ON 3-2-1 VALUES OF FINITE MULTIPLE HARMONIC q-SERIES AT ROOTS OF UNITY

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ABSTRACT. We mainly answer two open questions about finite multiple harmonic q-series on 3-2-1 indices at roots of unity, posed recently by H. Bachmann, Y. Takeyama, and K. Tasaka. Two conjectures regarding cyclic sums which generalize the given results are also provided.

1. INTRODUCTION.

For two r-tuples of non-negative integers $\mathbf{s} = (s_1, \dots, s_r)$ and $\mathbf{t} = (t_1, \dots, t_r)$ and a positive integer n, with a complex number q satisfying $q^m \neq 1$ for $n > m > 0$, we define two classes of multiple q-harmonic sums

$$H_n(\mathbf{s}; \mathbf{t}; q) = H_n(s_1, \dots, s_r; t_1, \dots, t_r; q) = \sum_{1 \le k_1 < \cdots < k_r \le n} \frac{q^{k_1 t_1 + \cdots + k_r t_r}}{[k_1]_q^{s_1} \cdots [k_r]_q^{s_r}},$$

$$H_n^\star(\mathbf{s}; \mathbf{t}; q) = H_n^\star(s_1, \dots, s_r; t_1, \dots, t_r; q) = \sum_{1 \le k_1 \le \cdots \le k_r \le n} \frac{q^{k_1 t_1 + \cdots + k_r t_r}}{[k_1]_q^{s_1} \cdots [k_r]_q^{s_r}},$$

where

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^{k-1}$$

is the q-analog of positive integer k. By convention, we put $H_n(\emptyset) = H_n^\star(\emptyset) = 1$, and $H_n(\mathbf{s}; \mathbf{t}; q) = 0$ if $n < r$. The number $w(\mathbf{s}) = \sum_{j=1}^r s_j$ is called the weight of the multiple harmonic sum.

For a primitive n-th root of unity ζ_n, the following work [2], we adopt the notation

$$z_n(\mathbf{s}; \zeta_n) = H_{n-1}(\mathbf{s}; \mathbf{s} - \{1\}^r; \zeta_n),$$
$$z_n^\star(\mathbf{s}; \zeta_n) = H_{n-1}^\star(\mathbf{s}; \mathbf{s} - \{1\}^r; \zeta_n),$$

where $\{a\}^r$ denotes the r-tuple with r consecutive copies of the letter a (note that we reversed the order of summation for convenience in our settings).

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In [2], Bachmann, Takeyama, and Tasaka studied special values of $z_n(\{k\}^r;\zeta_n)$ and in particular for $k = 1, 2, 3$, showed that

$$z_n(\{1\}^r;\zeta_n) = \frac{1}{n}\binom{n}{r+1}(1-\zeta_n)^r, \tag{1}$$

$$z_n(\{2\}^r;\zeta_n) = \frac{(-1)^r}{n(r+1)}\binom{n+r}{2r+1}(1-\zeta_n)^{2r}, \tag{2}$$

$$z_n(\{3\}^r;\zeta_n) = \frac{1}{n^2(r+1)}\left(\binom{n+2r+1}{3r+2} + (-1)^r\binom{n+r}{3r+2}\right)(1-\zeta_n)^{3r}. \tag{3}$$

The authors of [2] also formulated two open questions for finite multiple harmonic q-series z_n on 3-2-1 indices, namely,

$$\begin{aligned} z_n(\{1\}^a,2,\{1\}^b;\zeta_n) + z_n(\{1\}^b,2,\{1\}^a;\zeta_n) &\overset{?}{=} -\frac{1}{n}\binom{n+1}{a+b+3}(1-\zeta_n)^{a+b+2}, \\ z_n(\{2\}^a,3,\{2\}^b;\zeta_n) + z_n(\{2\}^b,3,\{2\}^a;\zeta_n) &\overset{?}{=} \frac{(-1)^{a+b}}{n(a+b+2)}\binom{n+a+b+1}{2(a+b)+3}(1-\zeta_n)^{2(a+b)+3}. \end{aligned} \tag{4}$$

In this paper, we prove the above relations and obtain related formulas for corresponding values of $\xi(\mathbf{s})$, which are defined as the limit values (see [1, Thm. 1.2])

$$\xi(\mathbf{s}) = \lim_{n\to\infty} z_n(\mathbf{s}; e^{\frac{2\pi i}{n}}).$$

Note that when n is a prime, formulas (4) as well as (2) and (3) follow from our results on q-congruences for multiple q-harmonic sums [3, Thm. 4.1, Thm. 5.1, Thm. 6.1, and Thm. 8.3], while formula (1) follows from [4, Cor. 2.2]. The methods of our paper [3] can be easily adjusted to prove (4) for arbitrary positive integer n.

Theorem 1.1. *For all non-negative integers a, b and any n-th primitive root of unity ζ_n,*

$$z_n(\{2\}^a,3,\{2\}^b;\zeta_n) + z_n(\{2\}^b,3,\{2\}^a;\zeta_n) = \frac{(-1)^{a+b}}{n(a+b+2)}\binom{n+a+b+1}{2(a+b)+3}(1-\zeta_n)^{2(a+b)+3}.$$

Theorem 1.2. *For all non-negative integers a, b and any n-th primitive root of unity ζ_n,*

$$z_n(\{1\}^a,2,\{1\}^b;\zeta_n) + z_n(\{1\}^b,2,\{1\}^a;\zeta_n) = -\frac{1}{n}\binom{n+1}{a+b+3}(1-\zeta_n)^{a+b+2}.$$

The complex numbers $\xi(\mathbf{s})$ are of interest in view of their connections to the finite and symmetric multiple zeta values as was shown in [1]. After letting $\zeta_n = e^{\frac{2\pi i}{n}}$ in Theorem 1.1 and Theorem 1.2, and by noting that for $j, k \in \mathbb{N}$,

$$\lim_{n\to\infty}\binom{n+j}{k}\frac{k!}{n^k} = 1, \quad \text{and} \quad \lim_{n\to\infty} n(1-e^{\frac{2\pi i}{n}}) = -2\pi i,$$

we obtain the following corollary.

Corollary 1.1. *For all non-negative integers a, b,*

$$\xi(\{1\}^a,2,\{1\}^b) + \xi(\{1\}^b,2,\{1\}^a) = -\frac{(-2\pi i)^{a+b+2}}{(a+b+3)!},$$

and

$$\xi(\{2\}^a,3,\{2\}^b) + \xi(\{2\}^b,3,\{2\}^a) = 0.$$

Note that the last relation can also be readily obtained from the definition of the symmetric multiple zeta values (see, for example, [1, Def. 2.5]).

Finally, we put forward the following conjectures regarding cyclic sums of multiple q-harmonic sums z_n at roots of unity, which generalize both of the theorems above.

Conjecture 1.1 (Cyclic-sum). *Let $d_0, d_1, \ldots, d_t$ be non-negative integers. Then*

(i) *For every integer $n > r$, where $r = \sum_{j=0}^{t} d_j + 2t$, and any primitive root of unity ζ_n,*

$$\sum_{j=0}^{t} z_n\left(\{1\}^{d_j}, 2, \{1\}^{d_{j+1}}, 2, \ldots, 2, \{1\}^{d_{j+t}}\right) = \frac{(-1)^t}{n}\binom{n+t}{r+1}(1-\zeta_n)^r.$$

(ii) *For every integer $n > r$, where $r = \sum_{j=0}^{t} 2d_j + 3t$, and any primitive root of unity ζ_n,*

$$\sum_{j=0}^{t} z_n\left(\{2\}^{d_j}, 3, \{2\}^{d_{j+1}}, 3, \ldots, 3, \{2\}^{d_{j+t}}\right) \in (1-\zeta_n)^r\mathbb{Q}.$$

In both sums above it is understood that $d_j = d_k$ if $j \equiv k$ modulo $t+1$.

Note that the case $t = 1$ follows from Theorem 1.1 and Theorem 1.2. The case of arbitrary t when all d_j are zeros follows from (2) and (3).

2. Proof of Theorem 1.1.

Let $\overline{\mathbf{s}} = (s_r, s_{r-1}, \ldots, s_1)$ denote the reverse of $\mathbf{s} = (s_1, \ldots, s_{r-1}, s_r)$. Then we have the following relations.

Lemma 2.1. *Let $\mathbf{s} = (s_1, \ldots, s_r)$ and $\mathbf{t} = (t_1, \ldots, t_r)$ be two r-tuples of non-negative integers, and ζ_n be an n-th primitive root of unity. Then*

$$H_{n-1}(\mathbf{s};\mathbf{t};\zeta_n) = (-1)^{w(\mathbf{s})} H_{n-1}(\overline{\mathbf{s}};\overline{\mathbf{s}}-\overline{\mathbf{t}};\zeta_n),$$
$$H^{\star}_{n-1}(\mathbf{s};\mathbf{t};\zeta_n) = (-1)^{w(\mathbf{s})} H^{\star}_{n-1}(\overline{\mathbf{s}};\overline{\mathbf{s}}-\overline{\mathbf{t}};\zeta_n),$$

and in particular,

$$z_n(\mathbf{s};\zeta_n) = (-1)^{w(\mathbf{s})} H_{n-1}(\overline{\mathbf{s}};\{1\}^r;\zeta_n), \tag{5}$$
$$z^{\star}_n(\mathbf{s};\zeta_n) = (-1)^{w(\mathbf{s})} H^{\star}_{n-1}(\overline{\mathbf{s}};\{1\}^r;\zeta_n).$$

Proof. Replacing each k_i by $n - k_i$ and reversing the order of summation, we get

$$\begin{aligned} H_{n-1}(\mathbf{s};\mathbf{t};\zeta_n) &= \sum_{0<k_1<\cdots<k_r<n} \frac{\zeta_n^{t_1k_1+\ldots+t_rk_r}}{[k_1]_{\zeta_n}^{s_1}\ldots[k_r]_{\zeta_n}^{s_r}} = \sum_{0<n-k_1<\cdots<n-k_r<n} \frac{\zeta_n^{t_1(n-k_1)+\ldots+t_r(n-k_r)}}{[n-k_1]_{\zeta_n}^{s_1}\ldots[n-k_r]_{\zeta_n}^{s_r}} \\ &= \sum_{0<k_r<k_{r-1}\cdots<k_1<n} \frac{\zeta_n^{-t_1k_1-\ldots-t_rk_r}}{[k_1]_{\zeta_n}^{s_1}\ldots[k_r]_{\zeta_n}^{s_r}} \times (-1)^{w(\mathbf{s})}\zeta_n^{k_1s_1+\ldots+k_rs_r} \\ &= (-1)^{w(\mathbf{s})} H_{n-1}(\overline{\mathbf{s}};\overline{\mathbf{s}}-\overline{\mathbf{t}};\zeta_n), \end{aligned}$$

where we used the identity

$$[n-k_i]_{\zeta_n} = \frac{1-\zeta_n^{n-k_i}}{1-\zeta_n} = \frac{1-\zeta_n^{-k_i}}{1-\zeta_n} = -\zeta_n^{-k_i}[k_i]_{\zeta_n}. \tag{6}$$

Setting $\mathbf{t} = \mathbf{s} - \{1\}^r$, we get (5). The proofs for the multiple harmonic star sums are similar. □

Proof of Theorem 1.1. We have

$$z_n(\{2\}^a,3,\{2\}^b;\zeta_n)+(1-\zeta_n)z_n(\{2\}^{a+b+1})$$
$$=\sum_{0<k_1<\cdots<k_a}\frac{\zeta_n^{k_1+\ldots+k_a}}{[k_1]_{\zeta_n}^2\cdots[k_a]_{\zeta_n}^2}\sum_{k_a<k_{a+1}<k_{a+2}}\left(\frac{\zeta_n^{2k_{a+1}}}{[k_{a+1}]_{\zeta_n}^3}+\frac{(1-\zeta_n)\zeta_n^{k_{a+1}}}{[k_{a+1}]_{\zeta_n}^2}\right)$$
$$\times\sum_{k_{a+1}<k_{a+2}<\cdots<k_{a+b+1}<n}\frac{\zeta_n^{k_{a+2}+\ldots+k_{a+b+1}}}{[k_{a+2}]_{\zeta_n}^2\cdots[k_{a+b+1}]_{\zeta_n}^2}$$
$$=H_{n-1}(\{2\}^a,3,\{2\}^b;\{1\}^{a+b+1};\zeta_n)=-z_n(\{2\}^b,3,\{2\}^a;\zeta_n),$$

where in the last equality we used (5). Hence

$$z_n(\{2\}^a,3,\{2\}^b;\zeta_n)+z_n(\{2\}^b,3,\{2\}^a;\zeta_n)=-(1-\zeta_n)z_n(\{2\}^{a+b+1}),$$

which, by (2), implies the theorem. □

3. Proof of Theorem 1.2.

The q-binomial coefficient, or Gaussian coefficient, when q is specified to a primitive root of unity has the following properties.

Lemma 3.1. *Let $n>1$ be a positive integer. Then for any primitive n-th root of unity ζ_n and $1\le k<n$,*

$$\begin{bmatrix}n-1\\k\end{bmatrix}_{\zeta_n}=(-1)^k\zeta_n^{-\binom{k+1}{2}}.$$

Proof. We have

$$\begin{bmatrix}n-1\\k\end{bmatrix}_{\zeta_n}=\prod_{j=1}^k\frac{[n-j]_{\zeta_n}}{[j]_{\zeta_n}}=\prod_{j=1}^k\frac{1-\zeta_n^{n-j}}{1-\zeta_n^j}=\prod_{j=1}^k\frac{1-\zeta_n^{-j}}{1-\zeta_n^j}=\prod_{j=1}^k(-\zeta_n^{-j})=(-1)^k\zeta_n^{-\binom{k+1}{2}}.\qquad\square$$

The proof of Theorem 1.2 is based on the following multiple q-binomial identity.

Theorem A ([3], Thm. 8.1) *Let $n,s_1,\ldots,s_r$ be positive integers. Then*

$$\sum_{k=1}^n\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k+1}{2}}\sum_{1\le k_1<k_2<\ldots<k_r=k}\prod_{i=1}^r\frac{q^{(s_i-1)k_i}}{[k_i]_q^{s_i}}=(-1)^r\sum_{\substack{1\le j_1\le j_2\le\ldots\le j_w\le n\\ j_i<j_{i+1},\,i\in I}}\prod_{i=1}^w\frac{q^{j_i}}{[j_i]_q},$$

where $w=w(\mathbf{s})=\sum_{i=1}^r s_i$, $I=\{s_1,s_1+s_2,\ldots,s_1+s_2+\cdots+s_{r-1}\}$, and the sum on the right is taken over all integers $j_1,\ldots,j_w$ satisfying the conditions $1\le j_i\le n$, $j_i<j_{i+1}$ for $i\in I$, and $j_i\le j_{i+1}$ otherwise.

From Theorem A we get a kind of duality for finite multiple q-harmonic sums z_n at roots of unity.

Theorem 3.1. *Let $n,s_1,\ldots,s_r$ be positive integers. Then*

$$z_n(\mathbf{s};\zeta_n)=(-1)^r\sum_{\substack{1\le j_1\le j_2\le\ldots\le j_w<n\\ j_i<j_{i+1},\,i\in I}}\prod_{i=1}^w\frac{\zeta_n^{j_i}}{[j_i]_{\zeta_n}},\tag{7}$$

where $w = w(\mathbf{s}) = \sum_{i=1}^{r} s_i$*,* $I = \{s_1, s_1+s_2, \ldots, s_1+s_2+\cdots+s_{r-1}\}$*, and the sum on the right is taken over all integers* $j_1, \ldots, j_w$ *satisfying the conditions* $1 \le j_i \le n$*,* $j_i < j_{i+1}$ *for* $i \in I$*, and* $j_i \le j_{i+1}$ *otherwise.*

Proof. To get (7), we replace n by $n-1$, and q by a primitive root of unity ζ_n in Theorem A, and apply Lemma 3.1. □

Proof of Theorem 1.2. Let $\mathbf{s} = (\{1\}^a, 2, \{1\}^b)$ and $w = w(\mathbf{s}) = a+b+2$. Applying Theorem 3.1 and noticing that $I = \{1, 2, \ldots, a, a+2, a+3, \ldots, a+b+1\}$, we get

$$\begin{aligned}
-z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) &= (-1)^{a+b} \sum_{1 \le j_1 < j_2 < \ldots < j_{a+1} \le j_{a+2} < j_{a+3} < \ldots < j_w < n} \prod_{i=1}^{w} \frac{\zeta_n^{j_i}}{[j_i]_{\zeta_n}} \\
&= (-1)^{a+b} \sum_{0 < n-j_1 < \ldots < n-j_{a+1} \le n-j_{a+2} < n-j_{a+3} < \ldots < n-j_w < n} \prod_{i=1}^{w} \frac{\zeta_n^{n-j_i}}{[n-j_i]_{\zeta_n}}.
\end{aligned}$$

Applying identity (6), we get

$$\begin{aligned}
-z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) &= \sum_{n > j_1 > \ldots > j_{a+1} \ge j_{a+2} > j_{a+3} > \ldots > j_w \ge 1} \frac{1}{[j_1]_{\zeta_n} \cdots [j_w]_{\zeta_n}} \\
&= \sum_{n > j_1 > \ldots > j_{a+1} > j_{a+3} > \ldots > j_w \ge 1} \frac{1}{[j_1]_{\zeta_n} \cdots [j_a]_{\zeta_n} [j_{a+1}]_{\zeta_n}^2 [j_{a+3}]_{\zeta_n} \cdots [j_w]_{\zeta_n}} + z_n(\{1\}^w; \zeta_n).
\end{aligned}$$

Noticing that

$$\frac{1}{[j_{a+1}]_{\zeta_n}^2} = \frac{(1-\zeta_n)[j_{a+1}]_{\zeta_n} + \zeta_n^{j_{a+1}}}{[j_{a+1}]_{\zeta_n}^2} = \frac{1-\zeta_n}{[j_{a+1}]_{\zeta_n}} + \frac{\zeta_n^{j_{a+1}}}{[j_{a+1}]_{\zeta_n}^2},$$

we obtain

$$-z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) = (1-\zeta_n) z_n(\{1\}^{w-1}; \zeta_n) + z_n(\{1\}^b, 2, \{1\}^a; \zeta_n) + z_n(\{1\}^w; \zeta_n).$$

Therefore, by (1),

$$\begin{aligned}
z_n(\{1\}^a, 2, \{1\}^b; \zeta_n) + z_n(\{1\}^b, 2, \{1\}^a; \zeta_n) &= -(1-\zeta_n) z_n(\{1\}^{w-1}; \zeta_n) - z_n(\{1\}^w; \zeta_n) \\
&= -\frac{1}{n}(1-\zeta_n)\binom{n}{w}(1-\zeta_n)^{w-1} - \frac{1}{n}\binom{n}{w+1}(1-\zeta_n)^w \\
&= -\frac{1}{n}\binom{n+1}{w+1}(1-\zeta_n)^w.
\end{aligned}$$

□

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