(0) = 0$ in $[-\lambda, \lambda]$. Observe that u satisfies the Emden-Fowler equation in $[-1, -\lambda]$, $u(x) > 0$ in $(-1, -\lambda]$ and $u(-1) = u'(-\lambda) = 0$. Therefore $u'(x) > 0$ in $[-1, -\lambda)$. Such a solution is unique and it is represented as

$$u(x) = \alpha^{2/(p-1)} \phi(\alpha(x + 1)) \quad \text{in } [-1, -\lambda] \quad \text{with } \alpha := \frac{1}{2(1 - \lambda)},$$

where $\phi(x)$ is defined by (2.3). Moreover $u(x) \equiv u(-\lambda)$ in $[-\lambda, 0]$ and $u(x)$ is extended in $[-1, 1]$ as an even function. Accordingly, the conclusion for $n = 0$ is valid.

Let $n \geq 2$ be an even integer. Put $n = 2m$. If u is an n-nodal even solution satisfying $u'(-1) > 0$, then u is periodic in $[-1, -\lambda]$, it has exactly m zeros in $(-1, -\lambda)$ and $u(-1) = u'(-\lambda) = 0$. Such a solution is unique. Moreover, the distance l between consecutive zeros of u is written as

$$l := 2(1 - \lambda)/(2m + 1).$$

Define $x_i := li-1$ with $1 \le i \le m$. Then x_i is the i-th zero of u in $(-1,-\lambda)$. Therefore $u(x)$ is written as $u(x) = l^{-2/(p-1)}\phi(l^{-1}(x+1))$ in $[-1,-\lambda]$. Moreover $u(x) \equiv u(-\lambda)$ in $[-\lambda, 0]$. Extend it as an even function on $[-1,1]$. This u is an n-nodal even solution with $u'(-1) > 0$. The proof is complete. □

Proof of Theorema 1.4. The theorem is derived directly from Theorems 4.4 and 4.5. □

Using the fact that any nontrivial solution of the Emden-Fowler equation is nondegenerate and employing the a priori estimates in Lemmas 3.10 and 3.11, we shall prove Theorem 1.5.

Proof of Theorem 1.5. It is enough to show the uniqueness of n-nodal solutions. Indeed, the symmetry of solution comes from Theorem 1.4. Fix $n \ge 0$ arbitrarily. We shall show the existence of ε_n such that an n-nodal solution with $u'(-1) > 0$ is unique when $\lambda \in (0, \varepsilon_n)$. We suppose that the claim above is false. Then there exist sequences $\{\lambda_k\}$, $\{u_k\}$ and $\{v_k\}$ such that $0 < \lambda_k < 1$, $\lambda_k \to 0$ as $k \to \infty$, u_k and v_k are different n-nodal solutions for $\lambda = \lambda_k$ satisfying $u_k'(-1) > 0$ and $v_k'(-1) > 0$. By Lemmas 3.10 and 3.11, there exist constants $c, C > 0$ independent of k such that

$$c \le \|u_k\|_{C^1} \le \|u_k\|_{W^{2,\infty}} \le C, \quad c \le \|v_k\|_{C^1} \le \|v_k\|_{W^{2,\infty}} \le C. \tag{4.5}$$

By the compact embedding, a subsequence (again denoted by u_k) of u_k converges to a limit u in $C^1[-1,1]$. Therefore $u(-1) = u(1) = 0$. Since u_k is a solution of (1.1), it satisfies

$$-\int_{-1}^{1} u_k'(x)\eta'(x)dx + \int_{-1}^{1} h(x,\lambda_k)|u_k|^{p-1}u_k\eta dx = 0, \tag{4.6}$$

for any test function $\eta \in C_0^\infty(-1,1)$. Since u_k converges to u in $C^1[-1,1]$ and $h(x,\lambda_k) \to 1$ a.e., the Lebesgue dominated convergence theorem shows that

$$-\int_{-1}^{1} u'(x)\eta'(x)dx + \int_{-1}^{1} |u|^{p-1}u\eta dx = 0. \tag{4.7}$$

Thus u satisfies the Emden-Fowler equation in the distribution sense. The standard regularity theorem ensures that $u \in C^2[-1,1]$. Since $\|u\|_{C^1} \ge c$ by (4.5), u is a nontrivial solution of (2.1). Since $u_k'(-1) > 0$, we have $u'(-1) \ge 0$. If $u'(-1) = 0$, then $u \equiv 0$. Therefore $u'(-1) > 0$. We shall show that u has exactly n zeros. Denote the i-th zero of u_k by $z_{k,i}$ with $1 \le i \le n$. Put $z_{k,0} := -1$ and $z_{k,n+1} := 1$. We shall show the existence of $c_0 > 0$ such that

$$z_{k,i} - z_{k,i-1} \ge c_0 \quad \text{for all } k \ge 1,\ 1 \le i \le n+1. \tag{4.8}$$

On the contrary, we suppose that (4.8) is wrong. Then there exist sequences u_j, x_j and y_j such that u_j is a subsequence of u_k, x_j and y_j are adjacent zeros of u_j, $-1 \le x_j < y_j \le 1$ and $y_j - x_j \to 0$ as $j \to \infty$. A subsequence of x_j converges to a limit x_0. There exists an $r_j \in (x_j, y_j)$ such that $u_j'(r_j) = 0$. Then $r_j \to x_0$. Hence $u'(x_0) = u(x_0) = 0$ and therefore $u \equiv 0$. A contradiction occurs. Thus (4.8) holds.

As $k \to \infty$ in (4.8), we see that the limit function u has exactly n zeros in $(-1, 1)$. Consequently, u_k converges to an n-nodal solution u of (2.8) with $u'(-1) > 0$. This solution u is unique by Proposition 2.8 and we denote it by $\phi_n(x)$. Since the limit ϕ_n is unique, we see that u_k itself (without extracting a subsequence) converges to $\phi_n(x)$. In the same way, v_k converges to $\phi_n(x)$.

We shall derive a contradiction under the assumption that there exist sequences $\{u_k\}$, $\{v_k\}$ and $\{\lambda_k\}$ such that $\lambda_k \to 0$ as $k \to \infty$, u_k and v_k are n-nodal solutions for $\lambda = \lambda_k$. Define

$$g(t) := |t|^{p-1}t, \quad \psi_k(x) := (u_k(x) - v_k(x))/\|u_k - v_k\|_\infty.$$

Then $\|\psi_k\|_\infty = 1$ and $\psi_k(x)$ satisfies

$$-\psi_k''(x) = h(x, \lambda_k)\frac{g(u_k) - g(v_k)}{u_k - v_k}\psi_k.$$

Since the right hand side is uniformly bounded and $\psi_k(-1) = \psi_k(1) = 0$, $\psi_k(x)$ is bounded in $W^{2,\infty}(-1, 1)$. By the compact embedding, along a subsequence, ψ_k converges to a limit ψ in $C^1[-1, 1]$. Using the same argument as in (4.6) and (4.7), we find that $\psi \in C^2[-1, 1]$ and

$$\psi'' + p|\phi_n|^{p-1}\psi = 0 \quad \text{in } (-1, 1), \quad \psi(-1) = \psi(1) = 0.$$

Since $\|\psi_k\|_\infty = 1$, it holds that $\|\psi\|_\infty = 1$. Thus ψ is a nontrivial solution. This contradicts Proposition 2.10. Consequently, there exists an ε_n such that an n-nodal solution with $u'(-1) > 0$ is unique when $\lambda \in (0, \varepsilon_n)$. $\square$

We shall show Theorem 1.7. To this end, let m be any nonnegative integer and we define u_{2m} by a unique $2m$-nodal even solution of (1.1) with $u_{2m}'(-1) > 0$. Then it has exactly m zeros in $(-1, -\lambda)$ and in $(\lambda, 1)$ also. Put $\alpha_m = \alpha_m(\lambda) := u_{2m}'(-1)$. It depends only on m and λ. We denote the unique solution of (3.6) by $u(x, \alpha)$ and its i-th zero in $(-1, \infty)$ by $z_i(\alpha)$. Recall that $u(x, \alpha)$ has infinitely many zeros in $(-1, \infty)$. We have $u_{2m}(x) = u(x, \alpha_m)$. If $m \ge 1$, it holds that

$$-1 < z_1(\alpha_m) < \cdots < z_m(\alpha_m) < -\lambda,$$

$$\lambda < z_{m+1}(\alpha_m) < \cdots < z_{2m}(\alpha_m) < 1.$$

If $m = 0$, then $z_1(\alpha_0) = 1$. Note that $u(x, \alpha)$ satisfies (2.1) in $(-1, -\lambda)$. By Lemma 2.5, for $\alpha \geq \alpha_m$, $u(x, \alpha)$ has at least m zeros in $(-1, -\lambda)$, i.e.,

$$-1 < z_1(\alpha) < \cdots < z_m(\alpha) < -\lambda \quad \text{for } \alpha \geq \alpha_m \text{ when } m \geq 1. \tag{4.9}$$

By Lemma 2.6, when α is large enough, $u(x, \alpha)$ has many zeros in $(-1, -\lambda)$. In particular, $z_{m+1}(\alpha) < -\lambda$ for $\alpha (> \alpha_m)$ large enough. Since $z_{m+1}(\alpha_m) > \lambda$, there exists a $\gamma_m = \gamma_m(\lambda)$ such that

$$\gamma_m > \alpha_m, \quad z_{m+1}(\gamma_m) = \lambda. \tag{4.10}$$

We choose the smallest γ_m satisfying the relation above. Then $\lambda < z_{m+1}(\alpha)$ for $\alpha \in [\alpha_m, \gamma_m)$. This inequality with (4.9) yields

$$-1 < z_1(\alpha) < \cdots < z_m(\alpha) < -\lambda \quad \text{for } \alpha \in [\alpha_m, \gamma_m] \text{ when } m \geq 1, \tag{4.11}$$

$$\lambda < z_{m+1}(\alpha) \quad \text{for } \alpha \in [\alpha_m, \gamma_m) \text{ when } m \geq 0. \tag{4.12}$$

By (4.11) and (4.12), we understand that a solution $u(x, \alpha)$ with $\alpha \in [\alpha_m, \gamma_m)$ has exactly m zeros in $(-1, -\lambda)$ and no zeros in $[-\lambda, \lambda]$. In the next lemma, we shall show the existence of $\beta_m \in (\alpha_m, \gamma_m)$ such that $u(x, \beta_m)$ has exactly m zeros in $(-1, -\lambda)$ and no zeros in $[-\lambda, 1]$. This result plays the most important role in the proof of Theorem 1.7.

Proposition 4.6. *Let m be any nonnegative integer and let $\alpha_m = \alpha_m(\lambda)$ and $\gamma_m = \gamma_m(\lambda)$ be defined as above. Then there exists a λ_m satisfying the following condition. For $\lambda \in (\lambda_m, 1)$, there exists a $\beta_m = \beta_m(\lambda) \in (\alpha_m(\lambda), \gamma_m(\lambda))$ such that*

$$z_m(\beta_m) < -\lambda < 1 < z_{m+1}(\beta_m) \quad \textit{for } m \geq 0, \tag{4.13}$$

where we have put $z_0(\beta_0) := -1$. Therefore $u(x, \beta_m)$ has exactly m zeros in $(-1, -\lambda)$ and no zeros in $[-\lambda, 1]$.

To prove the proposition above, we define some constants. Let c_1 and c_2 be defined by Proposition 2.3 and c_3 by Lemma 2.9. Put

$$a_1 := (c_1(p+1))^{1/(p+1)}, \quad a_2 := (c_1(p+1)/2)^{1/(p+1)}. \tag{4.14}$$

For a nonnegative integer m, we choose a constant $\varepsilon_m \in (0, 1)$ so small that

$$\frac{2}{p+1} \varepsilon_m^{p+1} a_1^{p+1} (m+1)^{2(p+1)/(p-1)} < 4^{-2(p+1)/(p-1)} c_1. \tag{4.15}$$

After fixing ε_m, we determine $\lambda_m \in (1/2, 1)$ close to 1 which satisfies

$$\left(\frac{p+1}{2}\right)^{2/(p-1)} < c_1 \left(\frac{2(1-\lambda_m)}{2m+1}\right)^{-2(p+1)/(p-1)}, \tag{4.16}$$

$$c_2(4\lambda_m \varepsilon_m c_3/(1-\varepsilon_m))^{-2/(p-1)} < \varepsilon_m a_2 (2(1-\lambda_m))^{-2/(p-1)}, \tag{4.17}$$

$$\frac{p+1}{2} < \varepsilon_m^{p+1} a_2^{p-1} \left(\frac{2(1-\lambda_m)}{2m+1}\right)^{-2}. \tag{4.18}$$

Observe that (4.16)–(4.18) remain valid with λ_m replaced by $\lambda \in (\lambda_m, 1)$.

We define $\xi(\alpha) := u(\lambda, \alpha)/u(-\lambda, \alpha)$. This is well defined for $\alpha \in [\alpha_m, \gamma_m]$ because of (4.9)–(4.12). Since $\xi(\alpha_m) = 1$ (because $u(x, \alpha_m)$ is even) and $\xi(\gamma_m) = 0$ by (4.10), there exists a $\beta_m = \beta_m(\lambda) \in (\alpha_m(\lambda), \gamma_m(\lambda))$ such that $\xi(\beta_m) = \varepsilon_m$, where we have already fixed ε_m satisfying (4.15). Then it holds that

$$\frac{u(\lambda, \beta_m)}{u(-\lambda, \beta_m)} = \varepsilon_m. \tag{4.19}$$

By (4.11) and (4.12), we obtain

$$-1 < z_1(\beta_m) < \cdots < z_m(\beta_m) < -\lambda \quad \text{when } m \geq 1, \tag{4.20}$$

$$\lambda < z_{m+1}(\beta_m) \quad \text{when } m \geq 0. \tag{4.21}$$

We shall show that Proposition 4.6 is valid for this β_m.

Proof of Proposition 4.6. Let $\lambda_m \in (1/2, 1)$ satisfy (4.16)–(4.18) and let $\lambda \in (\lambda_m, 1)$. Since the first inequality of (4.13) follows from (4.20), it is enough to show that $1 < z_{m+1}(\beta_m)$. We write $u(x) := u(x, \beta_m)$ for simplicity. By (4.20) and (4.21), $u(x)$ has no zeros in $[-\lambda, \lambda]$. We consider the case where $u(x) > 0$ in this interval (i.e., m is even by (4.20) and (4.21)) because the opposite case can be treated in the same way.

Put $H := u(-\lambda)(> 0)$. Then $u(\lambda) = \varepsilon_m H$ by (4.19). We abbreviate ε_m to ε. The graph of $u(x)$ in $[-\lambda, \lambda]$ is a straight line through two points $(\lambda, \varepsilon H)$ and $(-\lambda, H)$. We compute its slope as

$$u'(\lambda) = u'(-\lambda) = -\frac{1-\varepsilon}{2\lambda} H. \tag{4.22}$$

The energy $E(u)$ is equal to $E_-(x) \equiv E_-$ in $[-1, -\lambda]$ and $E_+(x) \equiv E_+$ in $[\lambda, 1]$. The energy at $-\lambda$ is given by

$$\begin{aligned} E_- &= E_-(-\lambda) = \frac{1}{2} u'(-\lambda)^2 + \frac{1}{p+1} |u(-\lambda)|^{p+1} \\ &= \frac{(1-\varepsilon)^2}{8\lambda^2} H^2 + \frac{1}{p+1} H^{p+1}. \end{aligned} \tag{4.23}$$

On the other hand, the energy E_+ is computed as

$$\begin{aligned} E_+ &= E_+(\lambda) = \frac{1}{2}u'(\lambda)^2 + \frac{1}{p+1}|u(\lambda)|^{p+1} \\ &= \frac{(1-\varepsilon)^2}{8\lambda^2}H^2 + \frac{\varepsilon^{p+1}}{p+1}H^{p+1}. \end{aligned} \tag{4.24}$$

We denote the distance between consecutive zeros of $u(x)$ in $[-1, -\lambda]$ by l. More strictly, if u has a zero in $(-1, -\lambda)$, we denote the smallest zero by z_1. Then $l = z_1 + 1$ because $u(-1) = 0$. If u has no zeros in $(-1, -\lambda)$, we extend $u|_{[-1,-\lambda]}$ as a solution of the Emden-Fowler equation into $(-\infty, -1]$. Then l is defined by the distance between consecutive zeros of $u(x)$. We shall show that

$$\left(m + \frac{1}{2}\right) l < 1 - \lambda < (m+1)l. \tag{4.25}$$

The second inequality follows from (4.20) and (4.21). We shall prove the first inequality. When $m = 0$, there exists a unique critical point t_1 of u in $(-1, -\lambda)$, i.e., $u'(t_1) = 0$. Indeed, this assertion comes from the fact that $u'(-\lambda) < 0$ by (4.22) and $u(x) > 0$ in $(-1, -\lambda)$. Then $l = 2(t_1+1) < 2(1-\lambda)$. Thus the first inequality of (4.25) holds for $m = 0$. If $m \geq 1$, there exists a unique $T_1 \in (z_m(\beta_m), -\lambda)$ such that $u'(T_1) = 0$. Since $T_1 - z_m(\beta_m) = l/2$, we have $ml + l/2 = T_1 + 1 < 1 - \lambda$ and the first inequality of (4.25) holds. We rewrite (4.25) as

$$\frac{1-\lambda}{m+1} < l < \frac{2(1-\lambda)}{2m+1}. \tag{4.26}$$

By Proposition 2.3 with (4.26), we have

$$c_1 \left(\frac{2(1-\lambda)}{2m+1}\right)^{-2(p+1)/(p-1)} \leq E_- \leq c_1 \left(\frac{1-\lambda}{m+1}\right)^{-2(p+1)/(p-1)}. \tag{4.27}$$

By (4.23), we obtain

$$\frac{1}{p+1}H^{p+1} \leq E_-,$$

which with the second inequality of (4.27) yields

$$\frac{1}{p+1}H^{p+1} \leq c_1 \left(\frac{1-\lambda}{m+1}\right)^{-2(p+1)/(p-1)}.$$

This is rewritten as

$$H \leq a_1 \left(\frac{1-\lambda}{m+1}\right)^{-2/(p-1)}, \tag{4.28}$$

where a_1 has been defined by (4.14).

We give a lower estimate of H. Since $\lambda \in (1/2, 1)$, (4.23) shows

$$E_- \leq \frac{1}{2}H^2 + \frac{1}{p+1}H^{p+1}. \tag{4.29}$$

Combining (4.16), (4.27) and (4.29), we have

$$\begin{aligned}\frac{1}{2}H^2 + \frac{1}{p+1}H^{p+1} &\geq E_- \geq c_1\left(\frac{2(1-\lambda)}{2m+1}\right)^{-2(p+1)/(p-1)}\\ > c_1\left(\frac{2(1-\lambda_m)}{2m+1}\right)^{-2(p+1)/(p-1)} &> ((p+1)/2)^{2/(p-1)}.\end{aligned} \tag{4.30}$$

Then $H > ((p+1)/2)^{1/(p-1)}$ because the opposite inequality contradicts (4.30). This inequality implies that

$$(1/2)H^2 < (1/(p+1))H^{p+1}.$$

From the inequality above with (4.29), we have $E_- \leq (2/(p+1))H^{p+1}$. This inequality with (4.27) shows that

$$a_2\left(\frac{2(1-\lambda)}{2m+1}\right)^{-2/(p-1)} \leq H, \tag{4.31}$$

where a_2 has been defined by (4.14).

For a positive integer $i (\geq 1)$, we denote the i-th zero of u in $(-1, \infty)$ by z_i and put $z_0 := -1$. Then $z_i = z_i(\beta_m)$ satisfies (4.20) and (4.21). We shall show the last inequality in (4.13): $1 < z_{m+1}(\beta_m)$. To the contrary, suppose that $z_{m+1} \leq 1$. Since $u'(\lambda) < 0$ and $u(x) > 0$ in (λ, z_{m+1}), $u'(x)$ is negative in $[\lambda, z_{m+1}]$. We define $U(x)$ by the restriction of $u(x)$ on $[\lambda, 1]$. Extend $U(x)$ into $\mathbb{R}$ as a solution of the Emden-Fowler equation. Then it holds that $U(x) = u(x)$ on $[\lambda, 1]$ and

$$U'' + |U|^{p-1}U = 0 \quad \text{in } \mathbb{R}.$$

We denote the largest critical point of $U(x)$ in $(-\infty, z_{m+1})$ by t_0. Hence $U'(t_0) = 0$ and $U'(x) < 0$ in (t_0, z_{m+1}). Define $V(x) := U(x + t_0)$ and $a := z_{m+1} - t_0$. Then V satisfies

$$V'' + |V|^{p-1}V = 0, \quad V > 0 \quad \text{in } (0, a), \quad V'(0) = V(a) = 0.$$

The equation above is the same as (2.9). Put $\mu := \lambda - t_0$. Since $u(\lambda) = \varepsilon H$, we use (4.22) to obtain

$$\frac{V'(\mu)}{V(\mu)} = \frac{u'(\lambda)}{u(\lambda)} = -\frac{1-\varepsilon}{2\lambda\varepsilon} = -M,$$

where we have put $M := (1-\varepsilon)/2\lambda\varepsilon$. Verify that the relation above coincides with the equation given in Lemma 2.9 (ii) with $\psi(x, a) = V(x)$. By (4.31), we have

$$\begin{aligned} V(\mu) = u(\lambda) = \varepsilon H &\geq \varepsilon a_2(2(1-\lambda)/(2m+1))^{-2/(p-1)} \\ &\geq \varepsilon a_2(2(1-\lambda))^{-2/(p-1)}, \end{aligned}$$

which shows that $\|V\|_\infty \geq V(\mu) \geq \varepsilon a_2(2(1-\lambda))^{-2/(p-1)}$. By (4.17), we have

$$\begin{aligned} &\varepsilon a_2(2(1-\lambda))^{-2/(p-1)} > \varepsilon a_2(2(1-\lambda_m))^{-2/(p-1)} \\ &\quad > c_2(4\lambda_m \varepsilon c_3/(1-\varepsilon))^{-2/(p-1)} > c_2(4\lambda\varepsilon c_3/(1-\varepsilon))^{-2/(p-1)} \\ &\quad = c_2(2c_3/M)^{-2/(p-1)}, \end{aligned}$$

with $M = (1-\varepsilon)/2\lambda\varepsilon$. Thus $\|V\|_\infty > c_2(2c_3/M)^{-2/(p-1)}$. We use Lemma 2.9 (iv) with $\psi = V$ to obtain $\mu < a/2$, and so $\lambda - t_0 \leq (z_{m+1} - t_0)/2$. This is rewritten as $z_{m+1} - t_0 \leq 2(z_{m+1} - \lambda)$. Since $z_{m+1} \leq 1$, we find that $a = z_{m+1} - t_0 \leq 2(1-\lambda)$. Since $2a$ is the distance between consecutive zeros of V, Proposition 2.3 shows that

$$E_+ = E(u)(\lambda) = E(V) = c_1(2a)^{-2(p+1)/(p-1)}.$$

Since $a \leq 2(1-\lambda)$, we have

$$E_+ \geq c_1(4(1-\lambda))^{-2(p+1)/(p-1)}. \tag{4.32}$$

On the other hand, since $\lambda \in (1/2, 1)$, by (4.24) we have

$$E_+ \leq \frac{1}{2}H^2 + \frac{\varepsilon^{p+1}}{p+1}H^{p+1}. \tag{4.33}$$

From (4.18) and (4.31), it follows that

$$\frac{p+1}{2} < \varepsilon^{p+1} a_2^{p-1}\left(\frac{2(1-\lambda_m)}{2m+1}\right)^{-2} < \varepsilon^{p+1} a_2^{p-1}\left(\frac{2(1-\lambda)}{2m+1}\right)^{-2} \leq \varepsilon^{p+1} H^{p-1},$$

which shows $(1/2)H^2 < (\varepsilon^{p+1}/(p+1))H^{p+1}$. By this inequality with (4.33) and (4.28), we have

$$E_+ < \frac{2\varepsilon^{p+1}}{p+1}H^{p+1} \leq \frac{2\varepsilon^{p+1}}{p+1}a_1^{p+1}\left(\frac{1-\lambda}{m+1}\right)^{-2(p+1)/(p-1)}. \tag{4.34}$$

Combining (4.32) with (4.34), we obtain

$$c_1(4(1-\lambda))^{-2(p+1)/(p-1)} < \frac{2\varepsilon^{p+1}}{p+1}a_1^{p+1}\left(\frac{1-\lambda}{m+1}\right)^{-2(p+1)/(p-1)},$$

which is reduced to

$$4^{-2(p+1)/(p-1)}c_1 < \frac{2\varepsilon^{p+1}}{p+1}a_1^{p+1}(m+1)^{2(p+1)/(p-1)}.$$

This contradicts (4.15). This conflict is caused by the assumption that $z_{m+1} \le 1$. Therefore $z_{m+1} > 1$ and the proof is complete. □

Recall that $u(x,\alpha)$ denotes the solution of (3.6) and $z_i(\alpha)$ is its i-th zero in $(-1,\infty)$. From the uniqueness of $(2m-1)$-nodal odd solution, we have the next lemma.

Lemma 4.7. *For each positive integer m, the equation $z_m(\alpha) = 0$ has a unique solution $\alpha > 0$. Moreover, for the solution α, $u(x,\alpha)$ is a $(2m-1)$-nodal odd solution satisfying $u'(-1) > 0$.*

Proof. Let m be a positive integer. If $z_m(\alpha) = 0$, then $u(x,\alpha)$ is an odd solution (see Lemma 3.4) having exactly $m-1$ zeros in $(-1,0)$. Hence it is a $(2m-1)$-nodal odd solution. From the uniqueness of such a solution, $z_m(\alpha) = 0$ has a unique solution. The proof is complete. □

Employing Proposition 4.6 and Lemma 4.7, we shall show Theorem 1.7.

Proof of Theorem 1.7. Let m be any nonnegative integer. Let ε_m satisfy (4.15) and let λ_m fulfill (4.16)–(4.18). For $\lambda \in (\lambda_m, 1)$, we shall prove that (1.1) has an (m,n)-solution for $0 \le n \le m$. We have already defined α_m, β_m and γ_m in Proposition 4.6 or before the proposition. Recall that $z_{m+1}(\gamma_m) = \lambda$ by (4.10). Since $z_{m+1}(\alpha) < -\lambda$ for $\alpha (> \gamma_m)$ large enough, there exists a $\delta_m \in (\gamma_m, \infty)$ such that $z_{m+1}(\delta_m) = 0$. By Lemma 4.7, δ_m is unique and $u(x,\delta_m)$ is a $(2m+1)$-nodal odd solution. Since $u(x,\delta_m)$ has exactly m zeros in $(0,1)$ and $u(0,\delta_m) = 0$, it has $m+1$ interior zeros in $(-\lambda,1)$. When α is sufficiently close to δ_m, $u(x,\alpha)$ has at least $m+1$ zeros in $(-\lambda,1)$. Thus we have

$$N[u(\cdot,\alpha),(-\lambda,1)] \ge m+1 \quad \text{when } |\alpha-\delta_m| \text{ is small enough.} \tag{4.35}$$

Since $z_{m+1}(\alpha) \ne 0$ in $(0,\delta_m)$ by Lemma 4.7 and $z_{m+1}(\gamma_m) = \lambda > 0$, we have

$$z_{m+1}(\alpha) > 0 \quad \text{for } \alpha \in (0,\delta_m). \tag{4.36}$$

Combining (4.9) with (4.36), we have

$$-1 < z_1(\alpha) < \cdots < z_m(\alpha) < -\lambda < 0 < z_{m+1}(\alpha) \quad \text{for } \alpha \in [\alpha_m,\delta_m). \tag{4.37}$$

Since $u(x, \beta_m)$ has no zeros in $[-\lambda, 1]$ by Proposition 4.6,

$$N[u(\cdot, \alpha), (-\lambda, 1)] = 0 \quad \text{when } \alpha \text{ is close to } \beta_m.$$

Noting this fact with (4.35), we define

$$b_0 := \sup\{\alpha \in (\beta_m, \delta_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = 0\}.$$

Then $\beta_m < b_0 < \delta_m$. In the same way as in the proof of Proposition 4.2, we can show that $u(x, b_0) \neq 0$ in $(-\lambda, 1)$ and $u(1, b_0) = 0$. Moreover $u(x, b_0)$ has exactly m zeros in $(-1, -\lambda)$ by (4.37). Thus $u(x, b_0)$ is an $(m, 0)$-solution.

When α is slightly larger than b_0, $u(x, \alpha)$ possesses exactly one zero in $(-\lambda, 1)$. Define

$$b_1 := \sup\{\alpha \in (\beta_m, \delta_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = 1\}.$$

Since $b_1 > b_0$, $u(x, b_1)$ has at least one zero in $(-\lambda, 1)$. If it has two or more zeros, we find a contradiction. Therefore $u(x, b_1)$ has exactly one zero in $(-\lambda, 1)$. By (4.37), this zero belongs to $(0, 1)$. Moreover, we can prove that $u(1, b_1) = 0$. Accordingly, $u(x, b_1)$ is an $(m, 1)$-solution. We can define b_i inductively. Suppose that b_{i-1} is well defined by

$$b_{i-1} := \sup\{\alpha \in (\beta_m, \delta_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = i - 1\},$$

and suppose that $u(x, b_{i-1})$ is an $(m, i-1)$-solution. When α is slightly larger than b_{i-1}, $u(x, \alpha)$ has exactly i zeros in $(-\lambda, 1)$. Therefore we can define

$$b_i := \sup\{\alpha \in (\beta_m, \delta_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = i\}.$$

Note that the zeros of $u(x, b_i)$ in $(-\lambda, 1)$ are in $(0, 1)$ because of (4.37). Therefore $u(x, b_i)$ is an (m, i)-solution. Consequently, we can define $b_0, b_1, \ldots, b_m$. By (4.35), $b_m < \delta_m$. Clearly, each (m, i)-solution is asymmetric when $i < m$.

$u(x, b_m)$ is an (m, m)-solution. We shall show that it is asymmetric. Recall that $u(x, \alpha_m)$ is a unique $2m$-nodal even solution with $u'(-1) > 0$ and $u(x, \delta_m)$ is a unique $(2m+1)$-nodal odd solution with $u'(-1) > 0$. Observe that symmetric (m, m)-solutions with $u'(-1) > 0$ are these two solutions only. Since $\alpha_m < b_m < \delta_m$, $u(x, b_m)$ is asymmetric. The proof is complete. □

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