C^m Semialgebraic Sections Over the Plane

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Abstract

In this paper we settle the two-dimensional case of a conjecture involving unknown semialgebraic functions with specified smoothness. More precisely, we prove the following result: Let $\mathcal{H}$ be a semialgebraic bundle with respect to $C^m_{loc}(\mathbb{R}^2, \mathbb{R}^D)$. If $\mathcal{H}$ has a section, then it has a semialgebraic section.

1 Introduction

In this paper we settle the two-dimensional case of a conjecture involving unknown semialgebraic functions with specified smoothness.

Recall that a *semialgebraic set* $E \subset \mathbb{R}^n$ is a union of finitely many sets of the form

$$\{x \in \mathbb{R}^n : P_1(x), P_2(x), \cdots, P_r(x) > 0, \text{ and } Q_1(x) = Q_2(x) = \cdots = Q_s(x) = 0\}$$

for polynomials $P_1, \cdots, P_r, Q_1, \cdots, Q_s$ on $\mathbb{R}^n$. (We allow the cases $r = 0$ or $s = 0$.) A *semialgebraic function* $\phi : E \to \mathbb{R}^D$ is a function whose graph $\{(x, \phi(x)) : x \in E\}$ is a semialgebraic set.

We define smoothness in terms of C^m and C^m_{loc}. Here, $C^m\left(\mathbb{R}^n, \mathbb{R}^D\right)$ denotes the space of all $\mathbb{R}^D$-valued functions on $\mathbb{R}^n$ whose derivatives up to order m are continuous and bounded on $\mathbb{R}^n$. $C^m_{loc}\left(\mathbb{R}^n, \mathbb{R}^D\right)$ denotes the space of $\mathbb{R}^D$-valued functions on $\mathbb{R}^n$ with continuous derivatives up to order m. If $D = 1$, we write $C^m\left(\mathbb{R}^n\right)$ and $C^m_{loc}\left(\mathbb{R}^n\right)$ in place of $C^m\left(\mathbb{R}^n, \mathbb{R}^D\right)$ and $C^m_{loc}\left(\mathbb{R}^n, \mathbb{R}^D\right)$, respectively.

To motivate our conjecture, we pose the following problems.

Problem 1 (Semialgebraic Whitney Problem; see [44].) *Fix $m \geq 0$. Let $\phi : E \to \mathbb{R}$ be semialgebraic. Suppose ϕ extends to a C^m_{loc} function on $\mathbb{R}^n$. Does it necessarily extend to a C^m_{loc} semialgebraic function on $\mathbb{R}^n$?*

Problem 2 (Linear Equations) *Fix $m \geq 0$. Consider the linear equation*

$$A_1 F_1 + \cdots + A_D F_D = f \tag{1}$$

for unknowns $F_1, \cdots, F_D$ on $\mathbb{R}^n$, where $A_1, \cdots, A_D$, f are given semialgebraic functions. If equation (1) admits a C^m_{loc} solution $F_1, \cdots, F_D$, does it necessarily admit a C^m_{loc} semialgebraic solution?

More generally, in place of (1) we can consider underdetermined systems of linear equations. Problem 1 was raised by Bierstone and Milman in [44]. Note that m is fixed in the above problems so we are not allowed to lose derivatives. Problems 1 and 2 are instances of a more general question. The purpose of this paper is to settle that question, and in particular provide affirmative answers to Problems 1 and 2, in the case of $C^m_{loc}(\mathbb{R}^2)$.

To pose our more general question, we set up notations and give a few basic definitions: Fix $m \geq 0$. If $F \in C^m_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we write $J_x(F)$ (the "jet" of F at x) to denote the m-th degree Taylor polynomial of F at x. Thus, $J_x(F)$ belongs to $\mathcal{P}$, the vector space of all such polynomials. For $x \in \mathbb{R}^n$, $P, Q \in \mathcal{P}$, we define $P \odot_x Q = J_x(PQ)$. The multiplication $\odot_x$ makes $\mathcal{P}$ into a ring, denoted by $\mathcal{R}_x$, the "ring of m-jets at x". We have $J_x(FG) = J_x(F) \odot_x J_x(G)$ for $F, G \in C^m_{loc}(\mathbb{R}^n)$.

We consider vector-valued functions $F = (F_1, \cdots, F_D) : \mathbb{R}^n \to \mathbb{R}^D$, and we write $F \in C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$ if each $F_i \in C^m_{loc}(\mathbb{R}^n)$. We define $J_xF = (J_xF_1, \cdots, J_xF_D) \in \mathcal{P} \oplus \cdots \oplus \mathcal{P}$. Under the natural multiplication

$$Q \odot_x (P_1, \cdots, P_D) := (Q \odot_x P_1, \cdots, Q \odot_x P_D),$$

the vector space $\mathcal{P} \oplus \cdots \oplus \mathcal{P}$ becomes an $\mathcal{R}_x$ module, which we denote by $\mathcal{R}^D_x$. We will discuss $\mathcal{R}_x$-submodules of $\mathcal{R}^D_x$; we allow both $\{0\}$ and $\mathcal{R}^D_x$ as submodules of $\mathcal{R}^D_x$. Fix m, n, D, and a subset $E \subset \mathbb{R}^n$. For each $x \in E$, let

$$H(x) = f(x) + I(x) \subset \mathcal{R}^D_x$$

be given, where $f(x) \in \mathcal{R}^D_x$ and $I(x) \subset \mathcal{R}^D_x$ is an $\mathcal{R}_x$-submodule. Then the family

$$\mathcal{H} = (H(x))_{x \in E} \tag{2}$$

is called a "bundle" over E. $H(x)$ is called the <u>fiber</u> of $\mathcal{H}$ at x.

Remark 1.1 *We remark that our notion of bundle differs from the notion of a bundle considered previously (e.g, [28]). In the present version, we do not require E to be compact and we require all the fibers $H(x)$ to be non-empty.*

When m, n, D are not clear from context, we speak of a "bundle with respect to $C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$". If $\mathcal{H}$ is given by (2) and $E' \subset E$, then we write $\mathcal{H}|_{E'}$ to denote the bundle $(H(x))_{x \in E'}$, and refer to $\mathcal{H}|_{E'}$ as the <u>restriction of $\mathcal{H}$ to E'</u>. A "section" of the bundle $\mathcal{H}$ in (2) is a vector-valued function $F \in C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$ such that $J_xF \in H(x)$ for all $x \in E$. Note that sections F belong to $C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$ by definition. The bundle (2) is called "semialgebraic" if

$$\{(x, P_1, \cdots, P_D) \in \mathbb{R}^n \oplus \mathcal{P} \oplus \cdots \oplus \mathcal{P} : x \in E, (P_1, \cdots, P_D) \in H(x)\}$$

is a semialgebraic set.

We can now state our general problem.

Problem 3 *Let $\mathcal{H} = (H(x))_{x \in E}$ be a semialgebraic bundle with respect to $C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$. If $\mathcal{H}$ has a section, does it necessarily have a semialgebraic section?*

Again, we note that sections of $\mathcal{H}$ must belong to C^m_{loc} for fixed m, so we are not allowed to lose derivatives.

One checks easily that Problems 1 and 2 are instances of Problem 3. Indeed, suppose $\phi : E \to \mathbb{R}$ is semialgebraic, as in Problem 1. Set $\mathcal{H} = (H(x))_{x \in E}$, where

$$H(x) = \{P \in \mathcal{P} : P(x) = \phi(x)\}.$$

Then $\mathcal{H}$ is a semialgebraic bundle, and a section of $\mathcal{H}$ is precisely a function $F \in C^m_{loc}(\mathbb{R}^n)$ such that $F = \phi$ on E. Similarly, given an equation (1) as in Problem 2, set $\mathcal{H} = (H(x))_{x \in \mathbb{R}^n}$ with

$$H(x) = \left\{(P_1, \cdots, P_D) \in \mathcal{P}^D : A_1(x) P_1(x) + \cdots + A_D(x) P_D(x) = f(x)\right\}.$$

Then $\mathcal{H}$ is a semialgebraic bundle, and a section of $\mathcal{H}$ is precisely a solution $F = (F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n, \mathbb{R}^D\right)$ of equation (1).

In this paper, we settle the two-dimensional case of Problem 3.

Theorem 1 *Let $\mathcal{H}$ be a semialgebraic bundle with respect to $C^m_{loc}\left(\mathbb{R}^2, \mathbb{R}^D\right)$. If $\mathcal{H}$ has a section, then it has a semialgebraic section.*

We sketch the proof of Theorem 1. By a change of coordinates and a partition of unity, we may localize the problem to a small thin wedge

$$\Gamma(c) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0, c], 0 \le x_2 \le x_1\}.$$

More precisely, it is enough to prove that $\mathcal{H}|_{\Gamma(c')}$ has a section for sufficiently small c'. We may assume also that our bundle $\mathcal{H} = (H(x_1, x_2))_{(x_1, x_2) \in \Gamma(c)}$ satisfies $H((0,0)) = \{0\}$.

There are two main steps in our proof.

<u>Step 1:</u>

We analyze what it means for a given $F = (F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n, \mathbb{R}^D\right)$ with $J_{(0,0)}F = 0$ to be a section of $\mathcal{H}$. Our analysis produces finitely many semialgebraic curves $\gamma_1, \gamma_2, \cdots, \gamma_{s_{\max}}$ in $\Gamma(c)$, and we find that F is a section of $\mathcal{H}$ if and only if

- $F(x_1, x_2)$ and its x_2-derivatives up to order m satisfy finitely many linear equations (with semialgebraic coefficients) on the γ_s and
- F satisfies finitely many linear equations on $\Gamma(c) \setminus (\gamma_1 \cup \cdots \cup \gamma_{s_{\max}})$.

The curves γ_s have the form $\gamma_s = \{(x, \psi_s(x)) : x \in [0, c]\}$ for semialgebraic functions $\psi_1, \cdots, \psi_{s_{\max}}$ of one variable.

<u>Step 2:</u>

The heart of our proof is to use the above characterization to produce finitely many linear equations and inequalities for unknown functions $\xi^l_{sk}(x)$ of one variable ($l = 0, \cdots, m; k = 1, \cdots, D; s = 1, \cdots, s_{\max}$) with the following properties:

(A) If $F = (F_1, \cdots, F_D) \in C^m_{loc}(\mathbb{R}^2, \mathbb{R}^D)$ is a section of $\mathcal{H}$ then the functions

$$\xi^l_{sk}(x_1) = \partial^l_{x_2} F_k(x_1, x_2)\big|_{x_2=\psi_s(x_1)} \tag{3}$$

satisfy the above equations and inequalities for $x \in [0, c]$; and conversely

(B) If semialgebraic functions $\xi^l_{sk}(x)$ satisfy the above equations and inequalities for $x \in [0, c]$, then for some small $c' < c$ there exists a semialgebraic section $F = (F_1, \cdots, F_D)$ of $\mathcal{H}|_{\Gamma(c')}$ such that (3) holds for $x \in [0, c']$.

The equations and inequalities in (A) and (B) above have the form

$$\sum_{l,k} G^{s\nu}_{lk}(x_1)\, \xi^l_{sk}(x_1) = H^{s\nu}(x_1) \tag{4}$$

for $s = 1, \cdots, s_{\max}, \nu = 1, \cdots, \nu_{\max}(s)$ and

$$\sum_{l,k} \tilde{G}^{s\mu}_{lk}(x_1)\, \xi^l_{sk}(x_1) = \tilde{H}^{s\mu}(x_1) + o(1) \text{ as } x_1 \to 0^+ \tag{5}$$

for $s = 1, \cdots, s_{\max}, \mu = 1, \cdots, \mu_{\max}(s)$, where the $G^{s\nu}_{lk}, H^{s\nu}, \tilde{G}_{lk}, \tilde{H}^{s\mu}$ are semialgebraic functions of one variable, together with

$$\xi^l_{s+1,k}(x_1) - \sum_{r=0}^{m-l} \frac{1}{r!} \xi^{l+r}_s(x_1) \cdot [\psi_{s+1}(x_1) - \psi_s(x_1)]^r = o\left([\psi_{s+1}(x_1) - \psi_s(x_1)]^{m-l} \right), \tag{6}$$

as $x \to 0^+$ $(l = 0, \cdots, m, s = 1, \cdots, s_{\max} - 1, k = 1, \cdots, D)$.

The equations and inequalities arising from Step 1 are particularly simple in the setting of Problem 1. Therefore, Problem 1 has a simple solution in the two dimensional case, avoiding the need for Step 2. We omit the details.

To help the reader understand our proof of Step 2, we now provide an informal discussion of a simple example. Suppose we learn from Step 1 that the sections of our bundle $\mathcal{H}$ are precisely those $\mathbb{R}^2$-valued functions $F = (F_1, F_2) \in C^m(\mathbb{R}^2)$ such that

$$J_{(0,0)} F_1 = J_{(0,0)} F_2 = 0 \tag{7}$$

and

$$F_2 = \phi F_1 + \psi \tag{8}$$

on the set

$$\Gamma_c = \left\{ (x, y) \in \mathbb{R}^2 : 0 < y < x < c \right\}, \tag{9}$$

where ϕ and ψ are semialgebraic functions satisfying the estimates

$$|\partial^\alpha \phi(x, y)|, |\partial^\alpha \psi(x, y)| \le C_\alpha y^{-|\alpha|} \text{ on } \Gamma_c. \tag{10}$$

We suppose ϕ and ψ are C^m smooth up to the upper boundary $\{(x,x) : 0 < x < c\}$; all trouble arises at the lower boundary $\{(x,0) : 0 < x < c\}$ and at the origin. We explain how to carry out Step 2 for this simple example. We begin with (A). Suppose $F = (F_1, F_2) \in C^m(\mathbb{R}^2)$ satisfies (7) and (8). We don't assume that F is semialgebraic. Set

$$\xi_k^l(x) = \partial_y^l F_k(x,0) \text{ for } l = 0, \cdots, m, k = 1, 2.$$

How can we deduce equations and inequalities of the form (4) and (5)?

Let us start with (4). Let $P_k^x = J_{(x,0)} F_k$ for $k = 1, 2$, Thus, each P_k^x belongs to $\mathcal{P}^x$, the space of all m-jets at $(x,0)$. From Taylor's theorem and (8), (10), we deduce that $P_1 = P_1^x$ and $P_2 = P_2^x$ satisfy

$$\partial_{x,y}^\alpha \{P_2 - \phi P_1 - \psi\}(x,y) = o\left(y^{m-|\alpha|}\right) \text{ as } y \to 0 \text{ for fixed } x. \tag{11}$$

Let $\hat{H}(x,0)$ be the set of all $(P_1, P_2) \in \mathcal{P}^x \oplus \mathcal{P}^x$ satisfying (11). Then $\widehat{\mathcal{H}} = \left(\hat{H}(x,0)\right)_{x \in [0,c)}$ is a bundle, and (F_1, F_2) is a section of $\widehat{\mathcal{H}}$. Applying the reasoning of Step 1 to the bundle $\widehat{\mathcal{H}}$, we find that (after we pass to a smaller c in (9) to avoid finitely many bad points), the sections of the bundle $\widehat{\mathcal{H}}$ are precisely the $\left(\tilde{F}_1, \tilde{F}_2\right) \in C^m(\mathbb{R}^2)$ such that

$$\sum_{l,k} G_{lk}^\nu(x)\, \partial_y^l \tilde{F}_k(x,0) = H^\nu(x)$$

for $x \in (0,c)$, $\nu = 1, \cdots, \nu_{\max}$, where the G_{kl}^ν and H^ν are semialgebraic functions of one variable.

In particular, our $\xi_k^l(x) = \partial_y^l F_k(x)$ satisfy

$$\sum_{l,k} G_{lk}^\nu(x)\, \xi_k^l(x) = H^\nu(x) \tag{12}$$

for $\nu = 1, \cdots, \nu_{\max}$. Equations (12) are of the form (4), with $s_{\max} = 1$. That is how we produce equations of the form (4) for our $\xi_k^l(x)$.

Next, we sketch how to produce inequalities of the form (5). (We won't encounter inequality (6) in our simple example, because $s_{\max} = 1$ here.) First of all, because our F_k satisfy (7), we have

$$\xi_k^l(x) = o\left(x^{m-l}\right) \text{ as } x \to 0^+, \tag{13}$$

$l = 0, \cdots, m, k = 1, 2$. To derive inequalities from our main assumptions (8), we define functions
$S_j\left(\left(\eta_k^l\right)_{l=0,\cdots,m,k=1,2}, \bar{x}, \bar{y}\right)$ on $\mathbb{R}^{2(m+1)+2}$ by setting

$$S_j\left(\left(\eta_k^l\right), \bar{x}, \bar{y}\right) = \frac{\partial_y^j \left\{\sum_{l=0}^m \frac{1}{l!}\eta_2^l y^l - \phi(\bar{x}, y) \sum_{l=0}^m \frac{1}{l!}\eta_1^l y^l - \psi(\bar{x}, y)\right\}\Big|_{y=\bar{y}}}{\bar{y}^{m-j}}. \tag{14}$$

Thanks to (8), (10) and Taylor's theorem, our $\xi_k^l(x) = \partial_y^l F_k(x,0)$ satisfy

$$\sup_{0<y<x} \left| S_j\left(\xi_k^l(x), x, y\right)\right| \to 0 \text{ as } x \to 0^+ \tag{15}$$

for each $j = 0, 1, \cdots, m$. This condition is not yet of the form (5), because it involves a sup over y. However, for each $x \in (0, c)$, we may pick out finitely many $y_1(x), \cdots, y_{\mu_{\max}}(x)$ such that

$$\sup_{0<y<x} \left| S_j\left(\left(\eta_k^l\right), x, y\right)\right| \le C \max_{\mu=1,\cdots,\mu_{\max}} \left| S_j\left(\left(\eta_k^l\right), x, y_\mu(x)\right)\right| \tag{16}$$

for any $\left(\eta_k^l\right) \in \mathbb{R}^{2(m+1)}$ for which the left hand side is finite. Here, the constants $\mu_{\max}$ and C are independent of x.

Once we establish (16), it is easy to show that $y_\nu(x)$ may be taken to depend semialgebraically on x. Consequently, condition (15) is equivalent to the assertion that $S_j\left(\left(\xi_k^l(x)\right), x, y_\mu(x)\right) \to 0$ as $x \to 0^+$ for each j, μ. That is,

$$(y_\mu(x))^{j-m}\, \partial_y^j \left\{ \sum_{l=0}^m \frac{1}{l!}\left(\xi_2^l(x)\right) y^l - \phi(x,y) \sum_{l=0}^m \frac{1}{l!}\left(\xi_1^l(x)\right) y^l - \psi(x,y) \right\}\Bigg|_{y=y_\mu(x)} \to 0 \tag{17}$$

as $x \to 0^+$ for each μ, j. These inequalities are of the form (5). We remark that (16) is achieved by considering convex sets

$$K_{\bar{y}}^{x} = \left\{ \begin{array}{c} \left[\left(\eta_k^l\right), t\right] \in \mathbb{R}^{2(m+1)} \oplus \mathbb{R} : \\ \left| \partial_y^j \left\{ \sum_{l=0}^m \frac{1}{l!}\eta_2^l y^l - \phi(\bar{x}, y) \sum_{l=0}^m \frac{1}{l!}\eta_1^l y^l - t\psi(\bar{x}, y) \right\}\Big|_{y=\bar{y}} \right| \le |\bar{y}|^{m-j} \end{array} \right\}$$

and applying an elementary result from convex geometry, namely,

Theorem 2 (Helly, see [39]) *Let $\mathcal{K}$ be a collection of compact convex subsets of $\mathbb{R}^D$. If $K_1 \cap \cdots \cap K_{D+1} \neq \emptyset$ for arbitrary $K_1, \cdots, K_{D+1} \in \mathcal{K}$, then $\bigcap_{K \in \mathcal{K}} K \neq \emptyset$.*

Therefore, for any $F = (F_1, F_2) \in C^m(\mathbb{R}^2)$ satisfying (7) and (8), we have seen that the $\xi_k^l(x) = \partial_y^l F_k(x,0)$ satisfy equations (12) and inequalities (13) and (17). These equations and inequalities are of the form (4) and (5). Thus, we have achieved (A) in Step 2. (Recall, $s_{\max} = 1$ here, so inequality (6) doesn't enter the picture.)

It remains to establish (B). Thus, let $\left(\xi_k^l(x)\right)_{l=0,\cdots,m,k=1,2}$ be semialgebraic functions satisfying (12), (13), and (17). We must produce semialgebraic C^m functions F_1, F_2 on $\mathbb{R}^2$ satisfying (7) and (8), with $\partial_y^l F_k(x,0) = \xi_k^l(x)$ $(x \in (0,c))$ for each l, k. We will simply set

$$F_1(x,y) = \sum_{l=0}^m \frac{\left(\xi_1^l(x)\right)}{l!} y^l \tag{18}$$

and

$$F_2 = \phi F_1 + \psi \tag{19}$$

on Γ_c. These functions are clearly semialgebraic C^m functions on the open wedge Γ_c.

We will prove that

(20) The F_k extend to C^m functions on $\Gamma_c^{\text{closure}}$, and $\partial_y^l F_k(x,0) = \xi_k^l(x)$ for each l, k.

Once we know (20), it is easy to extend the F_k to semialgebraic C^m functions $\tilde{F}_k$ on all of $\mathbb{R}^2$, and we see at once that the $\tilde{F}_k$ satisfy (7) and (8). Thus, (B) reduces at once to the task of establishing (20). In proving (20), we repeatedly allow ourselves to reduce the constant c in (9), to take advantage of an elementary property of semialgebraic functions $\varphi(t)$ of one variable, namely,

$$\left(\frac{d}{dt}\right)^j \varphi(t) = O\left(t^{-j}\varphi(t)\right) \text{ as } t \to 0^+, \tag{21}$$

as follows from the Puiseux expansion of φ.

To prove (20) we introduce the semialgebraic functions

$$F_k^{\#}(x,y) = \sum_{l=0}^{m} \frac{1}{l!}\left(\xi_k^l(x)\right) y^{m-l} \text{ for } k = 1,2. \tag{22}$$

In particular, $F_1^{\#} = F_1$. From estimates (13) and (21) we see that the $F_k^{\#}$ extend to C^m functions on $\Gamma_c^{\text{closure}}$ (for smaller c than before), and that

$$J_{(0,0)}F_k^{\#} = 0, \text{ and } \partial_y^l F_k^{\#}(x,0) = \xi_k^l(x). \tag{23}$$

To prove (20), we will show that

$$\partial_{x,y}^{\alpha}\left(F_k^{\#} - F_k\right)(x,y) \to 0 \text{ as } (x,y) \to \partial\Gamma_c^{\#} \text{ for } |\alpha| \le m. \tag{24}$$

Because our $\xi_k^l(x) = \partial_y^l F_k^{\#}(x,0)$ satisfy (12), the defining property of the G_{lk}^{ν}, H^{ν} in (12) tells us that $\left(F_1^{\#}, F_2^{\#}\right)$ is a section of the bundle $\widehat{\mathcal{H}}$ defined above. That means that

$$\partial_{x,y}^{\alpha}\left\{\left(J_{(\bar{x},0)}F_2^{\#}\right) - \phi(x,y)\left(J_{(\bar{x},0)}F_1^{\#}\right) - \psi(x,y)\right\}\Big|_{(\bar{x},y)} = o\left(y^{m-|\alpha|}\right) \tag{25}$$

as $y \to 0^+$, for each fixed $\bar{x} \in (0,c)$ and each $|\alpha| \le m$. From (10), (25) and Taylor's theorem, we easily deduce that

$$\partial_{x,y}^{\alpha}\left\{F_2^{\#} - \phi F_1^{\#} - \psi\right\}(x,y) = o\left(y^{m-|\alpha|}\right) \text{ for } |\alpha| \le m \tag{26}$$

as $y \to 0^+$ for each fixed x. Recalling the definitions of $F_k, F_k^{\#}$, we conclude that

$$\partial_{x,y}^{\alpha}\left\{F_2^{\#} - F_2\right\}(x,y) = o\left(y^{m-|\alpha|}\right) \text{ for } |\alpha| \le m \tag{27}$$

as $y \to 0^+$ for each fixed x.

We don't yet know that (27) holds uniformly in x. Special attention is needed to understand what happens as $(x, y) \in \Gamma_c$ tends to $(0,0)$. To deal with that, we next show that

$$\sup_{y \in (0,x)} \left| \frac{\partial_y^j \left(F_2^{\#} - F_2\right)(x,y)}{y^{m-j}} \right| \to 0 \text{ as } x \to 0^+ \ \ (j = 0, \cdots, m) . \tag{28}$$

To see (28), recall that we assume that our $\xi_k^l(x)$ satisfy (17). The defining property of the $y_\mu(x)$ therefore tells us that

$$\sup_{y \in (0,x)} \left| y^{j-m} \partial_y^j \left\{ \sum_{l=0}^{m} \frac{1}{l!} \left(\xi_2^l(x)\right) y^l - \phi(x,y) \sum_{l=0}^{m} \left(\xi_1^l(x)\right) y^l - \psi(x,y) \right\} \right| \to 0$$

as $x \to 0^+$ $(j = 0, 1, \cdots, m)$. That is,

$$\sup_{y \in (0,x)} y^{j-m} \left| \partial_y^j \left\{ F_2^{\#}(x,y) - F_2(x,y) \right\} \right| \to 0$$

as $x \to 0^+$ $(j = 0, 1, \cdots, m)$, proving (28).

Inequality (28) involves only y-derivatives. However, because $F_2^{\#}$ and F_2 are semialgebraic and C^m smooth in the open wedge, (28) implies the seemingly stronger estimate

$$\sup_{y \in (0,x)} \frac{\left| \partial_x^i \partial_y^j \left\{ F_2^{\#} - F_2 \right\}(x,y) \right|}{y^{m-i-j}} \to 0 \tag{29}$$

as $x \to 0^+$ $(j = 0, 1, \cdots, m)$.

To deduce (29) from (28), we proceed by induction on i. The base case $i = 0$ is (28). For the induction step, fix $\bar{\imath} \geq 1$, and assume (29) for (i, j) with $i < \bar{\imath}$. We will prove (29) for all (i, j) with $i = \bar{\imath}$, by arguing as follows. Given $(\bar{\imath}, j)$ with $\bar{\imath} + j \leq m$, we pick $y = \varphi(x) \in (0, x)$ to maximize

$$\frac{\left| \partial_x^{\bar{\imath}} \partial_y^j \left\{ F_2^{\#} - F_2 \right\}(x,y) \right|}{y^{m-\bar{\imath}-j}} \text{ over all } y \in (0,x) , \tag{30}$$

up to a factor of 2. (Note that (30) is bounded for each fixed $x \in (0, c)$, thanks to (27).) We can take $y = \varphi(x)$ to be semialgebraic. Thanks to (21), φ satisfies

$$\left(\frac{d}{dx}\right)^p \varphi(x) = O\left(x^{1-p}\right) \text{ on } (0,c) \text{, for any } p \geq 0, \tag{31}$$

after we pass to a smaller c in (9). By induction hypothesis, we have

$$\left. \frac{\partial_x^{\bar{\imath}-1} \partial_y^j \left\{ F_2^{\#} - F_2 \right\}}{y^{m-(\bar{\imath}-1)-j}} \right|_{(x,\varphi(x))} \to 0 \text{ as } x \to 0^+. \tag{32}$$

The left-hand side of (32) is a semialgebraic function of one variable. Another application of (21) therefore yields the estimate

$$\frac{d}{dx}\left[\left. \frac{\partial_x^{\bar{\imath}-1}\partial_y^j \left\{ F_2^{\#} - F_2 \right\}}{y^{m-(\bar{\imath}-1)-j}} \right|_{(x,\varphi(x))} \right] = o\left(x^{-1}\right) \text{ as } x \to 0^+, \tag{33}$$

after again reducing the constant c in (9). The left-hand side of (33) is equal to

$$\left. \frac{\partial_x^{\bar{\imath}}\partial_y^j \left\{ F_2^{\#} - F_2 \right\}}{y^{m-(\bar{\imath}-1)-j}} \right|_{(x,\varphi(x))} + \text{OTHER TERMS}, \tag{34}$$

where the OTHER TERMS involve $\partial_x^{\bar{\imath}-1}\partial_y^{j'} \left\{ F_2^{\#} - F_2 \right\}$ and $\frac{d\varphi(x)}{dx}$.

Using our induction hypothesis (estimate (29) for $i < \bar{\imath}$) and (31) to control the OTHER TERMS in (34), we deduce from (33) that

$$\left. \frac{\partial_x^{\bar{\imath}}\partial_y^j \left\{ F_2^{\#} - F_2 \right\}}{y^{m-\bar{\imath}-j}} \right|_{(x,\varphi(x))} \to 0 \text{ as } x \to 0^+. \tag{35}$$

Because we picked $y = \varphi(x)$ to maximize the left-hand side of (35) up to a factor of 2, we conclude that

$$\sup_{y \in (0,x)} \left| \frac{\partial_x^{\bar{\imath}}\partial_y^j \left\{ F_2^{\#} - F_2 \right\}}{y^{m-\bar{\imath}-j}} \right| \to 0 \text{ as } x \to 0^+,$$

completing our induction on i and proving (29). Our result (29) provides the control we need on what happens as $(x, y) \in \Gamma_c$ tends to $(0, 0)$.

Let us recall what we have learned.

- $F_1, F_2, F_1^{\#}, F_2^{\#}$ are semialgebraic.
- F_1, F_2 are C^m smooth on the open wedge Γ_c, and $F_2 = \phi F_1 + \psi$.
- $F_1^{\#}, F_2^{\#}$ are C^m smooth on the wedge $\Gamma_c^{\text{closure}}$, and $\partial_y^l F_k^{\#}(x, 0) = \xi_k^l(x)$ for $l = 0, \cdots, m, k = 1, 2$.
-
$$\partial_{x,y}^{\alpha} \left\{ F_k - F_k^{\#} \right\} = o\left(y^{m-|\alpha|}\right) (|\alpha| \le m, k = 1, 2) \text{ as } y \to 0^+ \text{ for each fixed } x. \tag{36}$$
- $\partial_{x,y}^{\alpha} \left\{ F_k - F_k^{\#} \right\} = o\left(y^{m-|\alpha|}\right)$ $(|\alpha| \le m, k = 1, 2)$ as $(x, y) \in \Gamma_c$ tends to $(0, 0)$.

There is still more work to do. We still don't know that condition (36) holds uniformly in x. However, we've made major progress. We refer the reader to Sections 5 and 6 below for complete Proofs of Step 1 and Step 2 (A) and (B). We can now easily deduce Theorem 1 from (A) and (B), as follows. Because $\mathcal{H}|_{\Gamma(c)}$ has a section, (A) tells us that the relevant equations and inequalities (4) and (5) for the ξ^l_{sk} admit a solution. Because the functions $G^{s\nu}_{lk}, H^{s\nu}, \tilde{G}_{lk}, \tilde{H}^{s\mu}$ are semialgebraic in (4), (5), it follows easily that we may take the $\xi^l_{sk}(x)$ to depend semialgebraically on x. Thanks to (B), we obtain a semialgebraic section of $\mathcal{H}|_{\Gamma(c')}$, completing the proof of Theorem 1. See Section 7 for details.

It will be interesting to understand whether our arguments and results hold more generally in the setting of polynomially bounded o-minimal structures.

Let us recall some of the literature regarding Problems 1, 2, 3. The literature on Whitney's extension problem goes back to the seminal works of H. Whitney [42,43], and includes fundamental contributions by G. Glaeser [31], Yu. Brudnyi and P. Shvartsman [8–11], E. Bierstone, P. Milman, and W. Pawłucki [3–5], as well as our own papers [13–26]. In the semialgebraic (and o-minimal) setting , the analogue of the classical Whitney extension theorem is due to K. Kurdyka and W. Pawłucki [34] and A. Thamrongthanyalak [40].

Problem 1 in the setting of $C^1_{loc}(\mathbb{R}^n)$ was settled affirmatively by M. Aschenbrenner and A. Thamrongthanyalak [1]. Our results on Problem 3 imply an affirmative solution for $C^m_{loc}(\mathbb{R}^2)$. For $C^m_{loc}(\mathbb{R}^n)$ with $m \geq 2$ and $n \geq 3$, Problems 1, 2, 3 remain open.

The problem of deciding whether a (possibly underdetermined) system of linear equations of the form (1) admits a C^0_{loc} solution was proposed by Brenner [7], and Epstein-Hochster [12]. Two independent solutions to this problem appear in Fefferman-Kollár [27]. Fefferman-Luli [30] solved the analogous problem for C^m_{loc} $(m \geq 1)$. See also [29].

Kollár-Nowak [33] proved by example that an equation of the form (1) may fail to admit a solution by C^0_{loc}-rational functions, even though $A_1, \cdots, A_D$ and f are polynomials and a C^0_{loc} solution $(F_1, \cdots, F_D)$ exists. They showed that $x_1^3x_2f_1 + (x_1^3 - (1+x_3^2)x_2^3)f_2 = x_1^4$ has a continuous semialgebraic solution but no continuous rational solution $(f_1, f_2) \in C^0_{loc}(\mathbb{R}^3, \mathbb{R}^2)$. However, [41] shows that a semialgebraic C^0_{loc} solution exists, and [33] shows that a solution by C^0_{loc} semialgebraic functions exists for Problems 1 and 2 posed over $\mathbb{R}^2$, again provided $A_1, \cdots, A_D, f$ are polynomials.

A recent paper of Bierstone-Campesato-Milman [2] shows that given a system of equations (1) with semialgebraic data A_i, f, there exists a function $r : \mathbb{N} \to \mathbb{N}$ independent of f such that if the system (1) admits a $C^{r(m)}_{loc}$ solution, then it admits a semialgebraic C^m_{loc} solution. The result of Bierstone-Campesato-Milman is more general than the version stated above; it applies to suitable o-minimal structures.

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2 Notation and Preliminaries

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a Nash function if it is real-analytic and semialgebraic. Write $B(x,r)$ to denote the ball of radius r about x in $\mathbb{R}^n$. The dimension of a semialgebraic set $E \subset \mathbb{R}^n$ is the maximum of the dimensions of all the imbedded (not necessarily compact) submanifolds of $\mathbb{R}^n$ that are contained in E.

We recall a few definitions from the Introduction. Fix m, n, D, and a subset $E \subset \mathbb{R}^n$. For each $x \in E$, let

$$H\left(x\right) = f\left(x\right) + I\left(x\right) \subset \mathcal{R}_x^D \tag{37}$$

be given, where $f\left(x\right) \in \mathcal{R}_x^D$ and $I\left(x\right) \subset \mathcal{R}_x^D$ is an $\mathcal{R}_x$-submodule. Then the family

$$\mathcal{H} = (H(x))_{x \in E}$$

is called a bundle over E. $H(x)$ is called the fiber of $\mathcal{H}$ at x.

When m, n, D are not clear from context, we speak of a "bundle with respect to $C^m_{loc}\left(\mathbb{R}^n, \mathbb{R}^D\right)$". If $\mathcal{H}$ is given by (37) and $E' \subset E$, then we write $\mathcal{H}|_{E'}$ to denote the bundle $\left(H\left(x\right)\right)_{x \in E'}$, and refer to it as the restriction of $\mathcal{H}$ to E'. If $\mathcal{H} = (H(x))_{x \in E}$ and $\mathcal{H}' = (H'(x))_{x \in E}$ are bundles, $\mathcal{H}'$ is called a subbundle of $\mathcal{H}$ if $H'(x) \subset H(x)$ for all $x \in E$. We write $\mathcal{H} \supset \mathcal{H}'$ to denote that $\mathcal{H}'$ is a subbundle of $\mathcal{H}$.

What we called a "bundle" in [28] we now call a "classical bundle". The definition is as follows. Fix m, n, D. Let $E \subset \mathbb{R}^n$ be compact. A classical bundle over E is a family $\mathcal{H} = \left(H\left(x\right)\right)_{x \in E}$ of (possibly empty) affine subspaces $H\left(x\right) \subset \mathcal{P}^D$, parametrized by the points $x \in E$, such that each non-empty $H\left(x\right)$ has the form

$$H\left(x\right) = \vec{P}^x + \vec{I}\left(x\right)$$

for some $\vec{P}^x \in \mathcal{P}^D$ and some $\mathcal{R}_x$-submodule $\vec{I}\left(x\right)$ of $\mathcal{P}^D$.

When m, n, D are not clear from context, we speak of a "classical bundle with respect to $C^m(\mathbb{R}^n, \mathbb{R}^D)$". We remark again that our notion of bundle differs from the notion of bundles considered previously (e.g., [28]). In the present version, we do not require that E be compact and we require all the fibers $H(x)$ to be non-empty. A section of the bundle $\mathcal{H}$ is a vector-valued function $F \in C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$ such that $J_x F \in H(x)$ for all $x \in E$. A section of a classical bundle $\mathcal{H}$ is a vector-valued function $F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ such that $J_x F \in H(x)$ for all $x \in E$.

3 Tools

3.1 Glaeser Refinements, Stable Glaeser Refinements

Given a bundle $\mathcal{H} = (H(x))_{x \in E}$ for $C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$ or a classical bundle $\mathcal{H} = (H(x))_{x \in E}$ for $C^m(\mathbb{R}^n, \mathbb{R}^D)$, we define the Glaeser refinement $\mathcal{H}' = (H'(x))_{x \in E}$ as follows:

> **(GR)** Let $x_0 \in E$. A given $P_0 \in H(x_0)$ belongs to $H'(x_0)$ if and only if the following holds. Given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_1, \cdots, x_k \in B(x_0, \delta) \cap E$, where k is a large enough constant depending only on m, n, and D, there exist $P_i \in H(x_i)$ $(i = 1, \cdots, k)$, such that
>
> $$|\partial^\alpha (P_i - P_j)(x_i)| \le \epsilon |x_i - x_j|^{m - |\alpha|},$$
>
> for all $|\alpha| \le m, 0 \le i, j \le k$.

A bundle or a classical bundle $\mathcal{H}$ is Glaeser stable if $\mathcal{H}' = \mathcal{H}$. Note that the Glaeser refinement $\mathcal{H}'$ of $\mathcal{H}$ may have empty fibers, even if $\mathcal{H}$ has none. In that case, we know that $\mathcal{H}$ has no sections. If $\mathcal{H}$ is a classical bundle, then so is $\mathcal{H}'$. If $\mathcal{H}$ is a bundle and no fibers of $\mathcal{H}'$ are empty, then $\mathcal{H}'$ is a bundle. Both for bundles and for classical bundles, every section of $\mathcal{H}$ is a section of $\mathcal{H}'$. (See [28] for the case of classical bundles; the elementary proofs carry over unchanged for bundles.) Note in particular that if a given bundle $\mathcal{H}$ has a section, then $\mathcal{H}'$ has no empty fibers, hence $\mathcal{H}'$ is a bundle and $\mathcal{H}'$ has a section.

Starting from a classical bundle $\mathcal{H}$, or a bundle $\mathcal{H}$ with a section, we can perform iterated Glaeser refinement to pass to ever smaller subbundles $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$, etc., without losing sections. We set $\mathcal{H}^{(0)} = \mathcal{H}$, and for $l \ge 0$, we set $\mathcal{H}^{(l+1)} = \left(\mathcal{H}^{(l)}\right)'$. Thus, by an obvious induction on l, we have $\mathcal{H} = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \cdots$, yet $\mathcal{H}$ and $\mathcal{H}^{(l)}$ have the same sections for all $l \ge 0$.

If $\mathcal{H} = (H(x))_{x \in E}$ is a semialgebraic bundle with respect to $C^m_{loc}(\mathbb{R}^n, \mathbb{R}^D)$, by an obvious induction on l, we have $H^{(l)}(x)$ depends semialgebraically on x, where $\mathcal{H}^{(l)} = (H^{(l)}(x))_{x \in E}$. In principle, each $\mathcal{H}^{(l)}$ can be computed from $\mathcal{H}$. We remark that iterated Glaeser refinement stabilizes after finitely many iterations (i.e. for a large enough integer l^* determined by m, n, D, we have $\mathcal{H}^{(l^*+1)} = \mathcal{H}^{(l^*)}$; thus $\mathcal{H}^{(l^*)}$ is Glaeser stable. See [28] for the case of classical bundles; the argument, which goes back to Glaeser [31] and Bierstone-Milman-Pawłucki [4, 5], applies unchanged for bundles. We call $\mathcal{H}^{(l^*)}$ the stable Glaeser refinement of $\mathcal{H}$.)

The main results of [28] give the following

Theorem 3 *For a large enough integer constant l_* determined by m, n, and D, the following holds. Let $\mathcal{H}$ be a classical bundle with respect to $C^m\left(\mathbb{R}^n, \mathbb{R}^D\right)$. Let $\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \cdots$ be its iterated Glaeser refinements. Then $\mathcal{H}$ has a section if and only if $\mathcal{H}^{(l_*)}$ has no empty fibers. Suppose $\mathcal{H}^{(l_*)}$ has no empty fibers. Let $x_0 \in E$ and let P_0 belong to the fiber of $\mathcal{H}^{(l_*)}$ at x_0. Then there exists a section F of the bundle $\mathcal{H}$, such that $J_{x_0}(F) = P_0$. Moreover, there exists a constant $k^\#$ depending only on m, n, and D such that the following holds: Suppose $\mathcal{H} = (H(x))_{x \in E}$ is a Glaeser stable classical bundle. Assume the following holds for some constant $M > 0$:*

- *Given $x_1, \cdots x_{k^\#} \in E$, there exist polynomials $P_1, \cdots, P_{k^\#} \in \mathcal{P}^D$, with $P_i \in H(x_i)$ for $1 \le i \le k^\#$; $|\partial^\alpha P_i(x_i)| \le M$ for all $|\alpha| \le m, 1 \le i \le k^\#$; and $|\partial^\alpha (P_i - P_j)(x_j)| \le M|x_i - x_j|^{m-|\alpha|}$ for all $|\alpha| \le m, 1 \le i, j \le k^\#$.*

Then there exists $F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ with $\|F\|_{C^m(\mathbb{R}^n,\mathbb{R}^D)} \le C(m,n,D)M$ and $J_x(F) \in H(x)$ for all $x \in E$.

3.2 Puiseux Series

We will use the following elementary result regarding semialgebraic functions. For a proof, see [32].

Lemma 3.1 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is semialgebraic. Then there exists a polynomial $P(z,x) \not\equiv 0$ on $\mathbb{R}^2$ such that $P(f(x), x) \equiv 0$. Moreover, for each $x_0 \in \mathbb{R}$ there exists $\delta > 0$ such that $f(x)$ for $x \in (x_0, x_0 + \delta)$ is given by a convergent Puiseux series.*

Corollary 3.1 *Let $F(x)$ be a semialgebraic function of one variable, satisfying $|F(x)| = O(x^p)$ on $(0, c]$ for some given p. Then the derivatives of F satisfy $|F^{(k)}(x)| = O(x^{p-k})$ on $(0, c']$ for some c'. Similarly, if $F(x) = o(x^p)$ for x in $(0, c)$, then $F^{(k)}(x) = o(x^{p-k})$ for x in $(0, c')$. More generally, $|F^{(k)}(x)| = O(|F(x)|/x^k)$ on $(0, c')$.*

Corollary 3.2 *Let F be a semialgebraic function in $C^m_{loc}(\Omega_1)$, where $\Omega_\delta = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x < \delta\}$ for $\delta > 0$. Then for small enough δ, $F|_{\Omega_\delta}$ extends to a C^m semialgebraic function on $\mathbb{R}^2$.*

Sketch of proof. The result follows in one line from known results, but we sketch an elementary proof. Without loss of generality, we may suppose that $J_{(0,0)}F = 0$. Then $\partial^k_{x_2} F(x_1, 0) = o(x_1^{m-k})$ for $k \le m$, hence $\partial^l_{x_1} \partial^k_{x_2} F(x_1, 0) = o(x_1^{m-k-l})$ for $0 \le k, l \le m$.

We set $\tilde{F}(x_1, x_2)$ equal to the m-th degree Taylor polynomial of $x_2 \mapsto F(x_1, x_2)$ about $x_2 = 0$ for each fixed x_1. The above estimates for derivatives of F show that $\tilde{F}$ is C^m on $\tilde{\Omega}_\delta = \{(x_1, x_2) : 0 \le -x_2 \le x_1 \le \delta\}$, and its x_2-derivatives up to order m agree with those of F on the x_1-axis. In particular, $J_{(0,0)}\tilde{F} = 0$.

Similarly, we set $F^\#(x_1, x_2)$ equal to the m-th degree Taylor polynomial of $x_2 \mapsto F(x_1, x_2)$ about $x_2 = x_1$ for each fixed x_1. Then $F^\#$ is C^m on $\Omega^\#_\delta = \{(x_1, x_2) : 0 \le x_1 \le x_2 \le 2x_1 \le 2\delta\}$, and its x_2-derivatives up to order m agree with those of F on the line $x_1 = x_2$. In particular, $J_{(0,0)}F^\# = 0$.

Setting $F^+ = \begin{cases} F & \text{on } \Omega_\delta \\ \tilde{F} & \text{on } \tilde{\Omega}_\delta \\ F^\# & \text{on } \Omega^\#_\delta \end{cases}$, we see that F^+ is a C^m semialgebraic function on $\{(x_1, x_2) : x_1 \in [0, \delta], -x_1 \le x_2 \le 2x_1\}$, $F^+ = F$ on Ω_δ, and $J_{(0,0)}F^+ = 0$.

Next, let $\theta(t)$ be a C^m semialgebraic function of one variable, equal to 1 in $[0,1]$ and supported in $[-1, 2]$. Then, for small enough δ, the function $F^{++}(x_1, x_2) = \theta(\frac{x_2}{x_1}) \cdot F^+(x_1, x_2)$ for $x_1 > 0$, $F^{++}(x_1, x_2) = 0$ otherwise, is a C^m semialgebraic function on the disc $B(0, \delta)$ that agrees with our given F on Ω_δ.

Finally, multiplying F^{++} by a semialgebraic cutoff function supported in a small disc about $(0,0)$ and equal to 1 in a smaller disc, we obtain a C^m semialgebraic function on $\mathbb{R}^2$ that agrees with F on Ω_δ for small enough δ. ■

3.3 Singularities of Semialgebraic Sets and Functions

We recall a few standard properties of semialgebraic sets and functions.

- Let $U \subset \mathbb{R}^n$ be an open semialgebraic set, and let $F : U \to \mathbb{R}^k$ be semialgebraic. Then there exists a semialgebraic subset $X \subset U$ of dimension less than n (the "singular set" of F) such that F is real-analytic on $U \setminus X$. (See Chapter 8 in [6].)
- A zero-dimensional semialgebraic set is finite. A one-dimensional semialgebraic set is a union of finitely many real-analytic arcs and finitely many points. (See Chapter 2 in [6].)

3.4 Existence of Semialgebraic Selections

For sets X, Y, we denote a map Ξ from X to the power set of Y by $\Xi : X \rightrightarrows Y$ and call such Ξ a set-valued map; a set-valued map Ξ is semialgebraic if $\{(x,y) : y \in \Xi(x)\}$ is a semialgebraic set. Let $E \subset \mathbb{R}^n$ and $\Xi : E \rightrightarrows \mathbb{R}^D$. A selection of Ξ is a map $f : E \to \mathbb{R}^D$ such that $f(x) \in \Xi(x)$ for every $x \in E$. We recall the following well-known result regarding semialgebraic selection (see, for example, [36]).

Theorem 4 *Let $\Xi : E \rightrightarrows \mathbb{R}^D$ be semialgebraic. If each $\Xi(x)$ is nonempty, then Ξ has a semialgebraic selection.*

3.5 Growth of Semialgebraic Functions

Recall from [30] the following result

Lemma 3.2 (Growth Lemma) *Let $E \subset \mathbb{R}^{n_1}$ and $E^+ \subset E \times \mathbb{R}^{n_2}$ be compact and semialgebraic, with $\dim E^+ \geq 1$. Let A be a semialgebraic function on E^+. Then there exist an integer $K \geq 1$, a semialgebraic function A_1 on E, and a compact semialgebraic set $\underline{E}^+ \subset E^+$, with the following properties.*

> **(GL1)** $\dim \underline{E}^+ < \dim E^+$.
>
> *For $x \in E$, set $E^+(x) = \{y \in \mathbb{R}^{n_2} : (x,y) \in E^+\}$ and $\underline{E}^+(x) = \left\{y \in \mathbb{R}^{n_2} : (x,y) \in \underline{E}^+\right\}$. Then, for each $x \in E$, the following hold.*
>
> **(GL2)** *If $\underline{E}^+(x)$ is empty, then*
>
> $$|A(x,y)| \leq A_1(x) \text{ for all } y \in E^+(x).$$

(GL3) *If* $\underline{E}^+(x)$ *is non-empty, then*

$$|A(x,y)| \leq A_1(x) \cdot \left[dist\left(y, \underline{E}^+(x)\right)\right]^{-K} \text{ for all } y \in E^+(x) \setminus \underline{E}^+(x).$$

The Growth Lemma follows easily from a special case of a theorem of Łojasiewicz and Wachta [35], as explained in [30]. We thank W. Pawłucki for teaching us that implication.

We will apply the Growth Lemma to prove the following.

Lemma 3.3 *Let* $F(x,y)$ *be a bounded semialgebraic function on* $[-1,1] \times (0,1]$, *and suppose that*

$$\lim_{y \to 0^+} F(x,y) = 0 \text{ for each } x \in [-1,1]. \tag{38}$$

Then there exist a positive integer N *and a semialgebraic function* $A(x)$ *on* $[-1,1]$ *such that*

$$F(x,y) \leq A(x)\, y^{\frac{1}{N}} \text{ for all } (x,y) \in [-1,1] \times (0,1].$$

Proof. It is enough to show that for some positive integer N we have

$$\sup_{y \in (0,1]} \frac{|F(x,y)|}{y^{1/N}} < \infty \text{ for all } x \in [-1,1], \tag{39}$$

for we may then set $A(x) = \sup_{y \in (0,1]} \frac{|F(x,y)|}{y^{1/N}}$, and $A(x)$ will depend semialgebraically on x.

For each fixed x, the function $y \mapsto F(x,y)$ is bounded and given near $(0,0)$ by a convergent Puiseux series that tends to zero as $y \to 0^+$. Hence, for some positive integer N_x we have

$$\sup_{y \in (0,1]} \frac{|F(x,y)|}{y^{1/N_x}} < \infty. \tag{40}$$

Our task is to show that N_x may be taken independent of x. Thanks to (40), we may exclude from consideration any given finite set of "bad" $x \in [-1,1]$.

We recall our main hypothesis (38). For each $(x,\varepsilon) \in [-1,1] \times (0,1]$ there exists $\delta \in (0,1]$ such that (x,ε,δ) belongs to the semialgebraic set

$$\{(x,\varepsilon,\delta) \in [-1,1] \times (0,1] \times (0,1] : |F(x,y)| \leq \varepsilon \text{ for all } y \in (0,\delta]\}.$$

Hence, there exists a semialgebraic function $\delta(x,\varepsilon)$ mapping $[-1,1] \times (0,1]$ into $(0,1]$ such that

$$|F(x,y)| \leq \varepsilon \text{ for } y \in (0,\delta(x,\varepsilon)], x \in [-1,1], \varepsilon \in (0,1]. \tag{41}$$

We set $\delta(x,0) = 1$ for $x \in [-1,1]$. Then $\delta : [-1,1] \times [0,1] \to (0,1]$ is semialgebraic and satisfies (41).

We now apply Lemma 3.2 to the function $\frac{1}{\delta(x,\varepsilon)}$. Thus, we obtain a semialgebraic set $\underline{E} \subset [-1,1] \times [0,1]$, a positive integer N, and a positive semialgebraic function $\underline{\delta}(x)$ on $[-1,1]$, with the following properties.

- $\dim \underline{E} \leq 1$.
- For $x \in [-1,1]$, let $\underline{E}(x) = \{\varepsilon : (x,\varepsilon) \in \underline{E}\}$.

Then

$$\delta(x,\varepsilon) \geq \underline{\delta}(x) \ \ (\text{all } \varepsilon > 0) \text{ if } \underline{E} = \emptyset \tag{42}$$

and

$$\delta(x,\varepsilon) \geq \underline{\delta}(x) \cdot [\mathrm{dist}(\varepsilon, \underline{E}(x))]^N \ \ (\text{all } \varepsilon \notin \underline{E}(x)) \text{ if } \underline{E} \neq \emptyset. \tag{43}$$

Because $\dim \underline{E} \leq 1$, there are at most finitely many $x \in [-1,1]$ for which $\underline{E}(x)$ is infinite.

As explained above, we may discard those "bad" x, it is enough to prove (39) for all x such that $\underline{E}(x)$ is finite. From now on, we restrict attention to "good" x, i.e., those x for which $\underline{E}(x)$ is finite.

Set

$$\underline{\varepsilon}(x) = \begin{cases} \frac{1}{2}\min(\underline{E}(x) \setminus \{0\}) & \text{if } \underline{E}(x) \text{ contains points other than } 0 \\ 1 & \text{otherwise} \end{cases}.$$

So $\underline{\varepsilon}(x) > 0$ for all "good" x.

If $\underline{E}(x) \neq \emptyset$, then $\mathrm{dist}(\varepsilon, \underline{E}(x)) \geq \varepsilon$ for $0 < \varepsilon \leq \underline{\varepsilon}(x)$, hence (43) gives

$$\delta(x,\varepsilon) \geq \underline{\delta}(x)\,\varepsilon^N \text{ for } 0 < \varepsilon \leq \underline{\varepsilon}(x). \tag{44}$$

If instead $\underline{E}(x) = \emptyset$, then because $\underline{\varepsilon}(x) = 1$, (42) again gives (44). Thus, (44) holds in all cases.

Now suppose $0 < y < \underline{\delta}(x) \cdot (\underline{\varepsilon}(x))^N$. Then, setting $\varepsilon = \left(\frac{y}{\underline{\delta}(x)}\right)^{1/N}$ and applying (44), we find that $\delta(x,\varepsilon) \geq y$. The defining property of $\delta(x,\varepsilon)$ therefore tells us that

$$|F(x,y)| \leq \varepsilon = \left(\frac{y}{\underline{\delta}(x)}\right)^{1/N}.$$

Thus, for any "good" x, we have shown that

$$\frac{|F(x,y)|}{y^{1/N}} \leq (\underline{\delta}(x))^{-1/N} \ \text{ for } 0 < y < \underline{\delta}(x) \cdot (\underline{\varepsilon}(x))^N. \tag{45}$$

On the other hand, recall that F is bounded; say, $|F(x,y)| \leq M$ for all $(x,y) \in [-1,1] \times (0,1]$. Hence,

$$\frac{|F(x,y)|}{y^{1/N}} \leq \frac{M}{(\underline{\delta}(x))^{1/N}\,\underline{\varepsilon}(x)} \text{ for } \underline{\delta}(x) \cdot (\underline{\varepsilon}(x))^N \leq y \leq 1. \tag{46}$$

Our desired estimate (39) is now immediate from (45) and (46).

The proof of Lemma 3.3 is complete. ■

Similar ideas can be used to prove an n-dimensional version of Lemma 3.3, but we don't discuss it here.

3.6 Logarithmic Derivatives of Semialgebraic Functions

Let V be a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$. Given $x \in \mathbb{R}^n$, we write $V(x)$ to denote the set of all $t \in \mathbb{R}^m$ such that $(x,t) \in V$. Given $(x,t) \in \mathbb{R}^n \times \mathbb{R}^m$, we write $\delta_V(x,t)$ to denote the distance from t to $V(x)$. We take $\delta_V(x,t) = +\infty$ if $V(x)$ is empty. For a smooth function $F(x,t)$ on $\mathbb{R}^n \times \mathbb{R}^m$, we write $\nabla_t F(x,t)$ to denote the gradient of the function $t \mapsto F(x,t)$.

The following theorem is proven by A. Parusinski in [37, 38]. We thank Edward Bierstone, Jean-Baptiste Campesato, Pierre Milman, and Wieslaw Pawłucki for pointing out the references, and thus helping us remove 10 pages from our paper.

Theorem 5 *Let $F(x,t)$ be a (real-valued) subanalytic function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}^m$. Then there exist a closed codimension 1 subanalytic set $V \subset \mathbb{R}^n \times \mathbb{R}^m$ and a constant $C > 0$ such that the function F is Nash outside V and moreover,*

$$|\nabla_t F(x,t)| \leq C \frac{|F(x,t)|}{\delta_V(x,t)}. \tag{47}$$

If F is semialgebraic, then we can take V to be semialgebraic.

As a special case of Theorem 5, we have the following.

Theorem 6 *Let $F(t)$ be a semialgebraic function on $\mathbb{R}^n$. Then there exist a closed semialgebraic $V \subset \mathbb{R}^n$ of dimension at most $(n-1)$, and a constant C, such that F is Nash outside V, and*

$$|\nabla F(t)| \leq C\, |F(t)| \cdot [dist(t,V)]^{-1}$$

for $t \in \mathbb{R}^n \setminus V$.

Remark 3.1 *Corollary 3.1 is a special case of Theorem 6. In Section 5.3, we will apply Theorem 6 to partition Ω into semialgebraic open subsets of $\mathbb{R}^2$, finitely many semialgebraic curves in $\mathbb{R}^2$, and finitely many points.*

3.7 Variant of Helly's Theorem

We recall the following result from convex geometry. Surely more precise versions of the result are well known, but we had trouble tracking down a reference so we will provide a proof.

Theorem 7 (Helly's Theorem Variant) *Let $(p_\omega)_{\omega \in \Omega}$ be a family of seminorms on a vector space V of dimension D. Assume that $\sup_{\omega\in\Omega} p_\omega(v) < \infty$ for every $v \in V$. Then there exist $\omega_1, \cdots, \omega_L \in \Omega$, with L depending only on D, such that*

$$\sup_{\omega\in\Omega} p_\omega(v) \leq C \cdot \max\{p_{\omega_1}(v), \cdots, p_{\omega_L}(v)\} \text{ for all } v \in V,$$

with C also depending only on D.

We use the following variant of the classical Helly theorem (see Section 3 in [14]) from elementary convex geometry.

Lemma 3.4 *Let $(K_\omega)_{\omega\in\Omega}$ be a collection of compact convex symmetric subsets of $\mathbb{R}^D$. Suppose the intersection of all the K_ω has nonempty interior. Then there exist $\omega_1, \cdots, \omega_L$ such that $K_{\omega_1} \cap \cdots \cap K_{\omega_L} \subset C \cdot \bigcap_{\omega\in\Omega} K_\omega$, where C and L depend only on D.*

The proof of the "Lemma on Convex Sets" in Section 3 of [14] applies here and proves Lemma 3.4, even though our present hypotheses differ slightly from those of [14]. We apply Lemma 3.4 to prove Theorem 7.

Proof of Theorem 7. Suppose first that each p_ω is a norm, not just a seminorm. Then the conclusion of Theorem 7 follows by applying Lemma 3.4 to the family of convex sets $K_\omega = \{v \in V : p_\omega(v) \leq 1\}$, $\omega \in \Omega$.

Now suppose each p_ω is a seminorm. Let $H(\omega) = \{v \in V : p_\omega(v) = 0\}$, and let H be the intersection of all the $H(\omega)$. Each $H(\omega)$ is a vector subspace of V. Consequently there exist $\lambda_1, \cdots, \lambda_s \in \Omega$, with $s \leq D$, such that $H = H(\lambda_1) \cap \cdots \cap H(\lambda_s)$. For $\omega \in \Omega$ and $v \in V$, set $p^*_\omega(v) = p_{\lambda_1}(v) + \cdots + p_{\lambda_s}(v) + p_\omega(v)$. Then p^*_ω is a seminorm on V, and $p^*_\omega(v) = 0$ if and only if $v \in H$. Regarding each p^*_ω as a norm on V/H, and applying Theorem 7 for collections of norms, we complete the proof of Theorem 7. ■

4 Preliminary Reductions

The purpose of this section is to reduce Theorem 1 to the following:

Lemma 4.1 (Main Lemma) *Let $\mathcal{H} = (H(x))_{x\in\mathbb{R}^2}$ be a semialgebraic bundle for $C^m_{loc}(\mathbb{R}^2, \mathbb{R}^D)$. Assume $\mathcal{H}$ is Glaeser stable. Assume $H(0) = \{0\}$. Then, for small enough $c > 0$, $\mathcal{H}|_{\Gamma(c)}$ has a semialgebraic section, where $\Gamma(c) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0, c], 0 \leq x_2 \leq x_1\}$.*

To deduce Theorem 1 from Lemma 4.1 we argue as follows: Suppose we are given a Glaeser stable bundle $\mathcal{H} = (H(x))_{x\in\mathbb{R}^2}$ for $C^m_{loc}(\mathbb{R}^2, \mathbb{R}^D)$ with $H(x) \subset \mathcal{P}^D$ depending semialgebraically on x. Assume $H(0) = \{0\}$. Let $\Gamma(c) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0, c], 0 \leq x_2 \leq x_1\}$. Theorem 3 tells us that $\mathcal{H}|_{\Gamma(c)}$ has a section F_c. The main lemma asserts that for c small enough $\mathcal{H}|_{\Gamma(c)}$ has a semialgebraic section. We will cover a full neighborhood of 0 by rotating wedges of the form $\Gamma(c)$. Using a partition of unity subordinate to the cover and the fact that $H(0) = \{0\}$, we can then patch together sections of $\mathcal{H}$, and obtain a semialgebraic section over a full neighborhood of 0. We may drop the restriction $H(0) = \{0\}$, because without loss of generality our given section F_c has jet 0 at the origin, so we may just cut down $H(0)$ to $\{0\}$. We can also drop the restriction that $\mathcal{H}$ is Glaeser stable (assuming $\mathcal{H}$ has a section) since we can always pass to the stable Glaeser refinement. Thus, any semialgebraic bundle having a section has a semialgebraic section over some neighborhood of 0. We can use compactness and a partition of unity to conclude that $\mathcal{H}$ admits a semialgebraic section over any given compact set.

Lemma 4.2 *Suppose $H(z)$ depends semialgebraically on $z \in \mathbb{R}^2$. If $\mathcal{H} = (H(z))_{z\in\mathbb{R}^2}$ has a section, then $\mathcal{H}$ has a section $F \in C^m_{loc}(\mathbb{R}^2, \mathbb{R}^D)$ such that for all $|\alpha| \leq m$, $|\partial^\alpha F(x)| \leq C(1+|x|)^K$ on $\mathbb{R}^2$, for some C and K.*

Proof. To prove this lemma, we may assume that $\mathcal{H}$ is Glaeser stable. Taking $E_R = \{x \in \mathbb{R}^2 : |x| \leq R\}$ with $R \geq 1$, and applying Theorem 3, we obtain a section F_R of $\mathcal{H}|_{E_R}$, with $||F_R||_{C^m} \leq C\,(R)^K$, because the "$M$ " in Theorem 3 applied to $\mathcal{H}|_{E_R}$ can be taken to depend semialgebraically on R. (That's where we use the fact that the bundle $\mathcal{H}$ is semialgebraic.) We can now easily use a partition of unity to patch together F_{2^k}, $k = 1, 2, 3, \cdots$, into a section F as in the conclusion of Lemma 4.2. ■

Fix K as in the conclusion of Lemma 4.2. Let $\Phi :$ Open Disc $\Delta \to \mathbb{R}^2$ be a semialgebraic diffeomorphism, for example, $\Phi(x) = \frac{x}{1-|x|^2}$. Let $\theta(x) > 0$ be a semialgebraic function on $\mathbb{R}^2$ that tends to zero so rapidly that

$$\partial^\alpha[(\theta F)\circ\Phi](y) \to 0, \text{ for all } |\alpha| \leq m \text{ as } y \to \partial\Delta,$$

whenever $|\partial^\alpha F(x)| \leq C(1+|x|)^K$ on $\mathbb{R}^2$, $|\alpha| \leq m$. We can now form a bundle $\mathcal{H}^*$ as follows: For x in Δ, the fiber $H^*(x)$ consists of all $J_x((\theta F)\circ\Phi)$ for sections F of the bundle $\mathcal{H}$. The fibers of $\mathcal{H}^*$ over points not in Δ are $\{0\}$. Then $\mathcal{H}^*$ is a semialgebraic bundle admitting a section. We have seen that semialgebraic bundles with sections have semialgebraic sections over any compact set. In particular, $\mathcal{H}^*$ has a semialgebraic section $\mathcal{F}$ over Δ^{closure}. Then $\frac{\mathcal{F}\circ\Phi^{-1}(x)}{\theta(x)}$ is a semialgebraic section of $\mathcal{H}$ over $\mathbb{R}^2$. Consequently, we can deduce Theorem 1 from Lemma 4.1.

The rest of the paper is devoted to the proof of Lemma 4.1.

5 Characterization of Sections

5.1 Semialgebraic Bundles

Fix $U \subset \mathbb{R}^n$ open, semialgebraic. Fix $\psi : U \to \mathbb{R}^k$ Nash. Let $\hat{\psi}(x) = (x, \psi(x)) \in \mathbb{R}^n \times \mathbb{R}^k$ for $x \in U$. We set $\hat{U} = \hat{\psi}(U)$. Let $\mathcal{P}$ denote the vector space of polynomials of degree at most m on $\mathbb{R}^n \times \mathbb{R}^k$. We write $z = (x, y)$ to denote a point of $\mathbb{R}^n \times \mathbb{R}^k$. We write $\mathcal{R}_z$ to denote the ring obtained from $\mathcal{P}$ by multiplication of m-jets at z. We fix a bundle $\mathcal{H} = (H(z))_{z\in\hat{U}}$, where, for each $z = \hat{\psi}(x) \in \hat{U}$ we have $H(z) = f^x + I(x)$, $f^x \in \mathcal{P}^D$, $I(x)$ an $\mathcal{R}_{\hat{\psi}(x)}$-submodule of $\mathcal{P}^D$. (We point out that $\mathcal{H}$ is a bundle, not a classical bundle, see Remark 1.1.) We suppose $\mathcal{H}$ is Glaeser stable. We assume that $H(z)$ depends semialgebraically on $z \in \hat{U}$. (We sometimes abuse notion by writing $I(z)$ for $I(x)$, where $z = \hat{\psi}(x)$.)

Under the above assumptions and definitions, we will prove the following result.

Lemma 5.1 *There exist a semialgebraic set $U_{bad} \subset \mathbb{R}^n$ of dimension less than n; Nash functions $A^i_{j\beta}, G^i$ on $U \setminus U_{bad}$ ($i = 1, \cdots, i_{\max}, j = 1, \cdots, D, \beta$ a multiindex of order $\leq m$ for $\mathbb{R}^k$) with the following property. Let $B \subset U \setminus U_{bad}$ be a closed ball. Set $\hat{B} = \hat{\psi}(B)$. Let $F = (F_1, \cdots, F_D) \in C^m_{loc}(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D)$. Then F is a section of $\mathcal{H}|_{\hat{B}}$ if and only if $\sum_{|\beta|\leq m}\sum_{j=1}^D A^i_{j\beta}(x)\cdot(\partial^\beta_y F_j(x, \psi(x))) = G^i(x)$ for all $x \in B$ (each i).*

Proof. We may suppose that f^x and $I(x)$ depend semialgebraically on $x \in U$. We write $f^x = (f_1^x, \cdots, f_D^x)$ and $\psi(x) = (\psi_1(x), \cdots, \psi_k(x)) \quad (x \in U)$. For $l = 1, \cdots, n$, we introduce the vector field

$$X_l = \frac{\partial}{\partial x_l} + \sum_{p=1}^{k} \frac{\partial \psi_p(x)}{\partial x_l} \frac{\partial}{\partial y_p} \text{on } U \times \mathbb{R}^k.$$

On $U \times \mathbb{R}^k$, then X_l are Nash, and $[X_l, X_{l'}] = 0$. For $\alpha = (\alpha_1, \cdots, \alpha_n)$, we write $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. The $X_1, \cdots, X_n, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_k}$ form a frame on $U \times \mathbb{R}^k$. Because $I(x)$ depends semialgebraically on $x \in U$, we may express

$$\text{(48)} \quad I(x) = \left\{ (P_1, \cdots, P_D) \in \mathcal{P}^D : \sum_{\substack{|\alpha|+|\beta| \le m \\ j=1,\cdots,D}} \tilde{A}^i_{j\alpha\beta}(x) \left(X^\alpha \partial_y^\beta P_j \right) \Big|_{\tilde{\psi}(x)} = 0, \text{ for } i = 1, \cdots, i_{\max} \right\}$$

for semialgebraic $\tilde{A}^i_{j\alpha\beta}$ on U.

We take U^1_{bad} to be the union of the singular sets of the $\tilde{A}^i_{j\alpha\beta}$. Then U^1_{bad} is a semialgebraic set of dimension $< n$ in $\mathbb{R}^n$, and the $\tilde{A}^i_{j\alpha\beta}$ are real-analytic on $U \setminus U^1_{\text{bad}}$. We may therefore rewrite the equation in (48) in the form

$$\sum_{\substack{|\alpha|+|\beta| \le m \\ j=1,\cdots,D}} \left(X^\alpha \left\{ A^i_{j\alpha\beta}(x) \partial_y^\beta P_j \right\} \right) \Bigg|_{\hat{\psi}(x)} = 0.$$

The $A^i_{j\alpha\beta}$ are Nash on $U \setminus U^1_{\text{bad}}$. Thus, for any closed ball $B \subset U \setminus U^1_{\text{bad}}$ the following holds. (We set $\hat{B} = \hat{\psi}(B)$.)

A given $F = (F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $(I(z))_{z \in \hat{B}}$ if and only if

$$\sum_{|\alpha| \le m} X^\alpha \left\{ \sum_{|\beta| \le m - |\alpha|} A^i_{j\alpha\beta}(x) \partial_y^\beta F_j(x, y) \right\} = 0 \text{ on } \hat{B} \text{ for all } i.$$

We look for integers $s \ge 0$ for which there exist Nash functions $A^i_{j\alpha\beta}$ on $U \setminus U^1_{\text{bad}}$ with the following property ("Property $\prod(s)$"): Let $B \subset U \setminus U^1_{\text{bad}}$ be a closed ball; set $\hat{B} = \hat{\psi}(B)$. Then $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $(I(z))_{z \in \hat{B}}$ if and only if

$$\text{(49)} \qquad \sum_{|\alpha| \le s} X^\alpha \left\{ \sum_{|\beta| \le m - |\alpha|} \sum_{j=1}^{D} A^i_{j\alpha\beta}(x) \partial_y^\beta F_j(x, y) \right\} = 0 \text{ on } \hat{B} \text{ for all } i.$$

We have seen that we can achieve Property $\prod(m)$.

Claim 5.1 *Let s be the smallest possible integer ≥ 0 for which we can achieve Property $\prod(s)$, and let $A^i_{j\alpha\beta}$ be as in Property $\prod(s)$. Then $s = 0$. In other words, Property $\prod(0)$ holds.*

Proof of Claim 5.1. Assuming $s \geq 1$, we will achieve Property $\prod(s-1)$, contradicting the fact that s is as small as possible.

Fix $B \subset U \setminus U^1_{\text{bad}}$ a closed ball, and let $(F_1, \cdots, F_D) \in C^m_{loc}(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D)$ be a section of $(I(z))_{z\in\hat{B}}$. (As always, $\hat{B} = \psi(B)$.) Fix $x_0 \in B$ and fix a multiindex α_0 with $|\alpha_0| = s$. For $j = 1, \cdots, D$, define functions on $\mathbb{R}^n \times \mathbb{R}^k$ by setting $F^{\#}_j(z) = \theta \cdot F_j(z)$ where $\theta \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^k)$ with jet $(J_{\hat{\psi}(x_0)}\theta)(x, y) = (x - x_0)^{\alpha_0}$. Then $(F^{\#}_1, \cdots, F^{\#}_D) \in C^m_{loc}(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D)$ is a section of $(I(z))_{z\in\hat{B}}$ because each $I(z)$ is an $\mathcal{R}_z$-submodule of $\mathcal{R}^D_z$.

Applying Property $\prod(s)$ to $(F^{\#}_1, \cdots, F^{\#}_D)$, we learn that

$$\sum_{|\beta|\leq m-|\alpha_0|} \sum_{j=1}^{D} A^i_{j\alpha_0\beta}(x_0)\left(\partial^\beta_y F_j\right)\Big|_{\hat{\psi}(x_0)} = 0 \ \text{ (all } i) .$$

This holds for all x_0 and for all $|\alpha_0| = s$. Thus, if $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $(I(z))_{z\in\hat{B}}$, then

$$\sum_{|\beta|\leq m-|\alpha|} \sum_{j=1}^{D} A^i_{j\alpha\beta}(x)\, \partial^\beta_y F_j(x, y) = 0 \tag{50}$$

on $\hat{B}$ for all $|\alpha| = s$ and for all i. Because the X_j are tangent to $\hat{B}$, it follows from (50) that

$$X^\alpha \left\{ \sum_{|\beta|\leq m-|\alpha|} \sum_{j=1}^{D} A^i_{j\alpha\beta}(x)\, \partial^\beta_y F_j(x, y) \right\} = 0 \tag{51}$$

on $\hat{B}$ for all $|\alpha| = s$ and for all i. From (49) and (51), we conclude that

$$\sum_{|\alpha|\leq s-1} X^\alpha \left\{ \sum_{|\beta|\leq m-|\alpha|} \sum_{j=1}^{D} A^i_{j\alpha\beta}(x)\, \partial^\beta_y F_j(x, y) \right\} = 0 \tag{52}$$

on $\hat{B}$ for all i. Thus, any section of $(I(z))_{z\in\hat{B}}$ satisfies (50) and (52). Conversely, suppose $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^k \times \mathbb{R}^k, \mathbb{R}^D\right)$ satisfies (50) and (52). Then, because (50) implies (51), it follows that (49) holds, and consequently $(F_1, \cdots, F_D)$ is a section of $(I(z))_{z\in\hat{B}}$. Thus, a given $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $(I(z))_{z\in\hat{B}}$ if and only if (50) and (52) hold. If $s \geq 1$, this implies that we have achieved Property $\prod(s-1)$, contradicting the minimal character of s, and establishing Claim 5.1. ■

We return to the proof of Lemma 5.1. Because Property $\prod(s)$ holds with $s = 0$, there exist Nash functions $A^i_{j\beta}$ on $U \setminus U^1_{\text{bad}}$, for which the following ("Property $\prod^*$") holds: Let $B \subset U \setminus U^1_{\text{bad}}$ be a closed ball. Set $\hat{B} = \hat{\psi}(B)$. Then a given $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $(I(z))_{z\in\hat{B}}$ if and only if

$$\sum_{|\beta|\leq m} \sum_{j=1}^{D} A^i_{j\beta}(x)\, \partial^\beta_y F_j(x, y) = 0 \text{ on } \hat{B} \text{ (all } i). \tag{53}$$

We fix $A^i_{j\beta}$ as above. We now return to our bundle $\mathcal{H} = (f^z + I(z))_{z\in\hat{U}}$. (We abuse notation by writing f^z for f^x where $z = \hat{\psi}(x)$.) Let $B \subset U \setminus U^1_{\text{bad}}$ be a closed ball, and let $\hat{B} = \hat{\psi}(B)$. Let $(F_1, \cdots, F_D)$ and $\left(\tilde{F}_1, \cdots, \tilde{F}_D\right) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ be any two sections of $\mathcal{H}|_{\hat{B}}$. Then $\left(F_1 - \tilde{F}_1, \cdots, F_D - \tilde{F}_D\right)$ is a section of $(I(z))_{z\in\hat{B}}$, and therefore by (53), we have

$$\sum_{\substack{|\beta|\leq m\\ j=1,\cdots,D}} A^i_{j\beta}(x)\,\partial^\beta_y F_j(x,y) = \sum_{\substack{|\beta|\leq m\\ j=1,\cdots,D}} A^i_{j\beta}(x)\,\partial^\beta_y \tilde{F}_j(x,y) \text{ on } \hat{B} \text{ for all } i. \tag{54}$$

Moreover, given $x_0 \in B$, we can take our section $\left(\tilde{F}_1, \cdots, \tilde{F}_D\right)$ above to satisfy

$$J_{\hat{\psi}(x_0)}\tilde{F}_j = f^{x_0}_j \;\; (j = 1, \cdots, D),$$

because $(f^{x_0}_1, \cdots, f^{x_0}_D) \in H\left(\hat{\psi}(x_0)\right)$ and $\mathcal{H}|_{\hat{B}}$ is Glaeser stable and has nonempty fibers. (See Theorem 3.) Therefore, (54) implies that

$$\sum_{|\beta|\leq m}\sum_{j=1}^{D} A^i_{j\beta}(x)\,\partial^\beta_y F_j(x,y) = G^i(x) \tag{55}$$

on $\hat{B}$ for each i, where

$$G^i(x) = \sum_{|\beta|\leq m}\sum_{j=1}^{D} A^i_{j\beta}(x)\left(\partial^\beta_y f^x\right)|_{\hat{\psi}(x)} \quad \left(x \in U \setminus U^1_{\text{bad}}\right).$$

Clearly, $G^i(x)$ is a semialgebraic function on $U \setminus U^1_{\text{bad}}$, and it is independent of the ball B in the above discussion. Thus, we have seen that any section $(F_1, \cdots, F_D)$ of $\mathcal{H}|_{\hat{B}}$ must satisfy (55).

Conversely, suppose $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ satisfies (55). Let $\left(\tilde{F}_1, \cdots, \tilde{F}_D\right) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ be a section of $\mathcal{H}|_{\hat{B}}$. (We know that a section exists because $\mathcal{H}|_{\hat{B}}$ is Glaeser stable and has nonempty fibers.) We know that $\left(\tilde{F}_1, \cdots, \tilde{F}_D\right)$ satisfies (55), hence

$$\sum_{|\beta|\leq m}\sum_{j=1}^{D} A^i_{j\beta}(x)\,\partial^\beta_y\left[F_j - \tilde{F}_j\right](x,y) = 0$$

on $\hat{B}$ for each i.

Recalling Property $\prod^*$, we now see that $\left(F_1 - \tilde{F}_1, \cdots, F_D - \tilde{F}_D\right)$ is a section of $(I(z))_{z\in\hat{B}}$. Because $\left(\tilde{F}_1, \cdots, \tilde{F}_D\right) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $\mathcal{H}|_{\hat{B}} = (f^z + I(z))_{z\in\hat{B}}$, we conclude that $(F_1, \cdots, F_D)$ is a section of $\mathcal{H}|_{\hat{B}}$. Thus, if $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$

satisfies (55), then it is a section of $\mathcal{H}|_{\hat{B}}$. We have now seen that a given $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $\mathcal{H}|_{\hat{B}}$ if and only if (55) holds. Thus, all the conclusions of Lemma 5.1 hold, except that perhaps the G^i are not real-analytic. We set U^2_{bad} =union of all the singular sets of the semialgebraic functions G^i. That's a semialgebraic set of dimension $< n$ in $\mathbb{R}^n$. We take $U_{\text{bad}} = U^1_{\text{bad}} \cup U^2_{\text{bad}}$, a semialgebraic set of dimension $< n$ in $\mathbb{R}^n$. The functions $A^i_{j\beta}$ and G^i are Nash on $U \setminus U_{\text{bad}}$.

If $B \subset U \setminus U_{\text{bad}}$ is a closed ball and $\hat{B} = \psi(B)$, then a given $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$ is a section of $\mathcal{H}|_{\hat{B}}$ if and only if

$$\sum_{|\beta| \le m} \sum_{j=1}^{D} A^i_{j\beta}(x) \left(\partial^\beta_y F_j\right)|_{\hat{\psi}(x)} = G^i(x)$$

on B for each i.

This completes the proof of Lemma 5.1. ∎

Remark 5.1 *Lemma 5.1 and its proof hold also for $k = 0$. In that case, $\hat{\psi}$ is the identity map and there are no y-variables, hence no y-derivatives in the conclusion of Lemma 5.1.*

Corollary 5.1 *Let $\mathcal{H}, U, \psi, \cdots$ be as in Lemma 5.1. Let $(F_1, \cdots, F_D) \in C^m_{loc}\left(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^D\right)$. Then $(F_1, \cdots, F_D)$ is a section of $\mathcal{H}|_{\hat{U} \setminus \hat{\psi}(U_{bad})}$ if and only if*

$$\sum_{|\beta| \le m} \sum_{j=1}^{D} A^i_{j\beta}(x)\, \partial^\beta_y F_j(x, y) = G^i(x)$$

on $\hat{U} \setminus \hat{\psi}(U_{bad})$, for all i.

Proof. $U \setminus U_{\text{bad}}$ is a union of (infinitely many overlapping) closed balls B. Applying Lemma 5.1 to each B, we obtain the desired conclusion. ∎

5.2 Gaussian Elimination with Parameters

Suppose we are given a system of linear equations

(56) $X_i + \sum_{j>k} A_{ij} X_j = b_i$, for $i = 1, \cdots, k$ with $|A_{ij}| \le 2^k$ for $i = 1, \cdots k$, $j = k+1, \cdots, M$, and

(57) $\sum_{j>k} C_{ij} X_j = g_i$, for $i = k+1, \cdots, N$,

where $0 \le k \le N, M$; the A_{ij}, C_{ij}, b_i, g_i are semialgebraic functions defined on a semialgebraic set $E \subset \mathbb{R}^n$; and $X_1, \cdots, X_M$ are unknowns. We say that this system is in <u>k-echelon form</u> on E

If $k = 0$, then we have simply (57) for $i = 1, \cdots, N$, so every system of linear equations with coefficient matrix and right-hand sides depending semialgebraically on $x \in E$ is in 0-echelon form on E. If also $C_{ij} \equiv 0$ on E for all $i = k+1, \cdots, N$, $j = k+1, \cdots, M$, then we say that our system of equations is in echelon form on E. In particular, a system in k-echelon form with $k = \min\{N, M\}$ is in echelon form on E. Suppose our system is in k-echelon form with $k < \min\{N, M\}$. We partition E as follows. Let $E_{\text{good}} = \{x \in E : \text{All the } C_{ij}(x) = 0\}$. For $\tilde{i} = k+1, \cdots, N$ and $\tilde{j} = k+1, \cdots, M$, we let $\tilde{E}(\tilde{i}, \tilde{j}) = \{x \in E : |C_{\tilde{i}\tilde{j}}| = \max_{ij} |C_{ij}| > 0\}$. The E_{good} and $\tilde{E}(i, j)$ form a covering of E.

We enumerate the pairs (i, j) in any order and then form sets $E(i, j)$ by removing from $\tilde{E}(i, j)$ all points contained in some $\tilde{E}(i', j')$ with (i', j') preceding (i, j). Then E_{good} and the $E(i, j)$ form a partition of E into semialgebraic sets. On E_{good}, our system is in echelon form. On each $E(a, b)$, we will exhibit a system of linear equations in $(k+1)$-echelon form, equivalent to the given system (56), (57). For fixed (a, b), we relabel equations and unknowns so that our system still has the form (56), (57), but with $|C_{k+1,k+1}| = \max_{ij} |C_{ij}| > 0$. Dividing equations (57) by $C_{k+1,k+1}$, we may assume that

$$C_{k+1,k+1} = 1 \tag{58}$$

and all

$$|C_{ij}| \leq 1. \tag{59}$$

Note that A_{ij}, C_{ij}, b_i, g_i still depend semialgebraically on x. From each equation (56), we subtract $A_{i(k+1)}$ times equation (57) with $i = k+1$. From each equation (57) $(i \neq k+1)$, we subtract $C_{i,k+1}$ times equation (57) with $i = k+1$. Thus, we obtain equations of the form

$$\left[\begin{array}{l} X_i + \sum_{j>k} \tilde{A}_{ij} X_j = \tilde{b}_i, \text{for } i = 1, \cdots, k \\ X_{k+1} + \sum_{j>k+1} C_{k+1,j} X_j = g_{k+1}, \\ \sum_{j \geq k+1} \tilde{C}_{ij} X_j = \tilde{g}_i, \text{ for } i > k+1. \end{array}\right. \tag{60}$$

Here, $\tilde{A}_{ij} = A_{ij} - A_{i(k+1)} C_{k+1,j}$ for $i = 1, \cdots, k$, $j \geq k+1$; and $\tilde{C}_{ij} = C_{ij} - C_{i,k+1} C_{k+1,j}$ for $i = k+2, \cdots, N$, $j > k+1$. In particular, $\tilde{A}_{i,k+1} = A_{i,k+1} - A_{i,k+1} \cdot C_{k+1,k+1} = 0$, and $\tilde{C}_{i,k+1} = C_{i,k+1} - C_{i,k+1} \cdot C_{k+1,k+1} = 0$, thanks to (58). Also, $\left|\tilde{A}_{ij}\right| \leq |A_{ij}| + |A_{i,k+1}| \cdot |C_{k+1,j}| \leq |A_{ij}| + |A_{i,k+1}|$ (by (59))$\leq 2^k + 2^k$ (because our system (56), (57) is in k-echelon form)$= 2^{k+1}$. Recall that $|C_{k+1,j}| \leq 1$.

These remarks show that the system of equations (60) is in $(k+1)$-echelon form.

We repeat this procedure, starting with a system in 0-echelon form, and partition E more and more finely into pieces E_ν, on each of which an equivalent system to (56), (57) is either in echelon form, or in k-echelon form for ever higher k. The procedure has to stop after at most $\min(N, M)$ steps, because a system in k-echelon form with $k = \min(N, M)$ is automatically in echelon form.

Thus, we have proven the following result

Lemma 5.2 *Consider a system of linear equations*

$$\sum_{j=1}^{M} C_{ij}(x) X_j = g_i(x) \quad (i = 1, \cdots, N) \tag{61}$$

where the $C_{ij}(x)$ *and* $g_i(x)$ *are semialgebraic functions defined on a semialgebraic set* $E \subset \mathbb{R}^n$*.*

Then we can partition E *into semialgebraic sets* E_ν $(\nu = 1, \cdots, \nu_{\max})$*, for which the following holds for each* ν*: There exist a permutation* $\pi : \{1, \cdots, M\} \to \{1, \cdots, M\}$ *and an integer* $0 \le k \le \min(N, M)$ *such that for each* $x \in E_\nu$*, the system* (61) *is equivalent to a system of the form*

$$\left[\begin{array}{l} X_{\pi i} + \sum_{j>k} \tilde{A}_{ij}(x) X_{\pi j} = \tilde{g}_i(x) \text{ for } i = 1, \cdots, k \\ \qquad 0 = \tilde{b}_i(x) \text{ for } i = k+1, \cdots, N. \end{array}\right. \tag{62}$$

That is, for each $x \in E_\nu$ *and each* $(X_1, \cdots, X_M) \in \mathbb{C}^M$*,* (61) *holds at* x *if and only if* (62) *holds at* x*. Here, the* $\tilde{A}_{ij}, \tilde{g}_i$*, and* $\tilde{b}_i$ *are semialgebraic functions on* E_ν*, and* $\left|\tilde{A}_{ij}(x)\right| \le 2^k$ *on* E_ν*.*

In essence, the method for solving the system (61) is just the usual Gaussian elimination, except that we take extra care to maintain the growth condition $\left|\tilde{A}_{ij}(x)\right| \le 2^k$.

5.3 What It Means to be a Section of a Semialgebraic Bundle

We work with a semialgebraic bundle $\mathcal{H} = (H(x))_{x \in \mathbb{R}^2}$. Each $H(x)$ is a coset of an $\mathcal{R}_x$-submodule of $(\mathcal{R}_x)^D$, depending semialgebraically on x. Here, $\mathcal{R}_x$ is the ring of the m-jets of functions at x. A function $F = (F_1, \cdots, F_D) \in C^m_{loc}(\Omega, \mathbb{R}^D)$ ($\Omega \subset \mathbb{R}^2$ open) is a section of $\mathcal{H}$ if for all $x \in \Omega$ the m-jet $J_x F$ belongs to $H(x)$. A function $F \in C^m_{loc}(\Omega, \mathbb{R}^D)$ is called a local section near x^0 ($x^0 \in \Omega$) if for some small disc $B \subset \Omega$ centered at x^0 we have $J_x F \in H(x)$ for all $x \in B$.

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x\}$. Let $\mathcal{H} = (H(x))_{x \in \mathbb{R}^2}$ be a semialgebraic bundle, with $H((0,0)) = \{0\}$. We assume that $\mathcal{H}$ has a section. We want a convenient condition on functions $F \in C^m_{loc}(\Omega, \mathbb{R}^D)$ that is equivalent to the assertion that $F|_{B \cap \Omega^{\text{interior}}}$ is a section of $\mathcal{H}$ for a small enough disc B centered at the origin. We achieve (approximately) that.

To do so, we partition Ω into semialgebraic open subsets of $\mathbb{R}^2$, finitely many semialgebraic curves in $\mathbb{R}^2$, and finitely many points. To start with, we partition Ω into the point $(0,0)$, the arcs $\{(x, 0) : x > 0\}, \{(x, x) : x > 0\}$, and Ω^{interior}. As we proceed, we will cut up each of our semialgebraic open sets into finitely many semialgebraic open subsets, finitely many semialgebraic arcs, and finitely many points. We won't keep track explicitly of the arcs and points at first; we just discard semialgebraic subsets of $\mathbb{R}^2$ of dimension ≤ 1. We apply Lemma 5.1 in the case $k = 0$ to Ω^{interior} and $\mathcal{H}$. (See Remark 5.1.) Thus, we obtain a semialgebraic $V_1 \subset \Omega^{\text{interior}}$ of dimension ≤ 1, outside of which the following holds for some semialgebraic functions $A^{\#}_{ij}(x), \phi^{\#}_i(x)$ for $1 \le i \le i_{\max}, 1 \le j \le D, x \in \Omega^{\text{interior}} \setminus V_1$: Let $F = (F_1, \cdots, F_D)$ belong to $C^m_{loc}(U, \mathbb{R}^D)$ where U is a neighborhood of $x^0 \in \Omega^{\text{interior}} \setminus V_1$. Then F is a local section of $\mathcal{H}$ near x^0 if and only if

(63) $\sum_{j=1}^D A^{\#}_{ij}(x) F_j(x) = \phi^{\#}_i(x)$, for $i = 1, \cdots, i_{\max}$, for all x in a neighborhood of x^0.

The equations (63) have a solution for each fixed x, because $\mathcal{H}$ has a section. Next, we apply Lemma 5.2 to the above system of linear equations. Thus, we obtain a partition of $\Omega^{\text{interior}} \setminus V_1$ into semialgebraic sets $E_\nu^\#$ ($\nu = 1, \cdots, \nu_{\max}^\#$), for which we have integers $\tilde{k}_\nu \geq 0$, permutations $\tilde{\pi}_\nu : \{1, \cdots, D\} \to \{1, \cdots, D\}$, and semialgebraic functions $\tilde{A}_{ij}^\nu(x)$ ($1 \leq i \leq \tilde{k}_\nu, \tilde{k}_\nu + 1 \leq j \leq D, x \in E_\nu^\#$), $\tilde{\phi}_i^\nu(x)$ such that for any $x \in E_\nu^\#$, the system of equations (63) is equivalent to

$$F_{\pi_\nu i}(x) + \sum_{j > \tilde{k}_\nu} \tilde{A}_{ij}^\nu(x) F_{\pi_\nu j}(x) = \tilde{\varphi}_i^\nu(x) \text{ for } i = 1, \cdots, \tilde{k}_\nu. \tag{64}$$

Moreover, the $\tilde{A}_{ij}^\nu(x)$ are bounded. Note that the functions $\tilde{b}_i$ in (62) are identically 0 because our equations (63) have a solution. Because $\mathcal{H}$ has a section, there exists $F = (F_1, \cdots, F_D) \in C_{loc}^m(\Omega, \mathbb{R}^D)$ satisfying (63) for all $x \in \Omega^{\text{interior}} \setminus V_1$, hence also satisfying (64) in $E_\nu^\#$. Consequently, the left-hand side of (64) is bounded (for bounded x), and thus also the $\tilde{\varphi}_i^D(x)$ are bounded (for bounded x). Applying Theorem 6, we obtain a semialgebraic $V_2 \subset \mathbb{R}^2$ of dimension ≤ 1, satisfying

$$|\partial^\alpha \tilde{\varphi}_i^\nu(x)|, \left|\partial^\alpha \tilde{A}_{ij}^\nu(x)\right| \leq C\left[\text{dist}(x, V_2)\right]^{-|\alpha|} \text{ for bounded } x \text{ outside } V_2\text{, for } |\alpha| \leq m. \tag{65}$$

By adding $\partial\Omega$ to V_2 and removing from V_2 all points outside Ω, we may assume $V_2 \subset \Omega$. (This operation does not increase the distance from V_2 to any point of Ω.) Let $\hat{E}_\nu$ ($\nu = 1, \cdots, \nu_{\max}$) be the connected components of the interiors of the sets $E_\nu^\# \setminus V_2$ ($\nu = 1, \cdots, \nu_{\max}^\#$). Then Ω is partitioned into the $\hat{E}_\nu$ and V_3, where V_3 is a semialgebraic subset of Ω of dimension ≤ 1. The $\hat{E}_\nu$ are pairwise disjoint open connected semialgebraic sets. Any path in Ω that does not meet V_3 stays entirely in a single $\hat{E}_\nu$. Indeed, suppose not: let $\gamma(t) \in \Omega$ ($t \in [0,1]$) be a path starting at $\gamma(0) \in \hat{E}_\nu$ not staying in $\hat{E}_\nu$ and not meeting V_3. Pick $t_* = \inf\left\{t > 0 : \gamma(t) \notin \hat{E}_\nu\right\}$. Then $t^* > 0$ since $\hat{E}_\nu$ is open. We can't have $\gamma(t_*) \in \hat{E}_{\nu'}$ with $\nu' \neq \nu$ else $\gamma(t) \in \hat{E}_{\nu'}$ (and $\in \hat{E}_\nu$) for $t \in [t_* - \varepsilon, t_*)$. We can't have $\gamma(t_*)$ in E_ν, since that would imply $\gamma(t)$ in E_ν for all t in $[t_*, t_* + \varepsilon]$. Thus, $\gamma(t_*) \in V_3$, contradicting the fact that γ does not meet V_3.

Moreover, there exist integers $\hat{k}_\nu \geq 0$, permutations $\hat{\pi}_\nu : \{1, \cdots, D\} \to \{1, \cdots, D\}$, and semialgebraic functions $\hat{A}_{ij}^\nu(x)$ $\left(1 \leq i \leq \hat{k}_\nu,\ \hat{k}_\nu + 1 \leq j \leq D\right)$ and $\hat{\varphi}_i^\nu(x)$ $\left(1 \leq i \leq \hat{k}_\nu\right)$ defined on $\hat{E}_\nu$, with the following properties

(66) $\left|\partial^\alpha \hat{A}_{ij}^\nu(x)\right|, |\partial^\alpha \hat{\varphi}_i^\nu(x)| \leq C\left[\text{dist}(x, V_3)\right]^{-|\alpha|}$ for bounded $x \in \hat{E}_\nu$, $|\alpha| \leq m$, and

(67) Let $x^0 \in \hat{E}_\nu$ and let $F = (F_1, \cdots, F_D)$ be C_{loc}^m in a neighborhood of x^0. Then F is a local section of $\mathcal{H}$ near x^0 if and only if

$$F_{\pi_\nu i}(x) + \sum_{j > \hat{k}_\nu} \hat{A}_{ij}^\nu(x) F_{\pi_\nu j}(x) = \hat{\varphi}_i^\nu(x)$$

in a neighborhood of x^0 for each $i = 1, \cdots, \hat{k}_\nu$.

We partition $V_3 \cup \{(x,0) : x \geq 0\} \cup \{(x,x) : x \geq 0\}$ into finitely many Nash open arcs (not containing their endpoints) and finitely many points. For small enough $\delta > 0$, $B(0,\delta) \subset \mathbb{R}^2$ avoids all the above arcs not containing 0 in their closure, and all the above points except possibly for the point 0. Taking δ small, we may assume that the remaining arcs have convergent Puiseux series in $B(0,\delta)$.

Notice that our semialgebraic one-dimensional sets are all contained in Ω; so no arcs have tangent lines at 0 lying outside the sector Ω. Thus, the remaining arcs have the form $\{y = \psi_s(x)\}$ in $B(0,\delta)$, where $\psi_1, \cdots, \psi_{s_{\max}}$ are semialgebraic functions of one variable, with convergent Puiseux expansion in $[0,\delta]$. We discard duplicates, i.e., we may assume ψ_s is never identically equal to $\psi_{s'}$ for $s' \neq s$. Note that the line segments $\{(x,0) : 0 < x < \delta\}$ and $\{(x,x) : 0 < x < \delta\}$ are among our arcs γ_s. Taking $\delta > 0$ smaller yet, we may assume that for each $s \neq s'$, either $\psi_s(x) < \psi_{s'}(x)$ for all $x \in (0,\delta)$, or $\psi_s(x) > \psi_{s'}(x)$ for all $x \in (0,\delta)$. (That's because the ψ_s are given by convergent Puiseux expansions.) Thus, in $B(0,\delta)$, our curves may be labelled so that $0 \equiv \psi_0(x) < \psi_1(x) < \cdots < \psi_{s_{\max}}(x) \equiv x$ for $x \in (0,\delta)$. The arcs are $\gamma_s = \{(x,\psi_s(x)) : x \in [0,\delta]\}$ for $s = 0, \cdots, s_{\max}$. (Here we have thrown in the point 0, and taken δ small to allow ourselves to include $x = \delta$, not just $x < \delta$.) The sets we discarded in passing from V_3 to the semialgebraic arcs $\gamma_0, \cdots, \gamma_{s_{\max}}$ are irrelevant in the sense that $V_3 \cap B(0,\delta) \subset (\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_{s_{\max}}) \cap B(0,\delta)$.

Let E_s $(s = 1, \cdots, s_{\max})$ be the part of the $B(0,\delta)$ lying between γ_{s-1} and γ_s, i.e., $E_s = \{(x,y) \in B(0,\delta) : 0 < x < \delta, \psi_{s-1}(x) < y < \psi_s(x)\}$. Any two points in a given E_s may be joined by a path in $B(0,\delta) \setminus \bigcup_{s=0}^{s_{\max}} \gamma_s \subset B(0,\delta) \setminus V_3$, hence all points in a given E_s lie in the same $\hat{E}_\nu$. Therefore, for $s = 1, \cdots, s_{\max}$, there exist $k_s \geq 0$, permutations $\pi_s : \{1, \cdots, D\} \to \{1, \cdots, D\}$, and semialgebraic functions $A^s_{ij}(x)$, $\psi^s_i(x)$ $(1 \leq i \leq k_s; j = k_s + 1, \cdots, D)$ on E_s, with the following properties

(68) Let $x^0 \in E_s$, and let $F = (F_1, \cdots, F_D)$ be C^m_{loc} in a neighborhood of x^0. Then F is a local section of $\mathcal{H}$ near x^0 if and only if

(69) $F_{\pi_s i}(x) + \sum_{j > k_s} A^s_{ij}(x) F_{\pi_s j}(x) = \psi^s_i(x)$ in a neighborhood of x^0 for each $i = 1, \cdots, k_s$.

Moreover,

(70) $|\partial^\alpha A^s_{ij}(x)|$, $|\partial^\alpha \psi^s_i(x)| \leq C\,[\mathrm{dist}(x, \gamma_s \cup \gamma_{s-1})]^{-|\alpha|}$ on E_s for $|\alpha| \leq m$.

In particular, if $F = (F_1, \cdots, F_D) \in C^m_{loc}(\Omega, \mathbb{R}^D)$, then $J_x F \in H(x)$ for all $x \in [\Omega \cap B(0,\delta)] \setminus (\gamma_0 \cup \cdots \cup \gamma_{s_{\max}})$ if and only if for each $s = 1, \cdots, s_{\max}$, (69) holds on all of E_s.

Next, we apply Lemma 5.1 to $\mathcal{H}_s = (H(x))_{x \in \gamma_s}$, $(s = 0, \cdots, s_{\max})$. We obtain semi-algebraic functions for which the following holds. Let $(x^0, \psi_s(x^0)) \in \gamma_s$ be given, and let $F = (F_1, \cdots, F_D) \in C^m_{loc}\left(U, \mathbb{R}^D\right)$, where U is a neighborhood of γ_s in $\mathbb{R}^2$. Then, except for finitely many bad x^0, we have the following equivalence: F is a local section of $\mathcal{H}_s$ near $(x^0, \psi_s(x^0))$ if and only if

$$\sum_{\substack{1 \leq j \leq D \\ 0 \leq l \leq m}} \Theta^{is}_{jl}(x)\, \partial^l_y F_j|_{(x,\psi_s(x))} = g^{si}(x) \quad (i = 1, \cdots, i_{\max}(s))$$

for all x in a neighborhood of x^0. Here, the Θ's and g's are semialgebraic functions of one variable. To say that F is a local section of $\mathcal{H}_s$ near $(x^0, \psi_s(x^0))$ means that $J_{(x,\psi_s(x))}F \in H(x, \psi_s(x))$ for all x in a neighborhood of x^0.

By restricting attention to $B(0,\delta)$ and taking $\delta > 0$ smaller, we may exclude from $B(0,\delta)$ all these bad x^0, except for $x^0 = 0$. Combining our results (68), (70) on the E_ν with the above result on the arcs γ_s, we obtain the following result.

Lemma 5.3 *Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$ and let $\mathcal{H} = (H(x))_{x\in\Omega}$ be a semialgebraic bundle, with each $H(x)$ consisting of m-jets at x of functions from $\mathbb{R}^2$ to $\mathbb{R}^D$.*

Assume $H((0,0)) = \{0\}$ and assume $\mathcal{H}$ has a section.

Then there exist the following objects, with properties to be specified below:

- *A positive number $\delta \in (0,1)$.*
- *Semialgebraic functions $0 = \psi_0(x) < \psi_1(x) < \cdots < \psi_{s_{\max}}(x) = x$ on $(0,\delta)$, all given by convergent Puiseux expansions on $(0,\delta)$.*
- *Integers k_s $(0 \le k_s \le D)$ and permutations $\pi_s : \{1,\cdots,D\} \to \{1,\cdots,D\}$ for $s = 1,\cdots,D$.*
- *Semialgebraic functions $A^s_{ij}(x,y)$ $(s = 1,\cdots,s_{\max}, 1 \le i \le k_s, k_s < j \le D)$ and $\varphi^s_i(x,y)$ $(s = 1,\cdots,s_{\max}, 1 \le i \le k_s)$ defined on $E_s = \{(x,y) : 0 < x < \delta, \psi_{s-1}(x) < y < \psi_s(x)\}$.*
- *Semialgebraic functions $\Theta^{si}_{jl}(x)$, $g^{si}(x)$ $(s = 0,\cdots,s_{\max}, i = 1,\cdots,i_{\max}(s), j = 1,\cdots,D,$ $l = 0,\cdots,m$ defined on $(0,\delta)$, and given there by there by convergent Puiseux expansions.*

The above objects have the following properties

- *(Estimates) For $(x,y) \in \Omega$ with $0 < x < \delta$ and $\psi_{s-1}(x) < y < \psi_s(x)$, we have $\left|\partial^\alpha A^s_{ij}(x,y)\right|, \left|\partial^\alpha \varphi^s_i(x,y)\right| \le C\left[\min\left(|y-\psi_s(x)|, |y-\psi_{s-1}(x)|\right)\right]^{-|\alpha|}$ for $|\alpha| \le m$.*
- *(Condition for sections) Let $F = (F_1,...,F_D) \in C^m_{loc}(\Omega,\mathbb{R}^D)$, and suppose $J_xF \in H(x)$ for all $x \in \Omega$.*

  (71) Then for $s = 1,\cdots,s_{\max}$, $i = 1,\cdots,k_s$, $x \in (0,\delta)$, $\psi_{s-1}(x) < y < \psi_s(x)$, we have

  $$F_{\pi_s i}(x,y) + \sum_{D\ge j>k_s} A^s_{ij}(x,y)\, F_{\pi_s j}(x,y) = \varphi^s_i(x,y)\,;$$

  and for $s = 0,1,\cdots,s_{\max}$, $i = 1,\cdots,i_{\max}(s)$, $x \in (0,\delta)$, we have

  $$\sum_{j=1}^{D}\sum_{l=0}^{m} \Theta^{si}_{jl}(x)\, \partial^l_y F_j(x,\psi_s(x)) = g^{si}(x)\,;$$

  and $J_{(0,0)}F_j = 0$ for all j.

  Conversely, if $F = (F_1,...,F_D) \in C^m_{loc}(\Omega,\mathbb{R}^D)$ and the conditions in (71) are satisfied, then $J_zF \in H(z)$ for all $z = (x,y) \in \Omega$ with $0 \le x < \delta$.

6 A Second Main Lemma

This section is devoted to the proof of the following lemma. See (A) and (B) in the Introduction.

Lemma 6.1 (Second Main Lemma) *Let $\mathcal{H} = (H(z))_{z\in\Omega}$ with $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$ and suppose $H(z)$ depends semialgebraically on z. (As usual, $H(z) \subset \mathcal{R}_z^D$ is a coset of an $\mathcal{R}_z$-submodule.) Suppose $\mathcal{H}$ has a section, and suppose $\mathcal{H}((0,0)) = \{0\}$. Then there exist semialgebraic functions $\theta_{jl}^{si}(x)$, $g^{si}(x)$, $\tilde{\theta}_{jl}^{si}(x)$, $\tilde{g}^{si}(x)$ of one variable, and $0 = \psi_0(x) < \cdots < \psi_{s_{\max}}(x) = x$, also semialgebraic, for which the following hold. Suppose $F = (F_1, \cdots, F_D) \in C^m(\Omega, \mathbb{R}^D)$ is a section of $\mathcal{H}$. Let $f_{jl}^s(x) = \partial_y^l F_j(x, \psi_s(x))$ for $0 \le s \le s_{\max}$, $0 \le l \le m$, $1 \le j \le D$.*

Then

(72) $\sum_{j,l} \theta_{jl}^{si}(x) f_{jl}^s(x) = g^{si}(x)$ on $(0,\delta)$ for some $\delta > 0$ for each s, i; and $\sum_{j,l} \tilde{\theta}_{jl}^{si}(x) f_{jl}^s(x) = \tilde{g}^{si}(x) + o(1)$ as $x \to 0^+$, each s, i; and $f_{jl}^s(x) = \sum_{k=0}^{m-l} \frac{1}{k!} f_{j(l+k)}^{s-1}(x) \cdot (\psi_s(x) - \psi_{s-1}(x))^k + o\left([\psi_s(x) - \psi_{s-1}(x)]^{m-l}\right)$ as $x \to 0^+$, each s, j, l.

(73) Conversely, if $f_{jl}^s(x)$ are semialgebraic functions satisfying (72), then there exists a semialgebraic C^m section $F = (F_1, \cdots, F_D)$ of $\mathcal{H}$ over $\Omega_{\delta'} = \{(x,y) : 0 \le y \le x \le \delta'\}$ (some $\delta' > 0$) such that $\partial_y^l F_j(x, \psi_s(x)) = f_{jl}^s(x)$ for $0 < x < \delta'$.

We call the curves $y = \psi_s(x)$ "critical curves".

6.1 The Jet of a Section at a Critical Curve

Fix $m \ge 1$. Recall that $\mathcal{P}$ denotes the space of polynomials of degree $\le m$ on $\mathbb{R}^2$, and $J_z F \in \mathcal{P}$ denotes the m-jet of F at $z \in \mathbb{R}^2$. $\odot_z$ denotes multiplication of jets at z. We write $\mathfrak{p}$ to denote the space of polynomials of degree $\le m$ on $\mathbb{R}$. If $F(x,y)$ is a C^m_{loc} function in a neighborhood of $(\bar{x}, 0)$, then $j_{\bar{x}} F \in \mathfrak{p}$ is the m-jet at 0 of the function $y \mapsto F(\bar{x}, y)$. We write $\boxdot$ to denote multiplication of m-jets at 0 of C^m_{loc} functions of one variable. If $\vec{F} = (F_1, \cdots, F_{j_{\max}})$ is a vector of C^m_{loc} functions on $\mathbb{R}^2$, then $J_z \vec{F}$ denotes

$$(J_z F_1, \cdots, J_z F_{j_{\max}}) \in \mathcal{P}^{j_{\max}}.$$

Similarly, $j_{\bar{x}} \vec{F}$ denotes $(j_{\bar{x}} F_1, \cdots, j_{\bar{x}} F_{j_{\max}}) \in \mathfrak{p}^{j_{\max}}$. A function $F^\# : (0,\delta) \to \mathfrak{p}$ may be regarded as a function of $(x,y) \in (0,\delta) \times \mathbb{R}$ such that for fixed x, the function $y \mapsto F^\#(x,y)$ is a polynomial of degree at most m.

Fix positive integers $i_{\max}, j_{\max}$. Let Aff denote the vector space of all affine functions defined on $\mathfrak{p}^{j_{\max}+i_{\max}}$. We make the following assumptions:

- We are given C^∞ semialgebraic functions $A_{ij}, B_i, (i = 1, \cdots, i_{\max}, j = 1, \cdots, j_{\max})$ defined on Ω_1, where for $\delta > 0$, $\Omega_\delta = \{(x,y) \in \mathbb{R}^2 : 0 < x < \delta, 0 < y < \psi(x)\}$, and $\psi : (0,1) \to (0,\infty)$ is a semialgebraic function satisfying $0 < \psi(x) \le x$ for $x \in (0,1)$.

- We assume that $\partial^\alpha A_{ij}, \partial^\alpha B_i$ extend to continuous functions on Ω_1^+ for $|\alpha| \leq m$, where, for $\delta > 0$, $\Omega_\delta^+ = \{(x,y) \in \mathbb{R}^2 : 0 < x \leq \delta, 0 < y \leq \psi(x)\}$.

- We suppose that

$$\begin{aligned} |\partial^\alpha A_{ij}(x,y)| &\leq Cy^{-|\alpha|}, \text{ and} \\ |\partial^\alpha B_i(x,y)| &\leq Cy^{-|\alpha|} \end{aligned}$$

  on Ω_1^+ for $|\alpha| \leq m$.

Lemma 6.2 *Under the above assumptions, there exist $\delta \in (0,1)$ and semialgebraic maps $\lambda_1, \cdots, \lambda_{k_{\max}}, \mu_1, \cdots, \mu_{l_{\max}} : (0,\delta) \to Aff$ such that the following hold:*

(74) *Suppose $\vec{F} = (F_1, \cdots, F_{j_{\max}})$ and $\vec{G} = (G_1, \cdots, G_{i_{\max}})$ belong to $C^m(\Omega_\delta^{closure}, \mathbb{R}^{j_{\max}})$ and $C^m(\Omega_\delta^{closure}, \mathbb{R}^{i_{\max}})$ respectively, with $J_{(0,0)}\vec{F} = 0, J_{(0,0)}\vec{G} = 0$. Suppose also that $G_i = \sum_j A_{ij}F_j + B_i$ for each i. Then $[\lambda_k(\bar{x})](j_{\bar{x}}\vec{F}, j_{\bar{x}}\vec{G}) = 0$ for $k = 1, \cdots, k_{\max}, \bar{x} \in (0,\delta)$, and $[\mu_l(\bar{x})](j_{\bar{x}}\vec{F}, j_{\bar{x}}\vec{G})$ is bounded on $(0,\delta)$ and tends to zero as $\bar{x} \to 0$, for each $l = 1, \cdots, l_{\max}$. We do not assume $\vec{F}$ or $\vec{G}$ is semialgebraic.*

(75) *Suppose there exists an $(\vec{F}, \vec{G})$ as in (74). Let $\vec{F}^\# = (F_1^\#, \cdots, F_{j_{\max}}^\#)$, $\vec{G}^\# = (G_1^\#, \cdots, G_{i_{\max}}^\#)$, where the $F_j^\#$ and $G_i^\#$ are semialgebraic maps from $(0,\delta) \to \mathfrak{p}$. Suppose that*

$$[\lambda_k(\bar{x})](\vec{F}^\#(\bar{x}), \vec{G}^\#(\bar{x})) = 0,$$

for $k = 1, \cdots, k_{\max}, \bar{x} \in (0,\delta)$; and that $[\mu_l(\bar{x})](\vec{F}^\#(\bar{x}), \vec{G}^\#(\bar{x}))$ is bounded on $(0,\delta)$ and tends to zero as $\bar{x} \to 0$. Then there exist $\delta' > 0$ and $\vec{F} = (F_1, \cdots, F_{j_{\max}})$, $\vec{G} = (G_1, \cdots, G_{i_{\max}})$ semialgebraic and in $C^m(\Omega_{\delta'}^{closure}, \mathbb{R}^{j_{\max}})$ and $C^m(\Omega_{\delta'}^{closure}, \mathbb{R}^{i_{\max}})$ respectively, with $J_{(0,0)}\vec{F} = 0, J_{(0,0)}\vec{G} = 0$, $G_i = \sum_j A_{ij}F_j + B_i$ and $j_{\bar{x}}\vec{F} = \vec{F}^\#(\bar{x}), j_{\bar{x}}\vec{G} = \vec{G}^\#(\bar{x})$, for all $\bar{x} \in (0,\delta')$. (Note that here we have passed from δ to a smaller δ'.)

The remainder of this section is devoted to a proof of Lemma 6.2. Let $\delta > 0$ be small enough to be picked below,

Definition 6.1 *We define a bundle $\mathcal{H}$ over $[0,1] \times \{0\} \subset \mathbb{R}^2$. Here, $\mathcal{H} = (H(\bar{x},0))_{\bar{x} \in [0,1]}$, with $H(\bar{x},0) \subset \mathcal{P}^{j_{\max}+i_{\max}}$ defined as follows.*

- *$H(0,0) = \{0\}$.*

- *If $\bar{x} \in (0,1]$, then $(\vec{P}, \vec{Q}) = (P_1, \cdots, P_{j_{\max}}, Q_1, \cdots, Q_{i_{\max}}) \in H(\bar{x},0)$ if and only if*

$$y^{|\alpha|-m}\partial^\alpha \left\{ \sum_j A_{ij}P_j + B_i - Q_i \right\} (\bar{x},y) \to 0$$

  as $y \to 0^+$, for each $|\alpha| \leq m$ and each i.

We will show that $\mathcal{H}$ is a bundle, i.e., $H(z)$ is a translate of an $\mathcal{R}_z$-submodule of $\mathcal{R}_z^{j_{\max}+i_{\max}}$ for each $z \in [0,\delta]\times\{0\}$; and we will show that $J_{(\bar{x},0)}(\vec{F},\vec{G}) \in H(\bar{x},0)$ (each $\bar{x}\in[0,\delta]$) if $\vec{F},\vec{G}$ are as in (74). Suppose $J_{(0,0)}(\vec{F},\vec{G})=0$, $\vec{F},\vec{G}$ are C^m on $\Omega_\delta^{\text{closure}}$, $G_i=\sum_j A_{ij}F_j+B_i$ on Ω_δ. Let $\bar{x}\in(0,\delta]$. Then

$$\partial^\alpha[A_{ij}(F_j - J_{(\bar{x},0)}F_j)](\bar{x},y) = o(y^{m-|\alpha|})$$

and

$$\partial^\alpha[G_i - J_{(\bar{x},0)}G_i](\bar{x},y) = o(y^{m-|\alpha|})$$

on Ω_δ for $|\alpha|\le m$, by Taylor's theorem and our estimates for $\partial^\alpha A_{ij}$. The above remarks imply that $\partial^\alpha\{\sum_j A_{ij}J_{(\bar{x},0)}F_j + B_i - J_{(\bar{x},0)}G_i\}(\bar{x},0) = o(y^{m-|\alpha|})$. Therefore, $J_{(\bar{x},0)}(\vec{F},\vec{G})\in H(\bar{x},0)$ for $\bar{x}\in(0,\delta]$. For $\bar{x}=0$, we just note that $J_{(0,0)}(\vec{F},\vec{G})=0\in H(0,0)$. That proves our assertion about $J_{(\bar{x},0)}(\vec{F},\vec{G})$.

Note that for $\bar{x}\neq 0$, $H(\bar{x},0)$ is a translate in $\mathcal{P}$ of

$$I(\bar{x}) = \left\{ \left(\vec{P},\vec{Q}\right) : \partial^\alpha\left(\sum_j A_{ij}P_i - Q_i\right)(\bar{x},y) = o\left(y^{m-|\alpha|}\right),\ \text{as } y\to 0^+,\ |\alpha|\le m \right\}.$$

Let $\left(\vec{P},\vec{Q}\right)\in I(\bar{x})$ and let $S\in\mathcal{P}$. Then for $|\alpha|\le m$, we have

$$\partial^\alpha\left(S\cdot\left[\sum_j A_{ij}P_j - Q_i\right]\right)(\bar{x},y) = o\left(y^{m-|\alpha|}\right),$$

hence

$$\partial^\alpha\left(\sum_j A_{ij}(SP_j) - (SQ_i)\right)(\bar{x},y) = o\left(y^{m-|\alpha|}\right),\ \text{as } y\to 0^+. \tag{76}$$

Also, our estimates on $\partial^\alpha A_{ij}$, together with Taylor's theorem, give

$$\partial^\alpha\left(A_{ij}\left(SP_i - J_{(\bar{x},0)}(SP_j)\right)\right)(\bar{x},0) = o\left(y^{m-|\alpha|}\right)$$

and

$$\partial^\alpha\left(SQ_i - J_{(\bar{x},0)}(SQ_i)\right)(\bar{x},0) = o\left(y^{m-|\alpha|}\right)\ \text{ as } y\to 0^+ \text{ for } |\alpha|\le m.$$

That is,

$$\partial^\alpha\left(A_{ij}\left(SP_j - S\odot_{(\bar{x},0)}P_j\right)\right)(\bar{x},y) = o\left(y^{m-|\alpha|}\right) \tag{77}$$

and

$$\partial^\alpha\left(SQ_i - S\odot_{(\bar{x},0)}Q_i\right)(\bar{x},0) = o\left(y^{m-|\alpha|}\right)\ \text{ as } y\to 0^+ \text{ for } |\alpha|\le m. \tag{78}$$

It now follows from (76), (77), and (78) that

$$\partial^\alpha\left(\sum_j A_{ij}\left[S\odot_{(\bar{x},0)}P_j\right] - \left[S\odot_{(\bar{x},0)}Q_i\right]\right)(\bar{x},y) = o\left(y^{m-|\alpha|}\right)$$

as $y \to 0^+$, for each $|\alpha| \leq m$. This completes the proof that the $I(\bar{x})$ is a submodule, when $\bar{x} \neq 0$.

For $\bar{x} = 0$, we just note that $\{0\}$ is an $\mathcal{R}_{(0,0)}$-submodule of $\mathcal{R}_{(0,0)}^{j_{\max}+i_{\max}}$. We have now shown that

- $\mathcal{H} = (H(\bar{x},0))_{\bar{x}\in[0,\delta]}$ is a bundle.
- If $(\vec{F},\vec{G})$ is as in (I) of Lemma 6.2, then $(\vec{F},\vec{G})$ is a section of $\mathcal{H}$.
- $H(\bar{x},0) \subset \mathcal{P}^{j_{\max}+i_{\max}}$ depends semialgebraically on $\bar{x}$, since A_{ij} and B_i are semialgebraic.

Lemma 6.3 *Let $\mathcal{H} = (H(\bar{x},0))_{(\bar{x},0)\in[0,\delta]\times\{0\}}$ be a semialgebraic bundle, $\mathcal{H} = H(\bar{x},0) \subset \mathcal{P}^{j_{\max}+i_{\max}}$. Then there exist semialgebraic functions $\lambda_1,\cdots,\lambda_{k_{\max}} : (0,\delta) \to$ Aff, and a finite set of bad points $\{\bar{\bar{x}}_1^{bad},\cdots,\bar{\bar{x}}_S^{bad}\}$ such that the following holds for any $\bar{\bar{x}} \in (0,\delta)$ other than the bad points. Let $\left(\vec{F},\vec{G}\right) = (F_1,\cdots,F_{j_{\max}},G_1,\cdots,G_{i_{\max}})$ be C^m in a neighborhood of $(\bar{\bar{x}},0)$ in $\mathbb{R}^2$. Then*

$$J_{(\bar{x},0)}\left(\vec{F},\vec{G}\right) \in H(\bar{x},0) \text{ for all } \bar{x} \text{ in some neighborhood of } \bar{\bar{x}}$$

if and only if

$$[\lambda_k(\bar{x})]\left(j_{\bar{x}}\vec{F}, j_{\bar{x}}\vec{G}\right) = 0 \text{ for all } \bar{x} \text{ in some neighborhood of } \bar{\bar{x}}, (k=1,\cdots,k_{\max}).$$

Proof. This is a 1 dimensional case of Lemma 5.1, whose proof can be found in Section 5.1. ■

Proof of Lemma 6.2. We apply Lemma 6.3 to the bundle $\mathcal{H}$ defined in Definition 6.1. By making δ smaller, we may assume there are no bad points $\bar{\bar{x}}_{\text{bad}}$. Thus, we have achieved the following: There exist semialgebraic functions $\lambda_1,\cdots,\lambda_{k_{\max}} : (0,\delta] \to$ Aff such that for any $\bar{\bar{x}} \in (0,\delta)$ and any $(\vec{F},\vec{G})$ that is C^m in a neighborhood of $(\bar{\bar{x}},0)$, we have

$$J_{(\bar{x},0)}(\vec{F},\vec{G}) \in H(\bar{x},0) \text{ for all } \bar{x} \text{ in some neighborhood of } \bar{\bar{x}}$$

if and only if

$$[\lambda_k(\bar{x})]\, j_{\bar{x}}\left(\vec{F},\vec{G}\right) = 0 \text{ for all } \bar{x} \text{ in some neighborhood of } \bar{\bar{x}}, (k=1,\cdots,k_{\max}).$$

In particular, if $\left(\vec{F},\vec{G}\right)$ is as in (74), then

$$[\lambda_k(\bar{x})]\, j_{\bar{x}}\left(\vec{F},\vec{G}\right) = 0 \text{ for all } \bar{x} \in (0,\delta), (k=1,\cdots,k_{\max}).$$

Next, we apply Theorem 7 in Section 3.7. Recall $H(\bar{x},0)$ is an affine space, so $\mathbb{R}\cdot H(\bar{x},0)$ is a vector space. We regard $\mathbb{R}\cdot H(\bar{x},0)$ as the space of all $(\vec{P},\vec{Q},t)$ such that $\partial^\alpha\{\sum_j A_{ij}P_j + tB_i - Q_i\}(\bar{x},y) = o(y^{m-|\alpha|})$ as $y \to 0^+$. We define seminorms on $\mathbb{R}\cdot H(\bar{x},0)$ by

$$|||(\vec{P},\vec{Q},t)|||_{\alpha,i,y} = \left| y^{|\alpha|-m}\partial^\alpha \left\{ \sum_j A_{ij}P_j + tB_i - Q_i \right\}(\bar{x},y) \right|$$

for fixed $\bar{x}$ and $0 < y < \psi(\bar{x})$. Notice that on $H(\bar{x}, 0)$, the seminorm agrees with

$$|||(\vec{P}, \vec{Q})|||_{\alpha,i,y} = \left| y^{|\alpha|-m} \partial^\alpha \left\{ \sum_j A_{ij} P_j + B_i - Q_i \right\} (\bar{x}, y) \right|$$

for fixed $\bar{x} \neq 0$ and $0 < y < \psi(\bar{x}), |\alpha| \leq m, i = 1, \cdots, i_{\max}$.

Note that

$$\sup_{\alpha,i,y} |||(\vec{P}, \vec{Q})|||_{\alpha,i,y}$$

is bounded for fixed $\left(\vec{P}, \vec{Q}\right) \in H(\bar{x}, 0)$, by definition of $H(\bar{x}, 0)$. Thus, by Theorem 7 in Section 3.7, for each $\bar{x} \in (0, \delta)$, there exist $y_\sigma \in (0, \psi(\bar{x}))$ $(\sigma = 1, \cdots, \sigma_{\max})$ with $\sigma_{\max}$ depending only on $i_{\max}, j_{\max}, m$ such that for any $\left(\vec{P}, \vec{Q}\right) \in H(\bar{x}, 0)$, we have

$$\begin{aligned} & \sup_{\substack{0<y<\psi(\bar{x}) \\ |\alpha| \leq m \\ i=1,\cdots,i_{\max}}} |y|^{|\alpha|-m} \left| \partial^\alpha \left\{ \sum_j A_{ij} P_j + B_j - Q_i \right\} (\bar{x}, y) \right| \\ \leq \; & C \max_{\substack{\sigma=1,\cdots,\sigma_{\max} \\ |\alpha| \leq m \\ i=1,\cdots,i_{\max}}} |y_\sigma|^{|\alpha|-m} \left| \partial^\alpha \left\{ \sum_j A_{ij} P_j + B_j - Q_i \right\} (\bar{x}, y_\sigma) \right|. \end{aligned} \tag{79}$$

Moreover, (79) is a semialgebraic condition. Therefore, we may take $y_1, \cdots, y_\sigma \in (0, \psi(\bar{x}))$ satisfying (79) to depend semialgebraically on $\bar{x} \in (0, \delta)$. Because $0 < y_\sigma(\bar{x}) < \psi(\bar{x}) \leq \bar{x}$ for $\bar{x} \in (0, \delta)$ and because $y_\sigma(\bar{x})$ depends semialgebraically on $\bar{x}$, we can take δ small to achieve the estimates

(80) $\left| \left(\frac{d}{dx}\right)^\alpha y_\sigma(\bar{x}) \right| \leq C \bar{x}^{1-\alpha}$ for $0 \leq \alpha \leq m$, $\sigma = 1, \cdots, \sigma_{\max}$, $\bar{x} \in (0, \delta)$.

(81) $0 < y_\sigma(\bar{x}) < \psi(\bar{x}) \leq \bar{x}$ for $\sigma = 1, \cdots, \sigma_{\max}$, $\bar{x} \in (0, \delta)$.

(82) $\bar{x} \mapsto y_\sigma(\bar{x})$ is a semialgebraic function.

(83) For any $\bar{x} \in (0, \delta)$ and any $\left(\vec{P}, \vec{Q}\right) = (P_1, \cdots, P_{j_{\max}}, Q_1, \cdots, Q_{i_{\max}}) \in H(\bar{x}, 0)$, we have

$$\begin{aligned} & \sup_{\substack{0<y<\psi(\bar{x}) \\ |\alpha| \leq m \\ i=1,\cdots,i_{\max}}} |y|^{|\alpha|-m} \left| \partial^\alpha \left\{ \sum_j A_{ij} P_j + B_j - Q_i \right\} (\bar{x}, y) \right| \\ \leq \; & C \max_{\substack{\sigma=1,\cdots,\sigma_{\max} \\ |\alpha| \leq m \\ i=1,\cdots,i_{\max}}} |y_\sigma(\bar{x})|^{|\alpha|-m} \left| \partial^\alpha \left\{ \sum_j A_{ij} P_j + B_j - Q_i \right\} (\bar{x}, y_\sigma(\bar{x})) \right|. \end{aligned}$$

with C depending only on $i_{\max}, j_{\max}, m$.

Fix $\bar{x} \in (0, \delta)$, and let $(\vec{p}, \vec{q}) = (p_1, \cdots, p_{j_{\max}}, q_1, \cdots, q_{i_{\max}}) \in \mathfrak{p}^{j_{\max}+i_{\max}}$. Thus, each p_j and q_i is a polynomial in y of degree at most m. For $0 \le a \le m$, $\sigma = 1, \cdots, \sigma_{\max}$, $i = 1, \cdots, i_{\max}$, let

$$\begin{aligned} &\mu^{\#}_{a,\sigma,i} [\bar{x}] (p_1, \cdots, p_{j_{\max}}, q_1, \cdots, q_{i_{\max}}) \\ = \quad & (y_\sigma (\bar{x}))^{a-m} \partial_y^a \left\{ \sum_j A_{ij} (\bar{x}, y) p_j (y) + B_i (\bar{x}, y) - q_i (y) \right\} \bigg|_{y=y_\sigma(\bar{x})} . \end{aligned}$$

Note that we don't take x-derivatives here, only y-derivatives. The $\mu^{\#}_{a,\sigma,i}(\bar{x})$ are affine functions from $\mathfrak{p}^{j_{\max}+i_{\max}}$ to $\mathbb{R}$; thus, each $\mu^{\#}_{a,\sigma,i}(\bar{x})$ belongs to Aff. Let $\mu_1 (\bar{x}), \cdots, \mu_{l_{\max}} (\bar{x})$ be an enumeration of the $\mu^{\#}_{a,\sigma,i} (\bar{x})$, together with the linear maps

$$\begin{aligned} (p_1, \cdots, p_{j_{\max}}, q_1, \cdots, q_{i_{\max}}) &\mapsto (\bar{x})^{a-m} \partial_y^a p_j (0) \\ (p_1, \cdots, p_{j_{\max}}, q_1, \cdots, q_{i_{\max}}) &\mapsto (\bar{x})^{a-m} \partial_y^a q_i (0) . \end{aligned}$$

We will prove the following

(84) Let $\vec{F}, \vec{G}$ be as assumed in (74). Then, as $\bar{x}$ varies over $(0, \delta)$, the $[\mu_l (\bar{x})] \left(j_{\bar{x}} \vec{F}, j_{\bar{x}} \bar{G} \right)$ remain bounded, and these quantities tend to zero as $\bar{x}$ tends to 0^+.

To prove (84), we recall that

$$\sum_j A_{ij} F_j + B_i - G_i = 0,$$

hence

$$\begin{aligned} (85) \quad & \mu^{\#}_{a,\sigma,i} (\bar{x}) \left(j_{\bar{x}} \vec{F}, j_{\bar{x}} \vec{G} \right) \\ = \quad & - (y_\sigma (\bar{x}))^{a-m} \partial_y^a \left[\sum_i A_{ij} (\bar{x}, y) \{ F_j (\bar{x}, y) - j_{\bar{x}} F_j (y) \} - \{ G_j (\bar{x}, y) - j_{\bar{x}} G_j (y) \} \right] \bigg|_{y=y_\sigma} . \end{aligned}$$

Let $w_F (\bar{x}) = \max_{|\beta|=m, j=1,\cdots,j_{\max}} \left(\sup_{0 \le y \le \psi(\bar{x})} \left[\partial^\beta F_j (\bar{x}, y) \right] - \inf_{0 \le y \le \psi(\bar{x})} \left[\partial^\beta F_j (\bar{x}, y) \right] \right)$ and similarly define $w_G (\bar{x})$ as above, with G in place of F. Because $\vec{F}, \vec{G}$ belong to $C^m \left(\Omega_\delta^{\text{closure}}, \mathbb{R}^{j_{\max}} \right)$ and $C^m \left(\Omega_\delta^{\text{closure}}, \mathbb{R}^{i_{\max}} \right)$ respectively, while $\psi (\bar{x}) \to 0$ as $\bar{x} \to 0$, we know that $w_F (\bar{x})$, $w_G (\bar{x})$ are bounded as $\bar{x}$ varies over $(0, \delta)$, and moreover $w_F (\bar{x}), w_G (\bar{x}) \to 0$ as $\bar{x} \to 0^+$. Taylor's theorem gives

(86) $\left| \partial_y^a [F_j (\bar{x}, y) - j_{\bar{x}} F_j (y)] \right| \le C w_F (\bar{x}) \cdot y^{m-a}$ for $0 \le a \le m$, $0 < y < \psi (\bar{x})$, $j = 1, \cdots, j_{\max}$.

(87) $\left| \partial_y^a \{ G_i (\bar{x}, y) - j_{\bar{x}} G_i (y) \} \right| \le C w_G (\bar{x}) \cdot y^{m-a}$ for $0 \le a \le m, 0 < y < \psi (\bar{x}), i = 1, \cdots, i_{\max}$.

We recall that

(88) $|\partial_y^a A_{ij}(\bar{x},y)| \leq Cy^{-a}$ for $0 \leq a \leq m, 0 < y < \psi(\bar{x}), i = 1,\cdots,i_{\max}, j = 1,\cdots,j_{\max}$.

Putting (86),(87),(88) into (85), we find that

$$\left|\mu^{\#}_{a,\sigma,i}(\bar{x})\left(j_{\bar{x}}\vec{F}, j_{\bar{x}}\vec{G}\right)\right| \leq Cw_F(\bar{x}) + Cw_G(\bar{x}),$$

hence the $\mu^{\#}_{a,\sigma,i}(\bar{x})\left(j_{\bar{x}}\vec{F}, J_{\bar{x}}\vec{G}\right)$ remain bounded as $\bar{x}$ varies over $(0,\delta)$, and these quantities tend to zero as $\bar{x} \to 0^+$. Also, because $J_{(0,0)}\vec{F} = 0, J_{(0,0)}\vec{G} = 0$, and $\vec{F},\vec{G}$ are in $C^m\left(\Omega_\delta^{\text{closure}}, \mathbb{R}^{j_{\max}}\right)$ and $C^m\left(\Omega_\delta^{\text{closure}}, \mathbb{R}^{i_{\max}}\right)$ respectively, we see that

$$(\bar{x})^{a-m}\partial_y^a F_j(\bar{x},0), (\bar{x})^{a-m}\partial_y^a G_i(\bar{x},0),$$

for $0 \leq a \leq m$, remain bounded as $\bar{x}$ varies over $(0,\delta)$, and these quantities tend to zero as $\bar{x} \to 0^+$. Thus, all the $\mu_l(\bar{x})\left(j_{\bar{x}}\vec{F}, j_{\bar{x}}\vec{G}\right)$ remain bounded on $(0,\delta)$ and tend to zero as $\bar{x} \to 0^+$. We have proven (84). Thus, we have defined our $\lambda_1,\cdots,\lambda_{k_{\max}}$ and $\mu_1,\cdots,\mu_{l_{\max}}$ and we have proven (74).

We now set out to prove (75). Thus, let $\vec{F}^{\#} = \left(F_1^{\#},\cdots,F_{j_{\max}}^{\#}\right)$ and $\vec{G}^{\#} = \left(G_1^{\#},\cdots,G_{i_{\max}}^{\#}\right)$ be as in (75). Recall, each $F_j^{\#}$ and $G_i^{\#}$ is a semialgebraic map from $(0,\delta)$ into $\mathfrak{p}$, and moreover

$$[\lambda_k(\bar{x})]\left(\vec{F}^{\#}(\bar{x}), \vec{G}^{\#}(\bar{x})\right) = 0 \text{ for } k = 1,\cdots,k_{\max}, \text{ all } \bar{x} \in (0,\delta); \text{ and}$$

$[\mu_l(\bar{x})]\left(\vec{F}^{\#}(\bar{x}), \vec{G}^{\#}(\bar{x})\right)$ is bounded as $\bar{x}$ varies over $(0,\delta)$ and tends to zero as $\bar{x} \to 0^+$ for each $l = 1,\cdots,l_{\max}$. Then

(89) $F_j^{\#}(\bar{x})$ has the form $y \mapsto \sum_{s=0}^m F_{js}(\bar{x})\, y^s$ and

(90) $G_i^{\#}(\bar{x})$ has the form $y \mapsto \sum_{s=0}^m G_{is}(\bar{x})\, y^s$,

with F_{js}, G_{is} semialgebraic functions of one variable. Taking δ small (depending on $\vec{F}^{\#},\vec{G}^{\#}$), we may assume the F_{js}, G_{is} are C^∞ on $(0,\delta)$.

Now, we define $\vec{F} = (F_1,\cdots,F_{j_{\max}})$, $\vec{G} = (G_1,\cdots,G_{i_{\max}}), \vec{G}^{\#\#} = \left(G_1^{\#\#},\cdots,G_{i_{\max}}^{\#\#}\right)$, where

$$F_j(\bar{x},y) = \sum_{s=0}^m F_{js}(\bar{x})\, y^s \tag{91}$$

for $(\bar{x},y) \in (0,\delta)\times\mathbb{R}$, $j = 1,\cdots,j_{\max}$,

$$G_i^{\#\#}(\bar{x},y) = \sum_{s=0}^m G_{is}(\bar{x})\, y^s \tag{92}$$

for $(\bar{x}, y) \in (0,\delta) \times \mathbb{R}$, $i = 1, \cdots, i_{\max}$,

$$G_i(\bar{x}, y) = \sum_j A_{ij}(\bar{x}, y) F_j(\bar{x}, y) + B_i(\bar{x}, y)$$

for $(\bar{x}, y) \in \Omega_\delta$, $i = 1, \cdots, i_{\max}$. Note that $F_j, G_i^{\#\#}$ are C^∞ functions on $(0,\delta) \times \mathbb{R}$ because the F_{js}, G_{is} are C^∞ functions on $(0,\delta)$. The functions $F_j, G_i^{\#\#}, G_i$ are semialgebraic because $F_j^{\#}, G_i^{\#}$ are semialgebraic. Let $\bar{x} \in (0,\delta)$. Then

$$\text{(93)} \qquad j_{\bar{x}} F_j = F_j^{\#}(\bar{x}) \in \mathfrak{p}, j_{\bar{x}} G_i^{\#\#} = G_i^{\#}(\bar{x}) \in \mathfrak{p}.$$

Therefore, for all $\bar{x}$ in a small neighborhood of a given $\overline{\overline{x}} \in (0,\delta)$, we have

$$\lambda_k(\bar{x}) \left(j_{\bar{x}} \vec{F}, j_{\bar{x}} \vec{G}^{\#\#} \right) = \lambda_k(\bar{x}) \left(\vec{F}^{\#}(\bar{x}), \vec{G}^{\#}(\bar{x}) \right) = 0$$

for $k = 1, \cdots, k_{\max}$; the last equality is an assumption made in (75). Because $\vec{F}, \vec{G}^{\#\#}$ are C^∞ in a neighborhood of $\left(\overline{\overline{x}}, 0\right)$, the defining property of the λ_k now tells us that

$$\left(J_{(\bar{x},0)} \vec{F}, J_{(\bar{x},0)} \vec{G}^{\#\#} \right) \in H(\bar{x}, 0)$$

for all $\bar{x}$ in a small neighborhood of $\overline{\overline{x}}$. Recalling that $\overline{\overline{x}} \in (0,\delta)$ is arbitrary, we conclude that

$$\text{(94)} \qquad \left(J_{(\bar{x},0)} \vec{F}, J_{(\bar{x},0)} \vec{G}^{\#\#} \right) \in H(\bar{x}, 0) \text{ for all } \bar{x} \in (0,\delta).$$

By definition of $H(\bar{x}, 0)$ and by the estimates

$$\begin{aligned} \partial^\alpha \left(F_j - J_{(\bar{x},0)} F_j \right)(\bar{x}, y) &= o\left(y^{m-|\alpha|} \right), \\ \partial^\alpha \left(G_i^{\#\#} - J_{(\bar{x},0)} G_i^{\#\#} \right)(\bar{x}, y) &= o\left(y^{m-|\alpha|} \right), \text{ and} \\ \left| \partial^\alpha A_{ij}(x, y) \right| &\le C y^{-|\alpha|}, \end{aligned}$$

we therefore have the following:

(95) For any $\bar{x} \in (0,\delta)$, any $i = 1, \cdots, i_{\max}$, and any $|\alpha| \le m$, the quantity

$$y^{|\alpha|-m} \partial^\alpha \left\{ \sum_j A_{ij} F_j + B_i - G_i^{\#\#} \right\} (\bar{x}, y)$$

is bounded as y varies over $(0, \psi(\bar{x}))$ and tends to zero as $y \to 0^+$.

We don't yet know that the above convergence is uniform in $\bar{x}$.

Next, we recall from (75) the assumption that the $\mu_l(\bar{x})\left(\vec{F}^{\#}(\bar{x}),\vec{G}^{\#}(\bar{x})\right)$ remain bounded as $\bar{x}$ varies over $(0,\delta)$ and moreover these quantities tend to zero as $\bar{x}\to 0^+$. Thus, the quantities

$$(96)\qquad (y_\sigma(\bar{x}))^{a-m}\,\partial_y^a\left\{\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right\}(\bar{x},y_\sigma(\bar{x}))$$

for $0\le a\le m, i=1,\cdots,i_{\max},\sigma=1,\cdots,\sigma_{\max}$, remain bounded as $\bar{x}$ varies over $(0,\delta)$, and tend to zero as $\bar{x}\to 0^+$. Because those quantities are semialgebraic functions of one variable, we may pass to a smaller δ and assert for any b, say $0\le b\le m$, that

$$(97)\qquad \left(\frac{d}{d\bar{x}}\right)^b\left\{y_\sigma(\bar{x})^{a-m}\,\partial_y^a\left[\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right](\bar{x},y_\sigma(\bar{x}))\right\}=o\left(\bar{x}^{-b}\right)$$

as $\bar{x}\to 0^+$ and this quantity is bounded for $\bar{x}$ bounded away from 0. For $0\le a+b\le m$, we will check that

$$(98)\qquad (\bar{x})^{a+b-m}\left(\frac{d}{d\bar{x}}\right)^b\left\{\partial_y^a\left[\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right](\bar{x},y_\sigma(\bar{x}))\right\}=o(1)$$

as $\bar{x}\to 0^+$ and the left-hand side is bounded. To see this, we write

$$\begin{aligned}
&\left(\frac{d}{d\bar{x}}\right)^b\left\{\partial_y^a\left[\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right](\bar{x},y_\sigma(\bar{x}))\right\}\\
=\;&\left(\frac{d}{d\bar{x}}\right)^b\left\{(y_\sigma(\bar{x}))^{m-a}(y_\sigma(\bar{x}))^{a-m}\,\partial_y^a\left[\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right](\bar{x},y_\sigma(\bar{x}))\right\}\\
=\;&\sum_{b'+b''=b}\operatorname{coeff}(b',b'')\underbrace{\left[\left(\frac{d}{d\bar{x}}\right)^{b'}(y_\sigma(\bar{x}))^{m-a}\right]}_{(\dagger)}\cdot\\
&\underbrace{\left[\left(\frac{d}{d\bar{x}}\right)^{b''}\left\{(y_\sigma(\bar{x}))^{a-m}\,\partial_y^a\left[\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right](\bar{x},y_\sigma(\bar{x}))\right\}\right]}_{(\ddagger)}.
\end{aligned}$$

Since $y_\sigma(\bar{x})$ is given by a Puiseux series for $\bar{x}\in(0,\delta)$ (small enough δ),

$$(\dagger)=O\left(y_\sigma(\bar{x})\right)^{m-a}\cdot\bar{x}^{-b'}=O\left(y_\sigma(\bar{x})^{m-a-b'}\right),$$

because $0<y_\sigma(\bar{x})<\psi(\bar{x})\le\bar{x}$. By (97), $(\ddagger)$ is $o\left(\bar{x}^{-b''}\right)$ as $\bar{x}\to 0^+$. So in fact, we get not only (98) but the stronger result

$$(99)\qquad \left(\frac{d}{d\bar{x}}\right)^b\left\{\partial_y^a\left[\sum_j A_{ij}F_j+B_i-G_i^{\#\#}\right](\bar{x},y_\sigma(\bar{x}))\right\}=o\left(y_\sigma(\bar{x})^{m-a}\cdot\bar{x}^{-b}\right)$$

as $\bar{x} \to 0^+$; the left-hand side is bounded.

Introduce the vector field $X_\sigma = \frac{\partial}{\partial x} + y'_\sigma(\bar{x}) \frac{\partial}{\partial y}$ on $\mathbb{R}^2$. We have

$$\left(\frac{d}{d\bar{x}}\right)^b \{\mathcal{F}(\bar{x}, y_\sigma(\bar{x}))\} = (X_\sigma)^b \mathcal{F}\Big|_{(\bar{x}, y_\sigma(\bar{x}))} \text{ for any } \mathcal{F} \in C^b_{loc}\left(\mathbb{R}^2\right).$$

Therefore, (99) yields

$$\text{(100)} \qquad \left(X^b_\sigma \partial^a_y\right) \left[\sum_j A_{ij} F_j + B_i - G_i^{\#\#}\right] (\bar{x}, y_\sigma(\bar{x})) = o\left(y_\sigma(\bar{x})^{m-a} \cdot \bar{x}^{-b}\right) \text{ as } \bar{x} \to 0^+$$

and the left-hand side is bounded for all $\bar{x}$, for $a + b \le m, \sigma = 1, \cdots, \sigma_{\max}, i = 1, \cdots, i_{\max}$. This implies that

(101) $(y_\sigma(\bar{x}))^{|\alpha|-m} \partial^\alpha \left[\sum_j A_{ij} F_j + B_i - G_i^{\#\#}\right] (\bar{x}, y_\sigma(\bar{x}))$ is bounded on $(0, \delta)$ and tends to zero as $\bar{x} \to 0^+$, for $|\alpha| \le m, i = 1, \cdots, i_{\max}, \sigma = 1, \cdots, \sigma_{\max}$.

Let $\alpha = (b, a)$, $\partial^\alpha = \partial^b_x \partial^a_y$. We deduce (101) from (100) by induction on b. For $b = 0$, (101) is the same as (100). Assume we know (101) for all $b' < b$. We prove (101) for the given b, using our induction hypothesis for b', together with (100). The quantity

$$\text{(102)} \qquad X^b_\sigma \partial^a_y \left\{\sum_j A_{ij} F_j + B_i - G_i^{\#\#}\right\} (\bar{x}, y_\sigma(\bar{x}))$$

is a sum of terms of the form

$$\text{(103)} \qquad \left(\partial^{b_1}_x y_\sigma(\bar{x})\right) \cdot \cdots \cdot \left(\partial^{b_\nu}_x y_\sigma(\bar{x})\right) \cdot \partial^{\bar{b}}_x \partial^{a+\nu}_y \left\{\sum_j A_{ij} F_j + B_i - G_i^{\#\#}\right\} (\bar{x}, y_\sigma(\bar{x}))$$

with $b_t \ge 1$ each t, $b_1 + \cdots + b_\nu + \bar{b} = b$. Note $\bar{b} + (a + \nu) = a + \bar{b} + b_1 + \cdots + b_\nu - (b_1 - 1) - \cdots - (b_\nu - 1) \le a + b$. We know that $(102) = o\left(y_\sigma(\bar{x})^{m-a-b}\right)$ by (100).

If $\bar{b} < b$, then by our induction hypothesis, the term (103) is dominated by

$$\begin{aligned} & O\left(\overbrace{y_\sigma(\bar{x})^{-[b_1-1]-\cdots-[b_\nu-1]}}^{\text{Here again we use } 0<y_\sigma<\bar{x}.}\right) \cdot o\left(y_\sigma(\bar{x})^{m-[a+\nu]-\bar{b}}\right) \\ = \; & o\left(y_\sigma(\bar{x})^{m-a-\bar{b}-b_1-\cdots-b_\nu}\right) = o\left(y_\sigma(\bar{x})^{m-a-b}\right). \end{aligned}$$

Therefore, in the equation $(102) = \sum (103)$, all terms are $o\left(y_\sigma(\bar{x})^{m-a-b}\right)$, except possibly the term arising from $\bar{b} = b$, which is

$$\partial^b_x \partial^a_y \left\{\sum_j A_{ij} F_j + B_i - G_i^{\#\#}\right\} (\bar{x}, y_\sigma(\bar{x})).$$

Therefore,

$$\partial_x^b \partial_y^a \left\{ \sum_j A_{ij} F_j + B_i - G_i^{\#\#} \right\} (\bar{x}, y_\sigma(\bar{x})) = o\left(y_\sigma(\bar{x})^{m-a-b} \right), \text{ as } \bar{x} \to 0^+.$$

This completes our induction on b, proving (101).

Thus,

(104) $\max_{\substack{\sigma=1,\cdots,\sigma_{\max} \\ i=1,\cdots,i_{\max} \\ |\alpha|\le m}} (y_\sigma(\bar{x}))^{|\alpha|-m} \left| \partial^\alpha \left\{ \sum_j A_{ij} F_j + B_i - G_i^{\#\#} \right\} (\bar{x}, y_\sigma(\bar{x})) \right|$ is bounded on $(0,\delta)$ and tends to zero as $\bar{x}$ tends to 0^+.

Recall that our $\mu_l(\bar{x})$ include the affine maps $(p_1, \cdots, p_{j_{\max}}, q_1, \cdots, q_{i_{\max}}) \mapsto \bar{x}^{a-m} \partial_y^a p_j(0)$ and $(p_1, \cdots, p_{j_{\max}}, q_1, \cdots, q_{i_{\max}}) \mapsto \bar{x}^{a-m} \partial_y^a q_i(0)$ for $0 \le a \le m$. Our assumption on the μ's made in (75) tells us therefore that $\bar{x}^{a-m} \partial_y^a \left(F_j^{\#}(\bar{x}) \right)(0)$ and $\bar{x}^{a-m} \partial_y^a \left(G_i^{\#}(\bar{x}) \right)(0)$ are bounded on $(0,\delta)$ and tend to zero as $\bar{x} \to 0^+$. That is,

(105) $\bar{x}^{s-m} F_{js}(\bar{x}), \bar{x}^{s-m} G_{is}(\bar{x})$ are bounded on $(0,\delta)$ and tend to zero as $\bar{x} \to 0^+$. $(0 \le s \le m)$.

(106) Because F_{js}, G_{js} are semialgebraic functions of one variable, it follows that, for $s, t \le m$, the functions

$$\left(\frac{d}{d\bar{x}} \right)^t F_{js}(\bar{x}), \left(\frac{d}{d\bar{x}} \right)^t G_{is}(\bar{x})$$

are bounded on $(0,\delta)$ if $s+t \le m$ and are $o\left(\bar{x}^{m-s-t}\right)$ as $\bar{x} \to 0^+$ (even if $s+t > m$).

Recalling now the definitions of the F_j and $G_i^{\#\#}$ in terms of the F_j, G_{is} (see (91), (92)), we conclude that

$$\begin{aligned} \partial_{\bar{x}}^t \partial_y^s F_j(\bar{x}, y) &= \sum_{m \ge \underline{s} \ge s} \left[\left(\frac{d}{d\bar{x}} \right)^t F_{j\underline{s}}(\bar{x}) \right] (\text{coefficient } (\underline{s}, s)) \cdot y^{\underline{s}-s} \\ &= \sum_{m \ge \underline{s} \ge s} o\left(\bar{x}^{m-t-\underline{s}} \right) \cdot y^{\underline{s}-s}. \end{aligned}$$

If $s+t=m$, then this is equal to $o\left(\frac{y}{\bar{x}}\right)^{\underline{s}-s} = o(1)$ for $0 < y < \psi(\bar{x}) \le \bar{x}$. Therefore, for $|\beta| = m$, we have $\left| \partial^\beta F_j(\bar{x}, y) \right| = o(1)$ as $(\bar{x}, y) \in \Omega_\delta$ tends to zero. Similarly, $\left| \partial^\beta G_i^{\#\#}(\bar{x}, y) \right| = o(1)$ as $(\bar{x}, y) \in \Omega_\delta$ tends to zero. That is, for $|\beta| = m$, the functions $\partial^\beta F_j(\bar{x}, y)$ and $\partial^\beta G_i^{\#\#}(\bar{x}, y)$ are bounded on Ω_δ and they tend to zero as $\bar{x} \to 0^+$ (keeping $(\bar{x}, y) \in \Omega_\delta$).

Let $\mathcal{E}(\bar{x}) = \sup \left\{ \left| \partial^\beta F_j(\bar{x}, y) \right|, \left| \partial^\beta G_i^{\#\#}(\bar{x}, y) \right| : |\beta| = m, 0 < y < \psi(\bar{x}) \text{ (all } i, j) \right\}$. Then

(107) $\mathcal{E}(\bar{x})$ is bounded on $(0,\delta)$ and tends to zero as $\bar{x} \to 0^+$.

By Taylor's theorem,

$$\left| \partial^\alpha \left\{ F_j - J_{(\bar{x},0)} F_j \right\} (\bar{x}, y) \right| \le C y^{m-|\alpha|} \mathcal{E}(\bar{x}) \text{ for } |\alpha| \le m, (\bar{x}, y) \in \Omega_\delta.$$

Recall that

$$|\partial^\alpha A_{ij}(\bar{x},y)| \le Cy^{-|\alpha|} \text{ for } |\alpha| \le m \text{ and } (\bar{x},y) \in \Omega_\delta.$$

Just as we estimated the functions F_j above, we have from Taylor's theorem that

$$\left|\partial^\alpha \left\{G_i^{\#\#} - J_{(\bar{x},0)} G_i^{\#\#}\right\}(\bar{x},y)\right| \le Cy^{m-|\alpha|}\mathcal{E}(\bar{x}) \text{ for } |\alpha| \le m, (\bar{x},y) \in \Omega_\delta.$$

Combining these estimates, we see that

$$\begin{aligned} &\left|\partial^\alpha \left\{\sum_j A_{ij}\left(F_j - J_{(\bar{x},0)}F_j\right) - \left(G_i^{\#\#} - J_{(\bar{x},0)}G_i^{\#\#}\right)\right\}(x,y)\right| \\ (108) \qquad &\le \quad Cy^{m-|\alpha|}\mathcal{E}(\bar{x}) \text{ for } |\alpha| \le m, (\bar{x},y) \in \Omega_\delta. \end{aligned}$$

Combining (104), (107), (108), we see that

$$(109) \qquad (y_\sigma(\bar{x}))^{|\alpha|-m}\,\partial^\alpha \left\{\sum_j A_{ij}\left[J_{(\bar{x},0)}F_j\right] + B_i - \left[J_{(\bar{x},0)}G_i^{\#\#}\right]\right\}(\bar{x}, y_\sigma(\bar{x}))$$

is bounded on $(0,\delta)$ and tends to 0 as $\bar{x}$ tends to 0^+.

Recall that $\left(J_{(\bar{x},0)}\vec{F}, J_{(\bar{x},0)}\vec{G}^{\#\#}\right) \in H(\bar{x})$ for all $\bar{x} \in (0,\delta]$ (see (94)). The above results, together with the property (83) of the $y_\sigma(\bar{x})$ now tells us that

(110) $y^{|\alpha|-m}\partial^\alpha \left\{\sum_j A_{ij}\left(J_{(\bar{x},0)}F_j\right) + B_i - \left(J_{(\bar{x},0)}G_i^{\#\#}\right)\right\}(\bar{x},y)$ is bounded on Ω_δ and tends to zero as $(\bar{x},y) \in \Omega_\delta$ tends to zero.

Together with (107), (108), this yields the following result

(111) $y^{|\alpha|-m}\partial^\alpha \left\{\sum_j A_{ij}F_j + B_i - G_i^{\#\#}\right\}(\bar{x},y)$ is bounded on Ω_δ and tends to zero as $(\bar{x},y) \in \Omega_\delta$ tends to zero. Here, $i = 1,\cdots,i_{\max}$ and $|\alpha| \le m$ are arbitrary.

From (95), we have

(112) $\lim_{y\to 0^+} y^{|\alpha|-m}\partial^\alpha \left(\sum_j A_{ij}F_j + B_i - G_i^{\#\#}\right)(x,y) = 0$ for each fixed $x \in (0,\delta)$.

The functions $A_{ij}, F_j, B_i, G_i^{\#\#}$ are semialgebraic. Therefore, by Lemma 3.3, there exist a positive integer K and a semialgebraic function of one variable $\mathcal{A}(x)$ such that

(113) $\left|y^{|\alpha|-m}\partial^\alpha \left(\sum_j A_{ij}F_j + B_i - G_i^{\#\#}\right)(x,y)\right| \le \mathcal{A}(x)\cdot y^{\frac{1}{K}}$ for all $(x,y) \in \Omega_\delta$, $|\alpha| \le m, i = 1,\cdots,i_{\max}$.

Taking δ smaller, we may assume $\mathcal{A}(x)$ is C^∞ on $(0,\delta]$. Consequently,

$$y^{|\alpha|-m}\partial^\alpha\left(\sum_j A_{ij}F_j + B_i - G_i^{\#\#}\right)(x,y)$$

tends to zero as $y \to 0^+$, uniformly as x varies over (ε,δ) for any $\varepsilon > 0$. Recalling that $G_i = \sum_j A_{ij}F_j + B_i$, we see that for $|\alpha| \le m, i = 1,\cdots,i_{\max}$,

$$y^{|\alpha|-m}\partial^\alpha\left\{G_i - G_i^{\#\#}\right\}(x,y) \to 0 \tag{114}$$

as $y \to 0^+$ uniformly for x in each interval (ε,δ). Recalling that $G_i^{\#\#}$ belongs to C^∞ in a neighborhood of $(x,0)$ (each $x \in (0,\delta)$), we conclude that the derivatives $\partial^\alpha G_i(x,y)$ ($|\alpha| \le m, i = 1,\cdots,i_{\max}$), initially defined on $\Omega_\delta = \{(x,y) : 0 < x < \delta, 0 < y < \psi(x)\}$ extend to continuous functions on

$$\Omega_\delta^{++} \equiv \{(x,y) : 0 < x < \delta, 0 \le y < \psi(x)\}. \tag{115}$$

Next, recall that F_{js} is C^∞ on $(0,\delta)$ and that we assume that $|\partial^\alpha A_{ij}(x,y)|, |\partial^\alpha B_i(x,y)| \le Cy^{-|\alpha|}$ on

$$\Omega^+ = \{(x,y) : 0 < x < \delta, 0 < y \le \psi(x)\} \tag{116}$$

on which the functions $\partial^\alpha A_{ij}, \partial^\alpha B_i$ are assumed to be continuous. We defined

$$\begin{aligned} G_i &= \sum_j A_{ij}F_j + B_i \\ &= \sum_j A_{ij}(x,y)\left[\sum_{s=0}^m F_{js}(x)\,y^s\right] + B_i(x,y). \end{aligned}$$

The above remarks (and the fact that $\psi(x) \ne 0$ for $x \in (0,\delta)$) show that $\partial^\alpha G_i$ extends to a continuous function on Ω^+ (see (116)), for $|\alpha| \le m, i = 1,\cdots,i_{\max}$. Combining our results for Ω^+ (see (116)) and for Ω^{++} (see (115)), we see that $\partial^\alpha G_i$ extends to a continuous function on $\Omega_{\frac{2\delta}{3}}^{\text{closure}} \setminus \{(0,0)\}$ for each $i = 1,\cdots,i_{\max}, |\alpha| \le m$. Also, $\partial^\alpha F_i$ is a continuous function on $\Omega_{\frac{2}{3}\delta}^{\text{closure}} \setminus \{(0,0)\}$ because F_i is C^∞ on $(0,\delta)\times\mathbb{R}$. By (105), we have $G_{is}(x) = o(x^{m-s})$ ($0 \le s \le m$) on $(0,\delta)$. Because G_{is} is semialgebraic, it follows that after possibly reducing δ, we have

$$\left(\frac{d}{dx}\right)^t G_{is}(x) = o\left(x^{m-s-t}\right) \text{ for } 0 \le t \le m, 0 \le s \le m, i = 1,\cdots,i_{\max}.$$

Because $G_i^{\#\#}(x,y) = \sum_{\underline{s}=0}^{m} G_{i\underline{s}}(x)\, y^{\underline{s}}$ and $0 < y < \psi(x) \le x$ on Ω_δ, we have on Ω_δ that

$$\begin{aligned}
\left|\partial_x^t \partial_y^s G_i^{\#\#}(x,y)\right| &= \left|\sum_{\underline{s}=s}^{m} \operatorname{coeff}(\underline{s},s) \cdot \left(\frac{d}{dx}\right)^t G_{i\underline{s}}(x) \cdot y^{\underline{s}-s}\right| \\
&= o\left(\sum_{\underline{s}=s}^{m} x^{m-\underline{s}-t} \cdot y^{\underline{s}-s}\right) \\
&= o\left(\sum_{\underline{s}=s}^{m} x^{m-\underline{s}-t} \cdot x^{\underline{s}-s}\right) \\
&= o\left(x^{m-s-t}\right) \text{ on } \Omega_\delta \text{ for } s,t \le m.
\end{aligned}$$

In particular,

(117) $\partial^\alpha G_i^{\#\#}(x,y) \to 0$ as $(x,y) \in \Omega_\delta$ tends to $(0,0)$ for $|\alpha| \le m, i = 1, \cdots, i_{\max}$.

On the other hand, recalling the definition $G_i = \sum_j A_{ij} F_j + B_i$, we see from (111) that $\partial^\alpha \left(G_i - G_i^{\#\#}\right)(x,y) \to 0$ as $(x,y) \in \Omega_\delta$ tends to $(0,0)$ for each $|\alpha| \le m$. Together with (117), this shows that $\partial^\alpha G_i(x,y) \to 0$ as $(x,y) \in \Omega_\delta$ tends to $(0,0)$ for each $|\alpha| \le m$.

Next, recall from (105) that $F_{js}(x) = o(x^{m-s})$ for $x \in (0,\delta)$, $j = 1, \cdots, j_{\max}, s = 0, \cdots, m$. Because the F_{jk} are semialgebraic functions of one variable, we conclude (after reducing δ) that $\left(\frac{d}{dx}\right)^t F_{js}(x) = o\left(x^{m-s-t}\right)$ on $(0,\delta)$ for $t \le m$.

Now, for $s + t \le m$ and $(x,y) \in \Omega_\delta$ (hence $0 < y < \psi(x) \le x$), we have

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial y}\right)^s \left(\frac{\partial}{\partial x}\right)^t F_j(x,y)\right| &= \left|\left(\frac{\partial}{\partial y}\right)^s \left(\frac{\partial}{\partial x}\right)^t \sum_{\underline{s}=0}^{m} F_{j\underline{s}}(x)\, y^{\underline{s}}\right| \\
&= \left|\sum_{\underline{s}=s}^{m} \operatorname{coeff}(\underline{s},s) \left[\left(\frac{d}{dx}\right)^t F_{j\underline{s}}(x)\right] \cdot y^{\underline{s}-s}\right| \\
&\le C \sum_{\underline{s}=s}^{m} \left|\left(\frac{d}{dx}\right)^t F_{j\underline{s}}(x)\right| \cdot x^{\underline{s}-s} \\
&= o\left(\sum_{\underline{s}=0}^{m} x^{m-\underline{s}-t} x^{\underline{s}-s}\right) = o\left(x^{m-s-t}\right).
\end{aligned}$$

Thus, for $|\alpha| \le m$, and $j = 1, \cdots, j_{\max}$, we have

$$\partial^\alpha F_j(x,y) \to 0 \text{ as } (x,y) \in \Omega_\delta \text{ tends to } (0,0).$$

We now know the following: $G_i = \sum_j A_{ij} F_j + B_i$ on Ω_δ. The F_j and G_i are semialgebraic on Ω_δ For $|\alpha| \le m$, the derivatives $\partial^\alpha F_j, \partial^\alpha G_i$ extend to continuous functions on $\Omega_{2\delta/3}^{\text{closure}} \setminus \{(0,0)\}$. For $|\alpha| \le m$, the derivatives $\partial^\alpha F_j(z)$, $\partial^\alpha G_i(z)$ tend to zero as $z \in \Omega_\delta$ tends to zero.

It follows that the F_j and G_i extend from $\Omega_{\delta/2}$ to semialgebraic functions in $C^m\left(\Omega_{\delta/2}^{\text{closure}}\right)$ and those functions all have m-jet zero at the origin. We extend F_j, G_i to semialgebraic C^m_{loc} functions on $\mathbb{R}^2$, using Corollary 3.2.

Next, we show that $j_{\bar{x}}\left(\vec{F},\vec{G}\right) = \left(\vec{F}^{\#}\left(\bar{x}\right),\vec{G}^{\#}\left(\bar{x}\right)\right)$ for $\bar{x} \in (0,\delta)$. From (93), we have

$$j_{\bar{x}}\left(\vec{F},\vec{G}^{\#\#}\right) = \left(\vec{F}^{\#}\left(\bar{x}\right),\vec{G}^{\#}\left(\bar{x}\right)\right).$$

From (114), we see that $j_{\bar{x}}\left(G_i - G_i^{\#\#}\right) = 0$ for all $\bar{x} \in (0,\delta)$. Therefore,

$$j_{\bar{x}}\left(\vec{F},\vec{G}\right) = j_{\bar{x}}\left(\vec{F},\vec{G}^{\#\#}\right) = \left(\vec{F}^{\#}\left(\bar{x}\right),\vec{G}^{\#}\left(\bar{x}\right)\right),$$

as desired. Thus, we have proven (75). The proof of Lemma 6.2 is complete. ■

6.2 Patching near a cusp

Lemma 6.4 *Let $\psi(x)$ be a semialgebraic function on $[0,\delta]$, satisfying $\psi(0) = 0, 0 < \psi(x) \leq x$ for all $x \in (0,\delta]$. We set*

$$E_\delta = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq \delta, 0 \leq y \leq \psi(x)\},$$

$$E_\delta^+ = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq \delta, \frac{1}{3}\psi(x) \leq y \leq \psi(x)\}, \text{ and}$$

$$E_\delta^- = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq \delta, 0 \leq y \leq \frac{2}{3}\psi(x)\}.$$

Fix a semialgebraic function of one variable, $\theta(t)$, satisfying $0 \leq \theta(t) \leq 1$, $\theta(t) = 1$ for $t \leq 1/3$, $\theta(t) = 0$ for $t \geq 2/3$, $\theta \in C^m$. Then set

$$\theta_-(x,y) = \theta\left(\frac{y}{\psi(x)}\right),\ \theta_+(x,y) = 1 - \theta_-(x,y) \text{ for } (x,y) \in E_\delta \setminus \{(0,0)\}.$$

Thus, $\theta_+, \theta_- \geq 0$ and $\theta_+ + \theta_- = 1$ on $E_\delta \setminus \{(0,0)\}$. Let $F^+ \in C^m(E_\delta^+)$ and $F^- \in C^m(E_\delta^-)$ be semialgebraic functions, with $J_{(0,0)}F^+ = J_{(0,0)}F^- = 0$. Suppose that

$$\partial_y^l F^+(x,\psi(x)) - \sum_{j=0}^{m-l} \frac{1}{j!}\partial_y^{l+j}F^-(x,0)\cdot(\psi(x))^j = o((\psi(x))^{m-l}) \tag{118}$$

as $x \to 0^+$ for each $l = 0,\cdots,m$. Define $F = \theta_+ \cdot F^+ + \theta_- \cdot F^-$ on $E_\delta \setminus \{(0,0)\}, F(0,0) = 0$.

Then F is a C^m semialgebraic function on $E_{\delta'}$ for some small δ'. The jet of F at the origin is zero. Moreover, $F = F^+$ in a neighborhood of any point $(x,\psi(x))$, $0 < x < \delta'$; and $F = F^-$ in a neighborhood of any point $(x,0), 0 < x < \delta'$.

Proof. Because $0 \leq \psi(x) \leq x$ and ψ is given near 0 by a convergent Puiseux series, we have $\psi^{(k)}(x) = O(x^{1-k})$ as $x \to 0^+$, for $k = 0,\cdots,m$. Also, because F^+, F^- have zero jet at $(0,0)$, we have, for $|\alpha| = m$, $\partial^\alpha F^+(x,y) = o(1)$ as $(x,y) \in E_\delta^+$ tends to zero and $\partial^\alpha F^-(x,y) = o(1)$ as $(x,y) \in E_\delta^-$ tends to zero. By induction on μ, we now prove that

(119) $\partial_x^\mu \partial_y^l F^+(x,\psi(x)) - \sum_{j=0}^{m-l-\mu} \frac{1}{j!}\partial_x^\mu \partial_y^{l+j} F^-(x,0)\cdot(\psi(x))^j = o((\psi(x))^{m-\mu-l})$ as $x \to 0^+$ for $\mu + l \le m$.

For $\mu = 0$, (119) is a hypothesis of our lemma. Assuming (119) for μ, we prove it for $\mu+1$. Thus, fix l satisfying $(\mu+1)+l \le m$. Recalling that $\partial_x^\mu \partial_y^{l+j} F^-(x,0) = o(1)$ when $\mu + (l+j) = m$, we conclude from (119) that

(120) $\partial_x^\mu \partial_y^l F^+(x,\psi(x)) - \sum_{j=0}^{m-l-\mu-1} \frac{1}{j!}\partial_x^\mu \partial_y^{l+j} F^-(x,0)\cdot(\psi(x))^j = o((\psi(x))^{m-\mu-l})$ as $x \to 0^+$.

Because the above functions are semialgebraic functions of one variable and thus given near 0 by convergent Puiseux series, it follows that $\frac{d}{dx}\{(120)\} = o((\psi(x))^{m-\mu-l}\cdot x^{-1})$, hence $\frac{d}{dx}\{(120)\} = o((\psi(x))^{m-\mu-l-1})$, because $0 < \psi(x) \le x$. Thus,

$$\begin{aligned}
&\left[(\partial_x + \psi'(x)\,\partial_y)\left(\partial_x^\mu \partial_y^l F^+\right)\right](x,\psi(x)) - \sum_{j=0}^{m-l-\mu-1} \frac{1}{j!}\partial_x^{\mu+1}\partial_y^{l+j} F^-(x,0)\,(\psi(x))^j \\
&- \sum_{j=1}^{m-l-\mu-1} \frac{1}{j!}\partial_x^{\mu}\partial_y^{l+j} F^-(x,0)\, j\,(\psi(x))^{j-1}\,\psi'(x) \\
=\;& o\left((\psi(x))^{m-\mu-l-1}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left[\partial_x^{\mu+1}\partial_y^l F^+(x,\psi(x)) - \sum_{j=0}^{m-l-(\mu+1)} \frac{1}{j!}\partial_x^{\mu+1}\partial_y^{l+j} F^-(x,0)\,(\psi(x))^j\right] \\
&+\psi'(x)\left[\partial_x^{\mu}\partial_y^{l+1} F^+(x,\psi(x)) - \sum_{j=0}^{m-l-\mu-2} \frac{1}{j!}\partial_x^{\mu}\partial_y^{l+1+j} F^-(x,0)\,(\psi(x))^j\right] \qquad (121)\\
=\;& o\left((\psi(x))^{m-(\mu+1)-l}\right).
\end{aligned}$$

For $j = m-l-\mu-1$, we have $\partial_x^\mu \partial_y^{l+1+j} F^-(x,0) = o(1)$, hence inductive hypothesis (119) for $(l+1)$ in place of l tells us that the second term in square brackets in (121) is $o\left((\psi(x))^{m-(\mu+1)-l}\right)$. Also, $|\psi'(x)| = O(1)$.

Consequently, the first term in square brackets in (121) is $o\left((\psi(x))^{m-(\mu+1)-l}\right)$, proving the analogue of (119) for $\mu+1$, thus completing the induction and establishing (119). We bring in the cutoff functions θ_+ and θ_-. Note that θ_+ is supported in E_δ^+ and θ_- is supported in E_δ^-. We will estimate the derivatives of θ_+, θ_- on E_δ. We have

$$\left(\frac{d}{dx}\right)^k \frac{1}{\psi(x)} = O\left(\frac{1}{\psi(x)}x^{-k}\right) \text{ as } x \to 0^+,$$

because ψ is given by a convergent Puiseux series. Because $0 < \psi(x) \le x$ for $x \in (0,\delta)$ and $0 \le y \le \psi(x)$ in E_δ, it follows that

$$\partial_x^l \partial_y^k \left(\frac{y}{\psi(x)}\right) = O\left((\psi(x))^{-k-l}\right)$$

as $(x,y)\in E_\delta\to 0$, for all $k,l\geq 0$.

Now, $\partial^\alpha_{x,y}\theta_-(x,y)$ is a sum of terms $\theta^{(s)}\left(\frac{y}{\psi(x)}\right)\cdot\prod_{\sigma=1}^s\left[\partial^{\alpha_\sigma}_{x,y}\left(\frac{y}{\psi(x)}\right)\right]$ with $\alpha_1+\cdots+\alpha_s=\alpha$, $s\leq|\alpha|$. Each such term is $O\left(\prod_{\sigma=1}^s\left(\frac{1}{\psi(x)}\right)^{|\alpha_\sigma|}\right)=O\left(\left(\frac{1}{\psi(x)}\right)^{|\alpha|}\right)$. Thus,

$$\left|\partial^\alpha_{x,y}\theta_-(x,y)\right|,\left|\partial^\alpha_{x,y}\theta_+(x,y)\right|\leq\frac{C_\alpha}{(\psi(x))^{|\alpha|}}\text{ on }E_\delta\text{ (smaller }\delta\text{) for }|\alpha|\leq m. \tag{122}$$

Next, we return to F^+,F^-, and prove the following estimate

$$\partial^\mu_x\partial^l_y\left(F^+-F^-\right)(x,y)=o\left([\psi(x)]^{m-\mu-l}\right)\text{ as }(x,y)\in E^+_\delta\cap E^-_\delta\to 0 \tag{123}$$

for each μ,l with $\mu+l\leq m$.

To see this, fix μ, $0\leq\mu\leq m$, and look at the polynomials

$$\begin{aligned}P^+_x(y) &= \sum_{j=0}^{m-\mu}\frac{1}{j!}\left[\partial^j_y\partial^\mu_x F^+(x,\psi(x))\right]\cdot(y-\psi(x))^j,\\ P^-_x(y) &= \sum_{j=0}^{m-\mu}\frac{1}{j!}\left[\partial^j_y\partial^\mu_x F^-(x,0)\right]\cdot y^j.\end{aligned}$$

Estimate (119) shows that

$$\partial^l_y\left(P^+_x-P^-_x\right)|_{y=\psi(x)}=o\left((\psi(x))^{m-\mu-l}\right)\text{ for }l=0,\cdots,m-\mu. \tag{124}$$

For y satisfying $(x,y)\in E^+_\delta\cap E^-_\delta$, we have $|y|,|y-\psi(x)|\leq\psi(x)$ and therefore (124) yields

$$\partial^l_y\left(P^+_x-P^-_x\right)(x,y)=o\left((\psi(x))^{m-\mu-l}\right)$$

as $(x,y)\in E^+_\delta\cap E^-_\delta$ tends to zero.

On the other hand, Taylor's theorem gives for $(x,y)\in E^+_\delta\cap E^-_\delta\setminus\{(0,0)\}$ the estimates

$$\partial^l_y\left[\partial^\mu_x F^+-P^+_x\right](x,y)=O\left((\psi(x))^{m-\mu-l}\cdot\max_{\bar{y}\in\left[\frac{1}{3}\psi(x),\psi(x)\right]}\left|\partial^{m-\mu}_y\partial^\mu_x F^+(x,\bar{y})\right|\right)$$

and

$$\partial^l_y\left[\partial^\mu_x F^--P^-_x\right](x,y)=O\left((\psi(x))^{m-\mu-l}\cdot\max_{\bar{y}\in\left[0,\frac{2}{3}\psi(x)\right]}\left|\partial^{m-\mu}_y\partial^\mu_x F^-(x,\bar{y})\right|\right).$$

The maxima in these last two estimates are $o(1)$, because $J_{(0,0)}F^+=J_{(0,0)}F^-=0$. Thus, as $(x,y)\in E^+_\delta\cap E^-_\delta\setminus\{(0,0)\}$ approaches zero, the quantities $\partial^l_y\left[\partial^\mu_x F^+-P^+_x\right](x,y)$, $\partial^l_y\left[\partial^\mu_x F^--P^-_x\right](x,y)$, $\partial^l_y\left[P^+_x-P^-_x\right](x,y)$ are all $o\left((\psi(x))^{m-\mu-l}\right)$. Consequently,

$$\left(\partial^l_y\partial^\mu_x F^+-\partial^l_y\partial^\mu_x F^-\right)(x,y)=o\left((\psi(x))^{m-\mu-l}\right)$$

as $(x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\}$ approaches zero, completing the proof of (123).

We now set $F = \theta_+ F^+ + \theta_- F^-$ on $E_\delta \setminus \{(0,0)\}$ and $F(0,0) = 0$. Evidently, F is C^m away from the origin, and semialgebraic; moreover, $F = F^+$ in a neighborhood of any point $(x^0, \psi(x^0))$ in E_δ $(x^0 \neq 0)$ and $F = F^-$ in a neighborhood of any point $(x^0, 0) \in E_\delta$ $(x^0 \neq 0)$.

It remains to check that $F \in C^m(E_\delta)$ near 0 and that $J_{(0,0)} F = 0$. That amounts to showing that

$$(125) \quad \partial_{x,y}^\alpha F(x,y) = o\left(x^{m-|\alpha|}\right) \text{ as } (x,y) \in E_\delta \setminus \{(0,0)\} \text{ approaches } (0,0) \text{ (all } |\alpha| \leq m).$$

To prove (125), we may assume $(x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\}$, because otherwise the left-hand side of (125) is $\partial_{x,y}^\alpha F^+$ for $(x,y) \in E_\delta^+ \setminus \{(0,0)\}$ or else $\partial_{x,y}^\alpha F^-$ for $(x,y) \in E_\delta^- \setminus \{(0,0)\}$, in which case (125) holds because $J_{(0,0)} F^+ = J_{(0,0)} F^- = 0$. For $(x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\}$, we have

$$(126) \qquad F = F^- + \theta_+ \left(F^+ - F^-\right).$$

Because $J_{(0,0)} F^- = 0$, we have

$$(127) \quad \partial_{x,y}^\alpha F^-(x,y) = o\left(x^{m-|\alpha|}\right) \text{ as } (x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\} \text{ tends to } (0,0), \text{ for } |\alpha| \leq m.$$

We recall that $\partial_{x,y}^\alpha \theta_+(x,y) = O\left((\psi(x))^{-|\alpha|}\right)$ for $|\alpha| \leq m$ and that $\partial_{x,y}^\alpha (F^+ - F^-)(x,y) = o\left((\psi(x))^{m-|\alpha|}\right)$ for $|\alpha| \leq m$ as $(x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\}$ tends to $(0,0)$, for $|\alpha| \leq m$.

Therefore, for $|\alpha| \leq m$, as $(x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\}$ tends to $(0,0)$, we have

$$\partial_{x,y}^\alpha \left\{\theta_+ \left(F^+ - F^-\right)(x,y)\right\} = o\left((\psi(x))^{m-|\alpha|}\right),$$

hence

$$(128) \qquad \partial_{x,y}^\alpha \left\{\theta_+ \left(F^+ - F^-\right)(x,y)\right\} = o\left(x^{m-|\alpha|}\right),$$

because $0 < \psi(x) \leq x$. Putting (127), (128) into (126), we see that

$$\partial_{x,y}^\alpha F(x,y) = o\left(x^{m-|\alpha|}\right)$$

as $(x,y) \in E_\delta^+ \cap E_\delta^- \setminus \{(0,0)\}$ tends to $(0,0)$, for $|\alpha| \leq m$. Thus, (125) holds. The proof of Lemma 6.4 is complete. ■

Next, we introduce a change of variables in a neighborhood of 0 in $\mathbb{R}_+^2 = \{(x,y) : x > 0\}$ of the form

$$(129) \qquad \bar{x} = x, \bar{y} = y + \tilde{\psi}(x),$$

where $\tilde{\psi}(x)$ is semialgebraic and satisfies $\left|\tilde{\psi}(x)\right| \leq Cx$ for $x \in (0,\delta)$. The inverse change of variables is of course

$$x = \bar{x}, y = \bar{y} - \tilde{\psi}(\bar{x}).$$

Note that $\partial^\alpha_{x,y}(\bar{x},\bar{y}) = O\left(x^{1-|\alpha|}\right)$ for $|y| \le Cx \ll 1$ because $\tilde{\psi}$ is given near 0 as a convergent Puiseux series, hence $\left|\tilde{\psi}(x)\right| \le Cx$ implies $\left|\tilde{\psi}^{(k)}\right| \le C_k x^{1-k}$ for small x. The change of variables (129) does not preserve C^m, but it does preserve C^m functions whose jets at 0 are equal to zero. Indeed, suppose $F(\bar{x},\bar{y}) \in C^m\left(\bar{E}\right)$ for $\bar{E} \subset \{(\bar{x},\bar{y}) : |\bar{y}| \le C\bar{x}\}$, with $0 \in \bar{E}$ and $J_0 F = 0$. Then $\bar{E}$ corresponds under (129) to a set $E \subset \{(x,y) : |y| \le C'x\}$, $0 \in E$. We may regard F as a function of (x,y), and for $|\alpha| \le m$, $\partial^\alpha_{x,y} F(x,y)$ is a sum of terms $\left|\partial^\beta_{\bar{x},\bar{y}} F(\bar{x},\bar{y})\right| \cdot \prod_{\nu=1}^{|\beta|} \left[\partial^{\alpha_\nu}_{x,y}(\bar{x},\bar{y})\right]$ with $|\beta| \le m$ and $\sum_\nu \alpha_\nu = \alpha$. If $J_{(0,0)}F = 0$ as a function of $(\bar{x},\bar{y})$, then $\partial^\beta_{\bar{x},\bar{y}} F(\bar{x},\bar{y}) = o\left(\bar{x}^{m-|\beta|}\right)$ on $\bar{E}$, hence $\partial^\beta_{\bar{x},\bar{y}} F(\bar{x},\bar{y}) = o\left(x^{m-|\beta|}\right)$ on E. Also, on E,

$$\prod_{\nu=1}^{|\beta|} \left[\partial^{\alpha_\nu}_{x,y}(\bar{x},\bar{y})\right] = \prod_{\nu=1}^{|\beta|} O\left(x^{1-|\alpha_\nu|}\right) = O\left(x^{|\beta| - \sum_\nu |\alpha_\nu|}\right) = O\left(x^{|\beta|-|\alpha|}\right).$$

Consequently, $\partial^\alpha_{x,y} F(x,y) = o\left(x^{m-|\alpha|}\right)$ on $E \setminus \{(0,0)\}$, for $|\alpha| \le m$. Thus, as claimed, $F \in C^m(E)$ and $J_{(0,0)}F = 0$.

The following generalization of Lemma 6.4 is reduced to Lemma 6.4 by means of the change of variables discussed above.

Lemma 6.5 *Let $0 \le \psi_-(x) \le \psi_+(x) \le x$ be semialgebraic functions on $[0,\delta]$, with $\psi_- < \psi_+$ on $(0,\delta]$. We set*

$$E_\delta = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le \delta, \psi_-(x) \le y \le \psi_+(x)\},$$

$$E_\delta^+ = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le \delta, 0 \le \psi_+(x) - y \le \frac{2}{3}(\psi_+(x) - \psi_-(x))\}, \text{ and}$$

$$E_\delta^- = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le \delta, 0 \le y - \psi_-(x) \le \frac{2}{3}(\psi_+(x) - \psi_-(x))\}.$$

Fix a semialgebraic function of one variable, $\theta(t)$, satisfying $0 \le \theta(t) \le 1$, $\theta(t) = 1$ for $t \le 1/3$, $\theta(t) = 0$ for $t \ge 2/3$, $\theta \in C^m$. Then set

$$\theta_-(x,y) = \theta\left(\frac{y - \psi_-(x)}{(\psi_+ - \psi_-)(x)}\right), \theta_+(x,y) = 1 - \theta_-(x,y) \text{ for } (x,y) \in E_\delta \setminus \{(0,0)\}.$$

Thus, $\theta_+, \theta_- \ge 0$ and $\theta_+ + \theta_- = 1$ on $E_\delta \setminus \{(0,0)\}$. Let $F^+ \in C^m(E_\delta^+)$ and $F^- \in C^m(E_\delta^-)$ be semialgebraic functions, with $J_{(0,0)}F^+ = J_{(0,0)}F^- = 0$. Suppose that

$$\partial_y^l F^+(x,\psi_+(x)) - \sum_{j=0}^{m-l} \frac{1}{j!} \partial_y^{l+j} F^-(x,\psi_-(x)) \cdot (\psi_+(x) - \psi_-(x))^j = o((\psi_+(x) - \psi_-(x))^{m-l})$$

as $x \to 0^+$ for each $l = 0, \cdots, m$. Define $F = \theta_+ \cdot F^+ + \theta_- \cdot F^-$ on $E_\delta \setminus \{(0,0)\}, F(0,0) = 0$.

Then F is a C^m semialgebraic function on $E_{\delta'}$ for some small δ'. The jet of F at $(0,0)$ is zero. Moreover, $F = F^+$ in a neighborhood of any point $(x,\psi_+(x))$, $0 < x < \delta'$, and $F = F^-$ in a neighborhood of any point $(x,\psi_-(x))$, $0 < x < \delta'$.

6.3 Proof of Lemma 6.1

Let $\mathcal{H} = (H(z))_{z\in\mathbb{R}^2}$ be a semialgebraic bundle with a C^m_{loc} section. Each $H(z)$ is a coset of an $\mathcal{R}_z$ submodule in $\mathcal{R}_z^D$. Assume $H((0,0)) = \{0\}$. Let $\Omega_\delta = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le \delta, 0 \le y \le x\}$ for $\delta > 0$. We look for semialgebraic C^m_{loc} sections of $\mathcal{H}|_{\Omega_\delta}$, for some small δ (which will keep shrinking as we discuss further). We apply Lemma 5.3. Thus, we obtain the following

- Semialgebraic functions $0 \le \psi_0(x) \le \psi_1(x) \le \cdots \le \psi_{s_{\max}}(x) = x$ on $(0,\delta)$, all given by convergent Puiseux expansions on $(0,\delta)$.
- Integers k_s $(0 \le k_s \le D)$ and permutations $\pi_s : \{1,\cdots,D\} \to \{1,\cdots,D\}$ for $s = 1,\cdots,s_{\max}$.
- Semialgebraic functions $A^s_{ij}(x,y)$ $(s = 1,\cdots,s_{\max}, 1 \le i \le k_s, k_s < j \le D)$ and $\varphi^s_i(x,y)$ $(s = 1,\cdots,s_{\max}, 1 \le i \le k_s)$ defined on $E_s = \{(x,y) : 0 < x < \delta, \psi_{s-1}(x) < y < \psi_s(x)\}$.
- Semialgebraic functions $\theta^{si}_{jl}(x)$, $g^{si}(x)$ $(s = 0,\cdots,s_{\max}, i = 1,\cdots,i_{\max}(s), j = 1,\cdots,D,$ $l = 0,\cdots,m)$ defined on $(0,\delta)$, and given there by convergent Puiseux expansions.

The above objects have the following properties

- (Estimates) For $(x,y) \in \Omega_1$ with $0 < x < \delta$ and $\psi_{s-1}(x) < y < \psi_s(x)$, we have $\left|\partial^\alpha A^s_{ij}(x,y)\right|$, $|\partial^\alpha \varphi^s_i(x,y)| \le C\left[\min\left(|y-\psi_s(x)|, |y-\psi_{s-1}(x)|\right)\right]^{-|\alpha|}$ for $|\alpha| \le m$.
- (Condition for sections) Let $F = (F_1,\cdots,F_D) \in C^m\left(\Omega_1,\mathbb{R}^D\right)$, and suppose $J_xF \in H(x)$ for all $x \in \Omega_1$.

  Then for $s = 1,\cdots,s_{\max}$, $i = 1,\cdots,k_s$, $x \in (0,\delta)$, $\psi_{s-1}(x) < y < \psi_s(x)$, we have

  $$F_{\pi_s i}(x,y) + \sum_{D\ge j>k_s} A^s_{ij}(x,y)\,F_{\pi_s j}(x,y) = \varphi^s_i(x,y)\,; \tag{130}$$

  and for $s = 0,1,\cdots,s_{\max}$, $i = 1,\cdots,i_{\max}(s)$, $x \in (0,\delta)$, we have

  $$\sum_{j=1}^{D}\sum_{l=0}^{m} \theta^{si}_{jl}(x)\,\partial^l_y F_j(x,\psi_s(x)) = g^{si}(x)\,; \tag{131}$$

  and

  $$J_{(0,0)}F_j = 0 \tag{132}$$

  for all j.

  Conversely, if $F = (F_j)_{j=1,\cdots,D} \in C^m_{loc}\left(\mathbb{R}^2,\mathbb{R}^D\right)$ satisfies (130), (131), (132), then F is a section of $\mathcal{H}$ over $\Omega_\delta^{\text{closure}}$.

Next, we set (for $s = 1, \cdots, s_{\max}$):

$$E_s^+ = \left\{ (x, y) \in \mathbb{R}^2 : 0 \le x \le \delta,\ 0 \le \psi_s(x) - y \le \frac{2}{3}(\psi_s - \psi_{s-1}(x)) \right\}$$

and

$$E_s^- = \left\{ (x, y) \in \mathbb{R}^2 : 0 \le x \le \delta, 0 \le y - \psi_{s-1}(x) \le \frac{2}{3}(\psi_s(x) - \psi_{s-1}(x)) \right\}.$$

Then $E_s^{+,\text{interior}} \cup E_s^{-,\text{interior}} = E_s$.

On $E_s^{+,\text{interior}}$ we have $\left|\partial^\alpha A_{ij}^s(x)\right|, \left|\partial^\alpha \varphi_i^s(x,y)\right| \le C(\psi_s(x) - y)^{-|\alpha|}$ for $|\alpha| \le m$, and on $E_s^{-,\text{ interior}}$ we have $\left|\partial^\alpha A_{ij}^s(x)\right|, \left|\partial^\alpha \varphi_i^s(x,y)\right| \le C(y - \psi_{s-1}(x))^{-|\alpha|}$ for $|\alpha| \le m$. We may apply Lemma 6.2 after a change of variables of the form $(\bar{x}, \bar{y}) = (x, \pm(y - \psi(x)))$. Thus, we obtain the following objects, with properties described below.

- Semialgebraic functions $\theta_{jl}^{+,si}(x)$, $g^{+,si}(x)$, $i = 1, \cdots, i_{\max}^+(s)$, $\theta_{jl}^{-,si}(x)$, $g^{-,si}(x)$, $i = 1, \cdots, i_{\max}^-(s)$, $l = 0, \cdots, m$, defined on $(0, \delta)$ (smaller δ).
- Semialgebraic functions $\tilde{\theta}_{jl}^{+,si}(x)$, $\tilde{g}^{+,si}(x)$, $i = 1, \cdots, \tilde{i}_{\max}^+(s)$, $\tilde{\theta}_{jl}^{-,si}(x)$, $\tilde{g}^{-,si}(x)$, $i = 1, \cdots, \tilde{i}_{\max}^-(s)$, $l = 0, \cdots, m$, defined on $(0, \delta)$ (smaller δ).

The properties for these functions are as follows: Let $F = (F_1, \cdots, F_D) \in C_{loc}^m(\mathbb{R}^2, \mathbb{R}^D)$ satisfy (130) in $E_s^{+,\text{interior}}$ and $J_{(0,0)}F = 0$. Then

(133) $$\sum_{\substack{1 \le j \le D \\ 0 \le l \le m}} \theta_{jl}^{+,si} \partial_y^l F_j(x, \psi_s(x)) = g^{+,si}(x)$$

for $x \in (0, \delta)$ and all i, and

(134) $$\sum_{\substack{1 \le j \le D \\ 0 \le l \le m}} \tilde{\theta}_{jl}^{+,si} \partial_y^l F_j(x, \psi_s(x)) = \tilde{g}^{+,si}(x) + o(1) \text{ as } x \to 0^+$$

for $x \in (0, \delta)$ and all i. Similarly, let $F = (F_1, \cdots, F_D) \in C_{loc}^m(\mathbb{R}^2, \mathbb{R}^D)$ satisfy (130) in $E_s^{-,\text{interior}}$ and $J_{(0,0)}F = 0$. Then

(135) $$\sum_{\substack{1 \le j \le D \\ 0 \le l \le m}} \theta_{jl}^{-,si} \partial_y^l F_j(x, \psi_{s-1}(x)) = g^{-,si}(x)$$

for $x \in (0, \delta)$ and all i, and

(136) $$\sum_{\substack{1 \le j \le D \\ 0 \le l \le m}} \tilde{\theta}_{jl}^{-,si} \partial_y^l F_j(x, \psi_{s-1}(x)) = \tilde{g}^{-,si}(x) + o(1) \text{ as } x \to 0^+$$

for all i.

(137) Conversely, fix s and suppose we are given semialgebraic functions $f_{jl}^{+,s}(x)$ on $(0,\delta)$ satisfying

$$\sum_{\substack{1\le j\le D\\0\le l\le m}} \theta_{jl}^{+,si} f_{jl}^{+,s}(x) = g^{+,si}(x) \text{ (all } i)$$

and

$$\sum_{\substack{1\le j\le D\\0\le l\le m}} \tilde{\theta}_{jl}^{+,si} f_{jl}^{+,s}(x) = \tilde{g}^{+,si}(x) + o(1) \text{ as } x\to 0^+ \text{ (all } i).$$

Then there exists a semialgebraic function $F=(F_1,\cdots,F_D)\in C^m\left(E_s^+,\mathbb{R}^D\right)$ such that (130) holds in $E_s^{+,\text{interior}}$ and $\partial_y^l F_j(x,\psi_s(x)) = f_{jl}^{+,s}(x)$ and $J_{(0,0)}F_j = 0$ for all j.

(138) Similarly, fix s and suppose we are given we are given semialgebraic functions $f_{jl}^{-,s}(x)$ on $(0,\delta)$ satisfying

$$\sum_{\substack{1\le j\le D\\0\le l\le m}} \theta_{jl}^{-,si} f_{jl}^{-,s}(x) = g^{-,si}(x) \text{ (all } i)$$

and

$$\sum_{\substack{1\le j\le D\\0\le l\le m}} \tilde{\theta}_{jl}^{-,si} f_{jl}^{-,s}(x) = \tilde{g}^{-,si}(x) + o(1) \text{ as } x\to 0^+ \text{ (all } i).$$

Then there exists a semialgebraic function $F=(F_1,\cdots,F_D)\in C^m\left(E_s^-,\mathbb{R}^D\right)$ such that (130) holds in $E_s^{-,\text{interior}}$ and $\partial_y^l F_j(x,\psi_s(x)) = f_{jl}^{-,s}(x)$ and $J_{(0,0)}F_j = 0$ for all j.

(139) Moreover, if $F=(F_1,\cdots,F_D)\in C^m\left(E_s^{\text{closure}},\mathbb{R}^D\right)$ with $J_{(0,0)}F=0$, then $f_{jl}^{+,s} = \partial_y^l F_j(x,\psi_s(x))$ and $f_{jl}^{-,s} = \partial_y^l F_j(x,\psi_{s-1}(x))$ satisfy the key hypothesis of Lemma 6.5, namely,

$$f_{jl}^{+,s}(x) - \sum_{k=0}^{m-l} \frac{1}{k!} f_{j(l+k)}^{-,s}(x)\left(\psi_s(x)-\psi_{s-1}(x)\right)^k = o\left(\left[\psi_s(x)-\psi_{s-1}(x)\right]^{m-l}\right) \text{ as } x\to 0^+$$

by Taylor's theorem.

Now, suppose $F=(F_1,\cdots,F_D)\in C^m_{loc}\left(\mathbb{R}^2,\mathbb{R}^D\right)$ is a section of $\mathcal{H}$ over Ω_δ. Then, setting $f_{jl}^s(x) = \partial_y^l F_j(x,\psi_s(x))$ for $x\in(0,\delta)$ (smaller δ), we learn that (because the F_j satisfy (130), (131), (132)), properties (131)$\cdots$(136) yield a collection of assertions of the form

(140) $$\sum_{\substack{j=1,\cdots,D\\l=0,\cdots,m}} \theta_{jl}^{\#,si}(x) f_{jl}^s(x) = g^{\#,si}(x) \text{ on } (0,\delta)$$

and

(141) $$\sum_{\substack{j=1,\cdots,D\\l=0,\cdots,m}} \tilde{\theta}_{jl}^{\#,si}(x) f_{jl}^s(x) = \tilde{g}^{\#,si}(x) + o(1) \text{ as } x\to 0^+;$$

and also from (139) we have

$$f^s_{jl}(x) = \sum_{k=0}^{m-l} \frac{1}{k!} f^{s-1}_{j(l+k)}(x) \left[\psi_s(x) - \psi_{s-1}(x)\right]^k + o\left(\left[\psi_s(x) - \psi_{s-1}(x)\right]^{m-l}\right) \text{ as } x \to 0^+. \tag{142}$$

Conversely, if the $f^s_{jl}(x)$ are semialgebraic functions of one variable, satisfying (140), (141), and (142), then for each $s = 1, \cdots, s_{\max}$ there exist $F^s_+ = (F^s_{+,1}, \cdots, F^s_{+,D}) \in C^m\left(E^{s,\text{ closure}}_+, \mathbb{R}^D\right)$, $F^s_- = (F^s_{-,1}, \cdots, F^s_{-,D}) \in C^m\left(E^{s,\text{ closure}}_-, \mathbb{R}^D\right)$ semialgebraic such that (130), (131), (132) hold in E^+_s, E^-_s, respectively and

- $\partial^l_y F^s_{+,j}(x, \psi_s(x)) = f^s_{jl}(x)$,
- $\partial^l_y F^s_{-,j}(x, \psi_{s-1}(x)) = f^{s-1}_{jl}(x)$, and
- $J_{(0,0)} F^s_+ = J_{(0,0)} F^s_- = 0$.

Note that F^s_+ is a section of $\mathcal{H}$ over E^+_s, and F^s_- is a section of $\mathcal{H}$ over E^-_s. Thanks to (142) and Lemma 6.5, we may patch together F^s_+, F^s_- into a semialgebraic $F_s = (F_{s,1}, \cdots, F_{s,D}) \in C^m(E^{\text{closure}}_s, \mathbb{R}^D)$ such that $J_{(0,0)} F_s = 0$, F_s is a section of $\mathcal{H}$ over E^{closure}_s, and $\partial^l_y F_{sj}(x, \psi(x)) = f^s_{jl}(x)$ and $\partial^l_y F_{sj}(x, \psi_{s-1}(x)) = f^{s-1}_{jl}(x)$. Because of these conditions, the F_s ($s = 1, \cdots, s_{\max}$) fit together (their transverse derivatives up to order m match at the boundaries where the E_s meet), so using also Corollary 3.2, we obtain from the F_s a single semialgebraic $F = (F_1, \cdots, F_D) \in C^m_{loc}(\mathbb{R}^2, \mathbb{R}^D)$ such that $J_{(0,0)} F = 0$, and F is a section of $\mathcal{H}$ over Ω_δ. Thus, we have proven Lemma 6.1.

7 Proof of Lemma 4.1 (Main Lemma)

From the Second Main Lemma (Lemma 6.1), we can easily deduce Lemma 4.1. Indeed, suppose $\mathcal{H} = (H(x,y))_{(x,y)\in\Omega_\delta}$ is as in the hypotheses of Lemma 4.1. Let $\theta^{si}_{jl}, g^{si}, \tilde{\theta}^{si}_{jl}, \tilde{g}^{si}, \psi_s$ be as in Lemma 6.1. For $x \in (0, \delta)$ with δ small enough, we introduce the following objects:

$$\begin{aligned}
W(x) &= \left\{ \left(\xi^s_{jl}\right)_{\substack{0\le s\le s_{\max}\\ 0\le l\le m\\ 1\le j\le D}} \in \mathbb{R}^{(s_{\max}+1)\cdot(m+1)\cdot D} : \sum_{j,l} \theta^{si}_{jl}(x)\, \xi^s_{jl} = g^{si}(x), \text{ each } s, i \right\}, \\
\mathcal{F}\left(\left(\xi^s_{jl}\right), x\right) &= \sum_{s,i} \left| \sum_{j,l} \tilde{\theta}^{si}_{jl}(x)\, \xi^s_{jl} - \tilde{g}^{si}(x) \right| \\
&\quad + \sum_{s\neq 0} \sum_{j,l} \frac{\left| \xi^s_{jl} - \sum_{k=0}^{m-l} \frac{1}{k!} \xi^{s-1}_{j(l+k)} \cdot \left(\psi_s(x) - \psi_{s-1}(x)\right)^k \right|}{\left[\psi_s(x) - \psi_{s-1}(x)\right]^{m-l}}, \\
\mathcal{F}_{\min}(x) &= \inf\left\{ \mathcal{F}\left(\left(\xi^s_{jl}\right), x\right) : \left(\xi^s_{jl}\right) \in W(x) \right\}, \text{ and} \\
\Xi_{OK}(x) &= \left\{ \left(\xi^s_{jl}\right) \in W(x) : \mathcal{F}\left(\left(\xi^s_{jl}\right), x\right) \le \mathcal{F}_{\min}(x) + x \right\}.
\end{aligned}$$

Because $\theta_{jl}^{si}, g^{si}, \tilde{\theta}_{jl}^{si}, \tilde{g}^{si}, \psi_s$ are semialgebraic, the objects defined above depend semialgebraically on x. Thanks to conclusion (72) of Lemma 6.1, each $W(x)$ and each $\Xi_{OK}(x)$ is non-empty, and

$$\mathcal{F}_{\min}(x) \to 0 \text{ as } x \to 0^{+}. \tag{143}$$

From Theorem 4 we obtain

(144) Semialgebraic functions $\xi_{jl}^{s}(x)$ on $(0,\delta)$ such that $\left(\xi_{jl}^{s}(x)\right) \in \Xi_{OK}(x)$ for each $x \in (0,\delta)$.

In particular, for $x \in (0,\delta)$, we have

$$\sum_{j,l} \theta_{jl}^{s,i}(x)\,\xi_{jl}^{s}(x) \;=\; g^{si}(x) \text{ for each } s,i,j; \tag{145}$$

$$\left|\sum_{j,l} \tilde{\theta}_{jl}^{si}(x)\,\xi_{jl}^{s}(x) - \tilde{g}^{si}(x)\right| \;\leq\; \left[\mathcal{F}_{\min}(x) + x\right] \text{ for each } s,i; \tag{146}$$

and

$$\begin{aligned} &\left|\xi_{jl}^{s}(x) - \sum_{k=0}^{m-l} \frac{1}{k!}\xi_{j(l+k)}^{s-1}(x)\cdot\left(\psi_s(x) - \psi_{s-1}(x)\right)^{k}\right| \\ \leq\;& \left[\mathcal{F}_{\min}(x) + x\right]\cdot\left(\psi_s(x) - \psi_{s-1}(x)\right)^{m-l}, \text{ for each } s,j,l \;\; (s \neq 0)\,. \end{aligned} \tag{147}$$

From (143), (146), (147), we see that

$$\sum_{j,l} \tilde{\theta}_{jl}^{si}(x)\,\xi_{jl}^{s}(x) = \tilde{g}^{si}(x) + o(1) \text{ as } x \to 0^{+}, \tag{148}$$

and

$$\begin{aligned} &\xi_{jl}^{s}(x) - \sum_{k=0}^{m-l} \frac{1}{k!}\xi_{j(l+k)}^{s-1}(x)\cdot\left(\psi_s(x) - \psi_{s-1}(x)\right)^{k} \\ =\;& o\left(\left[\psi_s(x) - \psi_{s-1}(x)\right]^{m-l}\right) \text{ as } x \to 0^{+}. \end{aligned} \tag{149}$$

Finally, from (144), (145), (148), (149), and the assertion (73) in Lemma 6.1, we conclude that $\mathcal{H}|_{\Omega_{\delta'}}$ has a C^{m}_{loc} semialgebraic section for some $\delta' < \delta$.

This completes the proof of Lemma 4.1 and that of Theorem 1.

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