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Canonical coverings of Enriques surfaces in characteristic 2

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Abstract. Let $\bar{Y}$ be a normal surface that is the canonical μ_2- or α_2-covering of a classical or supersingular Enriques surface in characteristic 2. We determine all possible configurations of singularities on $\bar{Y}$, and for each configuration we describe which type of Enriques surfaces (classical or supersingular) appear as quotients of $\bar{Y}$.

1. Introduction

Let X be an Enriques surface over an algebraically closed field k (see Section 2.1 for the definition). It is known that the torsion part Pic^τ_X of the Picard scheme of X is a finite group scheme of order 2, and thus there is a canonical G-covering $\bar{Y} \to X$, where $G := \mathcal{H}om(\mathrm{Pic}^\tau_X, \mathbb{G}_\mathrm{m})$ is the Cartier dual of Pic^τ_X. If $\operatorname{char} k \neq 2$, then Pic^τ_X and G are both isomorphic to $\mathbb{Z}/2\mathbb{Z}$, the covering is a finite étale $\mathbb{Z}/2\mathbb{Z}$-covering, and $\bar{Y}$ is a smooth K3 surface. If $\operatorname{char} k = 2$, then the situation is more complicated: there are three possibilities for G, namely $\mathbb{Z}/2\mathbb{Z}$, μ_2, and α_2. In this paper we study the *classical* and *supersingular Enriques surfaces* in characteristic 2, that is, $G = \mu_2$ or $G = \alpha_2$ respectively. In these cases the canonical G-covering $\bar{Y}$ is always singular, and $\bar{Y}$ may not be even birational to a K3 surface.

In this paper, we restrict our attention to the case where $\bar{Y}$ is normal, and we discuss two problems: to determine the possible configurations of singularities on $\bar{Y}$, and to describe all Enriques quotients of $\bar{Y}$.

1.1. Previous research

For the singularities, the following is known.

Theorem 1.1 ([CD89, Proposition 1.3.1 and Theorem 1.3.1]). *Let $\bar{Y}$ be the canonical covering of a classical or supersingular Enriques surface in characteristic 2. Then $\bar{Y}$ is K3-like (see Section 2.1). If $\bar{Y}$ is normal, then one of the following holds:*

- *$\bar{Y}$ has only rational double points (RDPs) as singularities. In this case $\bar{Y}$ is an RDP K3 surface (Section 2.1).*
- *$\bar{Y}$ has only isolated singularities and contains a non-RDP singularity. In this case there is exactly one non-RDP singularity and it is an elliptic double point (EDP), and $\bar{Y}$ is a rational surface.*

(However, the proof of [CD89, Proposition 1.3.1] contains a gap. See Remark 4.2.)
The following description is given by Ekedahl–Hyland–Shepherd-Barron.

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Theorem 1.2 (Ekedahl–Hyland–Shepherd-Barron [EHSB12, Corollary 3.7(3) and Corollary 6.16]). *Let $\bar{Y}$ be as in Theorem 1.1 and assume $\bar{Y}$ is normal. Then,*

- *The tangent sheaf $T_{\bar{Y}}$ is free.*
- *If $\mathrm{Sing}(\bar{Y})$ consists only of RDPs, then $\mathrm{Sing}(\bar{Y})$ is one of*

$$12A_1,\ 8A_1 + D_4^0,\ 6A_1 + D_6^0,\ 5A_1 + E_7^0,\ 3D_4^0,\ D_4^0 + D_8^0,\ D_4^0 + E_8^0,\ D_{12}^0.$$

They also claimed that all global derivations $D \in H^0(\bar{Y}, T_{\bar{Y}})$ are p-closed ([EHSB12, Corollary 7.3]), but their proof covers only the generic case.

Recently Schröer proved the following results.

Theorem 1.3 (Schröer [Sch21, Theorems 6.3–6.4 and Sections 13–15]). *Let $\bar{Y}$ be as in Theorem 1.1 and assume $\bar{Y}$ is normal.*

(1) *The quotient of $\bar{Y}$ by a global fixed-point-free derivation that is either of multiplicative type or of additive type is an Enriques surface.*

(2) *Under a mild assumption on $\bar{Y}$, all global derivations $D \in H^0(\bar{Y}, T_{\bar{Y}})$ are p-closed, and most of them (those belonging to the complement of finitely many lines) are fixed-point-free. Hence $\bar{Y}$ admits a 1-dimensional family of Enriques quotients parametrized by a nonempty open subscheme of $\mathbb{P}(H^0(\bar{Y}, T_{\bar{Y}})) \cong \mathbb{P}^1$.*

(3) *There exists an example of $\bar{Y}$ with an EDP.*

(4) *If $\mathrm{Sing}(\bar{Y})$ contains an EDP, then $\mathrm{Sing}(\bar{Y})$ consists precisely of that point.*

(5) *There is a method of a construction, from a given rational elliptic surface $J \to \mathbb{P}^1$ satisfying certain assumptions on singular fibers, of a J-torsor $X \to \mathbb{P}^1$ that is an Enriques surface whose canonical covering $\bar{Y}$ is birational to the Frobenius base change $J^{(2/\mathbb{P}^1)}$ and moreover having the same type of singularities as $J^{(2/\mathbb{P}^1)}$.*

Ekedahl–Hyland–Shepherd-Barron did not show whether every configuration in Theorem 1.2 is actually possible. The method of Theorem 1.3(5) applied to various rational elliptic surfaces would construct examples for all configurations in Theorem 1.2, but this is not explicitly mentioned, and classical and supersingular surfaces are not explicitly distinguished.

Katsura–Kondo ([KK15, Section 4] and [KK18, Section 3]) (resp. Kondo ([Kon21, Section 3])) described the families of Enriques quotients of two (resp. one) explicit examples of canonical coverings $\bar{Y}$ with $\mathrm{Sing}(\bar{Y}) = 12A_1$ (resp. $\mathrm{Sing}(\bar{Y}) = 8A_1 + D_4^0$). They consist of both classical and supersingular Enriques quotients (resp. only classical ones).

1.2. Our results

Now we shall state the main results of this paper. Let $\bar{Y} \to X$ be as above and assume $\bar{Y}$ is normal. We show that the conclusion of Theorem 1.3(2) holds unconditionally. We describe the (2-dimensional) restricted Lie algebra $H^0(\bar{Y}, T_{\bar{Y}})$ and the (1-dimensional family of) Enriques quotients of $\bar{Y}$. The answers depend on the configuration of singularities on $\bar{Y}$ and, perhaps surprisingly, if $\mathrm{Sing}(\bar{Y})$ is other than $12A_1$, then the Enriques quotients of $\bar{Y}$ are either all classical or all supersingular. We also determine which configurations of singularities actually occur. It turns out that in the RDP case every configuration in Theorem 1.2 is possible. In the EDP case there is only one possible configuration: one EDP of type E_{12} (the singularity defined by $z^2 + x^3 + y^7 = 0$, see Section 3.1 for the precise definition and properties).

THEOREM 1.4. *Let $\bar{Y}$ be a normal surface that is the canonical covering of some classical or supersingular Enriques surface in characteristic $p = 2$. (Then, by Theorem 1.2, the tangent sheaf $T_{\bar{Y}}$ is free and hence $\mathfrak{g} := H^0(\bar{Y}, T_{\bar{Y}})$ is 2-dimensional.)*

Then all element $D \in \mathfrak{g}$ are p-closed and most of them (those belonging to the complement of finitely many lines) are fixed-point-free. Hence, as above, $\bar{Y}$ admits a 1-dimensional family of Enriques quotients, parametrized by a nonempty open subscheme of $\mathbb{P}(\mathfrak{g}) \cong \mathbb{P}^1$. Moreover, according to the singularities of $\bar{Y}$, the following assertions hold.

(1) *Suppose* $\mathrm{Sing}(\bar{Y})$ *is* $12A_1$. *Then the subset $\mathfrak{l} := \{D \in \mathfrak{g} \mid D^p = 0\}$ is a line, and each nonzero element of $\mathfrak{l}$ is fixed-point-free. Hence there is exactly one Enriques quotient of $\bar{Y}$ that is supersingular and all other Enriques quotients are classical.*

(2) *Suppose* $\mathrm{Sing}(\bar{Y})$ *is one of* $8A_1 + D_4^0$, $6A_1 + D_6^0$, *or* $5A_1 + E_7^0$. *Then the subset $\mathfrak{l} := \{D \in \mathfrak{g} \mid D^p = 0\}$ is a line, and each nonzero element of $\mathfrak{l}$ is not fixed-point-free. Hence all Enriques quotients of $\bar{Y}$ are classical.*

(3) *Suppose* $\mathrm{Sing}(\bar{Y})$ *is one of* $3D_4^0$, $D_4^0 + D_8^0$, $D_4^0 + E_8^0$, *or* D_{12}^0. *Then all $D \in \mathfrak{g}$ satisfy $D^p = 0$. Hence all Enriques quotients of $\bar{Y}$ are supersingular.*

(4) *Suppose* $\mathrm{Sing}(\bar{Y})$ *contains an EDP. Then the EDP is of type E_{12} and this is the only singularity of $\bar{Y}$, and all $D \in \mathfrak{g}$ satisfy $D^p = 0$. Hence all Enriques quotients of $\bar{Y}$ are supersingular.*

In cases (1) and (2), the restricted Lie algebra $\mathfrak{g}$ is non-abelian and the image of the bracket is $\mathfrak{l}$. In cases (3) and (4), $\mathfrak{g}$ is abelian.

THEOREM 1.5. *The* 9 *configurations of* $\mathrm{Sing}(\bar{Y})$ *mentioned in Theorem 1.4 are precisely the ones that can occur for the normal canonical coverings of classical or supersingular Enriques surfaces in characteristic* 2.

As explained above, Theorem 1.5 follows implicitly from Theorem 1.3(5) of Schröer in the RDP cases, and is explicitly proved by Schröer (Theorem 1.3(3)) in the EDP case (modulo the assertion that the EDP is E_{12}).

COROLLARY 1.6. *The possible configurations of singularities on the normal canonical coverings of classical (resp. supersingular) Enriques surfaces in characteristic* 2 *are*

$$12A_1,\ 8A_1 + D_4^0,\ 6A_1 + D_6^0,\ \textit{and}\ 5A_1 + E_7^0$$
$$\textit{(resp.}\ 12A_1,\ 3D_4^0,\ D_4^0 + D_8^0,\ D_4^0 + E_8^0,\ D_{12}^0,\ \textit{and}\ E_{12}\textit{)}.$$

This paper is organized as follows. In Section 2, we introduce some notions and basic facts on K3 and Enriques surfaces, derivations, and restricted Lie algebras. In Section 3 we discuss p-closed derivation quotients of rational double point (RDP) singularities, elliptic double point (EDP) singularities, and K3-like surfaces (mainly in characteristic 2).

In Section 4 we prove Theorem 1.4. Our proof relies on the previous works of Ekedahl–Hyland–Shepherd-Barron [EHSB12] and Schröer [Sch21], and techniques from the recent preprint [Mat21] of the author on μ_p- and α_p-actions on K3 surfaces.

In Section 5, we recall the examples of $12A_1$ and $8A_1 + D_4^0$ given by Katsura–Kondo and Kondo, and give examples of the remaining configurations, thus proving Theorem 1.5. Our constructions for the RDP cases are either straight generalizations of Kondo's (for classical cases) or influenced by his (for supersingular cases). A difference is that our presentation deals with regular derivations on RDP K3 surfaces, which would be easier to compute than rational derivations on smooth K3 surfaces used in Katsura–Kondo's and Kondo's. Also, most of our constructions can be viewed as explicit special cases of Schröer's constructions.

2. Preliminaries

Throughout the paper we work over an algebraically closed field k of characteristic $p \geq 0$.

2.1. Enriques surfaces and K3-like surfaces

A *K3 surface* is a proper smooth surface X with $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$. An *Enriques surface* is a proper smooth surface X with K_X numerically trivial and $b_2(X) = 10$. Here $b_i(X) := \dim H^i_{\text{ét}}(X, \mathbb{Q}_l)$ is the l-adic Betti number for an auxiliary prime $l \neq \operatorname{char} k$.

Suppose X is an Enriques surface. In characteristic $\neq 2$, we have $K_X \not\sim 0$, $2K_X \sim 0$, $\operatorname{Pic}^\tau_X \cong \mathbb{Z}/2\mathbb{Z} \cong \mu_2$. Here $\sim$ is the linear equivalence. In characteristic 2, exactly one of the following holds ([BM76, Section 3]).

- $K_X \sim 0$, $\operatorname{Pic}^\tau_X \cong \mu_2$. In this case X is called *singular*.
- $K_X \not\sim 0$, $2K_X \sim 0$, $\operatorname{Pic}^\tau_X \cong \mathbb{Z}/2\mathbb{Z}$. In this case X is called *classical*.
- $K_X \sim 0$, $\operatorname{Pic}^\tau_X \cong \alpha_2$. In this case X is called *supersingular*.

In any case, the isomorphism $H^1_{\text{fl}}(X, G) \cong \operatorname{Hom}(G^\vee, \operatorname{Pic}^\tau_X)$ of [Sch21, Proposition 4.1] (where G is a finite commutative group scheme and $G^\vee = \mathcal{H}om(G, \mathbb{G}_{\mathrm{m}})$ is its Cartier dual) induces a canonical $(\operatorname{Pic}^\tau_X)^\vee$-torsor $\bar{Y} \to X$, which we call the *canonical covering* of X.

An *RDP K3 surface* (resp. *RDP Enriques surface*) is a proper surface with only RDPs as singularities (if any) whose minimal resolution is a smooth K3 (resp. Enriques) surface.

We say that an RDP Enriques surface is classical or supersingular if its minimal resolution is so.

A *K3-like surface*, following [BM76], is a proper reduced Gorenstein surface X, not necessarily normal, whose dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X$ and satisfying $h^i(X, \mathcal{O}_X) = 1, 0, 1$ for $i = 0, 1, 2$. Any RDP K3 surface is K3-like. Any K3-like surface with $b_1 = 0$ is either an RDP K3 surface or a (normal or non-normal) rational surface by [CD89, proof of Theorem 1.3.1].

A *genus one fibration* on a smooth proper surface X is a morphism $X \to \mathbb{P}^1$, not necessarily with a section, whose generic fiber is a curve of arithmetic genus one. It is called an *elliptic fibration* (resp. a *quasi-elliptic fibration*) if the generic fiber is a smooth elliptic curve (resp. a cuspidal rational curve). We do not use quasi-elliptic fibrations in this paper.

Proposition 2.1 ([CD89, Theorems 5.7.2 and 5.7.5]). *Let $X \to \mathbb{P}^1$ be a genus one fibration on a classical Enriques surface in characteristic* 2. *Then there are exactly* 2 *multiple fibers, and each is either a smooth ordinary elliptic curve or a singular fiber of additive type.*

2.2. Derivations

A *(regular) derivation* on a scheme X is a k-linear endomorphism D of $\mathcal{O}_X$ satisfying $D(ab) = aD(b) + D(a)b$.

The *fixed locus* $\mathrm{Fix}(D)$ of a derivation D is the closed subscheme of X corresponding to the ideal $(\mathrm{Im}(D))$ generated by $\mathrm{Im}(D) = \{D(a) \mid a \in \mathcal{O}_X\}$. If X is normal and $D \neq 0$, then the *divisorial part* of $\mathrm{Fix}(D)$ is denoted by (D).

Assume X is a smooth integral surface and $D \neq 0$. Then we define the *isolated part* of $\mathrm{Fix}(D)$, denoted $\langle D \rangle$, as follows. If we write $D = f(g\partial/\partial x + h\partial/\partial y)$ with g, h coprime for some local coordinate x, y, then (D) and $\langle D \rangle$ correspond to the ideal (f) and (g, h) respectively.

Suppose for simplicity that X is integral. Then a *rational derivation* on X is a global section of $\mathrm{Der}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} k(X)$, where $\mathrm{Der}(\mathcal{O}_X)$ is the sheaf of derivations on X. Thus, a rational derivation is locally of the form $f^{-1}D$ with f a regular function and D a regular derivation. We extend the notion of divisorial and isolated parts to rational derivations by $(f^{-1}D) = (D) - \mathrm{div}(f)$ and $\langle f^{-1}D \rangle = \langle D \rangle$.

Suppose $\mathrm{char}\, k = p > 0$. A derivation D is said to be *of multiplicative type* (resp. *of additive type*) if $D^p = D$ (resp. $D^p = 0$). Such derivations correspond to actions of the group scheme μ_p (resp. α_p) on the scheme. More generally, D is said to be *p-closed* if there exists $h \in k(X)$ with $D^p = hD$.

We recall the Rudakov–Shafarevich formula and the Katsura–Takeda formula.

Theorem 2.2 (Rudakov–Shafarevich [RS76, Corollary 1 to Proposition 3]). *Let D be a nonzero p-closed rational derivation on a smooth variety X in characteristic $p > 0$. Denote by $\pi \colon X \to X^D = Y$ the quotient morphism. Then we have*

$$K_X \sim \pi^* K_Y + (p-1)(D),$$

where $\sim$ is the linear equivalence.

Theorem 2.3 (Katsura–Takeda [KT89, Proposition 2.1]). *Let D be a nonzero rational derivation on a smooth proper surface X. Then*

$$\deg c_2(X) = \deg\langle D \rangle - K_X \cdot (D) - (D)^2.$$

In characteristic $p = 2$ we have the following corollary of the Rudakov–Shafarevich formula.

Proposition 2.4 ([EHSB12, Lemma 3.14]). *Let D be a nonzero p-closed rational derivation on a smooth variety X in characteristic $p = 2$. Then $K_X - (D)$ is divisible by* 2 *in* $\mathrm{Pic}(X)$.

Proof. Let $\pi\colon X \to X^D = Y$ the quotient morphism. Then the "dual" morphism $\pi'\colon Y \to X^{(2)}$ is purely inseparable of degree 2, hence is the quotient by some rational derivation D'. By replacing with a multiple we may assume $D'^2 = 0$. Then $I := \mathrm{Im}(D')$ is a fractional ideal of $X^{(2)}$. By removing a closed subscheme of X of codimension at least 2 (which does not change the Picard group), we may assume I is principal, thus identified with a divisor Δ. Then $(D') = \pi'^*(\Delta)$. By the Rudakov–Shafarevich formula, we have

$$K_X - (D) \sim \pi^* K_Y \sim \pi^*((D') + \pi'^*(K_{X^{(2)}})) \sim \pi^*\pi'^*(\Delta + K_{X^{(2)}}),$$

and the image of $\pi^*\pi'^* = F^*$ is divisible by p. □

2.3. Restricted Lie algebras of dimension 2

Recall that a *restricted Lie algebra* over a field k of characteristic $p > 0$ is a k-vector space $\mathfrak{g}$ together with two operation, the bracket $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and the p-th power map $-^{(p)}\colon \mathfrak{g} \to \mathfrak{g}$, satisfying certain conditions. An example is $H^0(X, T_X)$ for a scheme X, where the bracket is the usual one ($[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$) and the p-th power $D^{(p)}$ of D is the p-th iterate $D^p = D \circ \cdots \circ D$. (In this example, $\circ$ is defined only on $\mathcal{E}nd(\mathcal{O}_X) \supset T_X$, but $[D_1, D_2]$ and D^p belong to T_X.)

We say that an element x of a restricted Lie algebra $\mathfrak{g}$ is *p-closed* if it satisfies $x^{(p)} = \lambda x$ for some scalar $\lambda \in k$, and that it is *of multiplicative type* (resp. *of additive type*) if we can take $\lambda \neq 0$ (resp. $\lambda = 0$). We also say that a line $[x] = kx \subset \mathfrak{g}$ generated by a nonzero element x is *p-closed*, *of multiplicative type*, or *of additive type* if it contains a nonzero element with those properties.

Note that if X is proper and T_X is free, then a line of $\mathfrak{g} = H^0(X, T_X)$ is p-closed (in this sense, where the ratio is a scalar) if and only if some, equivalently any, nonzero element in the line is p-closed (in the sense of Section 2.1, where the ratio can be any rational function).

Proposition 2.5 ([Wan13, Proposition A.3]). *There are exactly* 5 *isomorphism classes of restricted Lie algebras $\mathfrak{g}$ of dimension* 2 *(over a fixed algebraically closed field k of characteristic $p > 0$). In each case there is a basis x, y satisfying the following properties.*

(1) $[x, y] = y$, $x^{(p)} = x$, $y^{(p)} = 0$.

(2) $[x, y] = 0$, $x^{(p)} = 0$, $y^{(p)} = 0$.

(3) $[x, y] = 0$, $x^{(p)} = x$, $y^{(p)} = 0$.

(4) $[x, y] = 0$, $x^{(p)} = y$, $y^{(p)} = 0$.

(5) $[x, y] = 0$, $x^{(p)} = x$, $y^{(p)} = y$.

We will use the following observations to describe the restricted Lie algebra of the canonical coverings.

Corollary 2.6. *Let $\mathfrak{g}$ be as in Proposition 2.5.*

(1) *Suppose $\mathfrak{g}$ has at least 3 p-closed lines, among which at least 1 is of multiplicative type and at least 1 is of additive type. Then $\mathfrak{g}$ is of type (1), all lines are p-closed, and exactly 1 is of additive type and all others are of multiplicative type.*

(2) *Suppose $\mathfrak{g}$ has at least 2 lines of additive type. Then $\mathfrak{g}$ is of type (2), and all lines are p-closed of additive type.*

Proof. Given the classification, we can describe the p-closed lines in each case by a straightforward calculation (see below). We conclude that if $\mathfrak{g}$ is of type (1) or (2) in Proposition 2.5 then the p-closed lines are as described in the statement of this corollary; and if $\mathfrak{g}$ is of type (3) (resp. (4), resp. (5)), then exactly 1 (resp. 0, resp. $p+1$) line is of multiplicative type, exactly 1 (resp. 1, resp. 0) line is of additive type, and no other lines are p-closed. The assertions follow.

For example, if $\mathfrak{g}$ is of type (1) and $v = ax + by$, then $v^{(p)} = a^p x + a^{p-1} by = a^{p-1}(ax + by)$ is always proportional to v, and $v^{(p)} = 0$ if and only if $a = 0$. If $\mathfrak{g}$ is of type (5) and $v = ax + by \neq 0$, then $v^{(p)} = a^p x + b^p y$ is never 0, and it is proportional to v if and only if $\det \begin{pmatrix} a^p & b^p \\ a & b \end{pmatrix} = ab(a^{p-1} - b^{p-1}) = 0$. □

3. p-closed derivations and quotients

3.1. Derivations on RDPs and EDPs

Definition 3.1. An *elliptic singularity* is an isolated surface singularity $x \in X$ with $\mathrm{length}(R^1 f_* \mathcal{O})_x = 1$, where f is a resolution of singularity. An *elliptic double point* (EDP) is an elliptic singularity that is a double point.

Definition 3.2. In this paper, we say that a 2-dimensional local k-algebra in characteristic $p = 2$ is an *EDP of type E_{12}* if its completion is isomorphic to $k[[x,y,z]]/(z^2 + x^3 + y^7)$.

This is the quotient of $k[[X,Y]]$ by the derivation D defined by $D(X) = Y^6$ and $D(Y) = X^2$, with $x = X^2$, $y = Y^2$, $z = X^3 + Y^7$.

It is easy to see that it is an EDP whose minimal resolution consists of a rational cuspidal curve of self-intersection -1. We observe that $k[[x,y,z]]/(z^2 + x^3 + y^7 + \varepsilon)$ is also an EDP of type E_{12} if $\varepsilon \in (x^5, x^3y, x^2y^3, xy^4, y^9) \subset k[[x,y]]$.

This symbol E_{12} is used for the (exceptional unimodal) singularity in characteristic 0 defined by the same equation, and the index 12 stands for the Milnor number (i.e. $\dim_k k[[x,y,z]]/(F_x, F_y, F_z)$ for $k[[x,y,z]]/(F)$) in characteristic 0, although in characteristic 2 this is not the Milnor number (nor the Tjurina number). Instead we have the equality between the index and the degree $\deg\langle D \rangle$ of the derivation. The same equality also holds for RDPs of type A_1, D_{2n}^0, E_7^0, and E_8^0 ([Mat21, Corollary 3.9]).

Proposition 3.3. *Let $\bar{W} = \operatorname{Spec} B$ be an EDP of type E_{12} in characteristic 2 and D a p-closed derivation on $\bar{W}$ with $\operatorname{Fix}(D) = \varnothing$. Then $Z = \bar{W}^D$ is smooth.*

Proof. We may assume $B = k[[x,y,z]]/(z^2 + x^3 + y^7)$. The derivation D satisfies $x^2 D(x) + y^6 D(y) = 0$, hence $D(x) = y^6 b$ and $D(y) = x^2 b$ for some $b \in B$. In particular

$D(x)$ and $D(y)$ belong to the maximal ideal $\mathfrak{m}$ of B. Since $\mathrm{Fix}(D) = \varnothing$ we have $D(z) \in B^*$. Then the maximal ideal $\mathfrak{n}$ of B^D is generated by three elements

$$x' := x - D(z)^{-1}D(x)z,\ y' := y - D(z)^{-1}D(y)z,\ z' = z^2,$$

and since we have a relation

$$\begin{aligned} z' = z^2 = x^3 + y^7 &= x^2(x' + D(z)^{-1}D(x)z) + y^6(y' + D(z)^{-1}D(y)z) \\ &= x^2x' + y^6y' \in \mathfrak{n}^2, \end{aligned}$$

it is in fact generated by the two elements x' and y'. Thus $\bar{W}^D$ is smooth. $\square$

LEMMA 3.4 (cf. [Sch21, Propositions 2.3–2.4]). *Suppose B is the localization or the completion at a closed point of a normal surface in characteristic $p > 0$. Assume the closed point is a singularity with $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3$, where $\mathfrak{m} \subset B$ is the maximal ideal. Suppose B admits a p-closed derivation D with $\mathrm{Fix}(D) = \varnothing$. Then,*

(1) *The tangent module $T_B = \mathrm{Der}(B)$ is a free B-module (of rank 2).*

(2) *An element $D' \in T_B$ has no fixed points if and only if the projection of D' to $T_B \otimes B/\mathfrak{m}$ belongs to the complement of a certain line.*

(3) *Assume $p = 2$. Suppose $D'_1, D'_2 \in T_B$ generate T_B and that $\mathrm{Fix}(D'_1) \neq \varnothing$. If B is an RDP of type A_{2n-1} for some $n \geq 1$ (resp. any other singularity), then D'_1 is not of additive type (resp. not of multiplicative type).*

(1) and (2) slightly generalize the results of Schröer [Sch21, Propositions 2.3–2.4], in which B is assumed to be of the form $k[[x, y, z]]/(z^p - f(x, y))$, and the proof is parallel. (1) also follows from [EHSB12, Corollary 3.7(2)]. (3) generalizes the case of A_1 proved in [EHSB12, Lemma 7.5].

PROOF. By [Mat21, Lemma 2.8], we can take $x, y, z \in \mathfrak{m}$ generating $\mathfrak{m}$ and satisfying $D(x) = D(y) = 0$. We may assume B is complete. Hence we may assume $B = k[[x, y, z]]/(F)$ with $F \in k[[x, y, z^p]]$. The tangent module T_B can be identified with the B-module $\{(a, b, c) \in B^3 \mid aF_x + bF_y + cF_z = 0\}$ by $D \mapsto (D(x), D(y), D(z))$. Here F_x, F_y, F_z are the images in B of the partial derivatives of F.

Since $F \in k[x, y, z^p]$ we have $F_z = 0$. Since the singularity is isolated (since B is normal), the ideal $(F_x, F_y, F_z) = (F_x, F_y)$ is of height 2. Since B is a hypersurface singularity, hence Cohen–Macaulay, this implies that F_x, F_y is a regular sequence. Hence T_B has a basis $D_1 = (0, 0, 1), D_2 = (F_y, -F_x, 0)$. This shows (1). Clearly $g_1D_1 + g_2D_2$ ($g_1, g_2 \in B$) has no fixed points if and only if $g_1 \in B^*$. This shows (2).

Now assume $p = 2$ and let D'_1, D'_2 be as in (3). We have $D'_1 = g_1D_1 + g_2D_2$ with $g_1 \in \mathfrak{m}$, $g_2 \in B^*$. Then we have $D'_1(x) = g_2F_y \neq 0$ and

$$\begin{aligned} (D'_1)^2(x) = D'_1(g_2F_y) &= (g_2F_y)_x g_2F_y + (g_2F_y)_y g_2F_x + (g_2F_y)_z g_1 \\ &= \big(g_2F_{yx} + (g_2)_xF_y + (g_2)_yF_x + (g_2)_z g_2^{-1} g_1\big) D'_1(x). \end{aligned}$$

Note that B is of type A_{2n-1} for some $n \geq 1$ if and only if $F_{xy} \in B^*$ (by [Mat21, Theorem 3.3(1)], B cannot be of type A_{2n}). Assume this is the case (resp. not the case). Then since $F_x, F_y, g_1 \in \mathfrak{m}$ and $g_2 \in B^*$, the coefficient of $D_1'(x)$ is an element of B^* (resp. $\mathfrak{m}$), in particular not equal to 0 (resp. 1). $\square$

3.2. Derivations on K3-like surfaces

REMARK 3.5. The derivation corresponding to the canonical μ_2- or α_2-covering of a classical or supersingular Enriques surface is fixed-point-free. This follows from Bombieri–Mumford's construction [BM76, Corollary in Section 3].

We also have a partial converse:

PROPOSITION 3.6 (cf. [Mat21, Sections 3–4] and [Sch21, Proposition 5.1]). *Let $\bar{Y}$ be a normal K3-like surface in characteristic* 2 *with only RDPs or EDPs of type E_{12} as singularities. Let D be a derivation of multiplicative (resp. additive) type satisfying* $\mathrm{Fix}(D) = \varnothing$. *Then,*

(1) *The quotient $\bar{X} := \bar{Y}^D$ is a classical (resp. supersingular) RDP Enriques surface, and $\bar{Y} \times_{\bar{X}} X$ is the canonical covering of the minimal resolution X of $\bar{X}$.*

(2) *Let $\bar{\pi} \colon \bar{Y} \to \bar{X}$ be the quotient map. If $w \in \bar{Y}$ is a closed point that is either a smooth point, an RDP of type A_1, D_{2n}^0, E_7^0, or E_8^0, or an EDP of type E_{12}, then $\bar{\pi}(w)$ is smooth. If w is an RDP of type A_{2n-1} $(n \geq 2)$, D_{2n+1}^0 $(n \geq 2)$, or E_6^0, then $\bar{\pi}(w)$ is an RDP of type A_{n-1}, A_1, A_2 respectively. No other types of RDPs can appear on $\bar{Y}$.*

(3) *If $\mathrm{Sing}(\bar{Y})$ has only RDPs, then the total index of the RDPs on $\bar{Y}$ is ≥ 12. and the equality holds if and only if $\bar{X}$ is a smooth Enriques surface.*

PROOF. (2) This follows from [Mat21, Theorem 3.3(1)] if w is a smooth point or an RDP, and from Proposition 3.3 if w is an EDP.

(1) By (2), $\bar{X}$ has only RDPs as singularities (if any). Let $X \to \bar{X}$ be the minimal resolution. By [Mat21, Theorem 3.3(1)], $\bar{Y} \times_{\bar{X}} X$ is also a normal K3-like surface with only RDPs and EDPs of type E_{12}. Moreover D extends to a regular derivation $\tilde{D} := D \otimes 1$ on $\bar{Y} \times_{\bar{X}} X$ of multiplicative (resp. additive) type with $\mathrm{Fix}(\tilde{D}) = \varnothing$ and with quotient $(\bar{Y} \times_{\bar{X}} X)^{\tilde{D}} = X$. Hence we may assume $X = \bar{X}$ is smooth. As in [Mat21, proof of Proposition 4.1], we have $K_X \equiv 0$, where $\equiv$ is the numerical equivalence. Hence, to show X that is an Enriques surface, it suffices to show $\chi(\mathcal{O}_X) = 1$.

Suppose D is of multiplicative type. Then we have a decomposition $\bar{\pi}_* \mathcal{O}_{\bar{Y}} = \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} (\bar{\pi}_* \mathcal{O}_{\bar{Y}})_i$ to eigenspaces of D of eigenvalues $i \in \mathbb{Z}/2\mathbb{Z}$. Since $\mathrm{Fix}(D) = \varnothing$, $(\bar{\pi}_* \mathcal{O}_{\bar{Y}})_1 = \mathrm{Im}(D)$ is an invertible sheaf locally generated by an element of $\bar{\pi}_* \mathcal{O}_{\bar{Y}}^* \cap (\bar{\pi}_* \mathcal{O}_{\bar{Y}})_1$ and satisfies $((\bar{\pi}_* \mathcal{O}_{\bar{Y}})_1)^{\otimes 2} \cong \mathcal{O}_X$, hence $(\bar{\pi}_* \mathcal{O}_{\bar{Y}})_1$ is a 2-torsion class in $\mathrm{Pic}(X)$. In particular we have $\chi((\bar{\pi}_* \mathcal{O}_{\bar{Y}})_1) = \chi((\bar{\pi}_* \mathcal{O}_{\bar{Y}})_0) = \chi(\mathcal{O}_X)$ by Riemann–Roch, hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\bar{Y}})/2 = 1$ and X is an Enriques surface. If the class $(\bar{\pi}_* \mathcal{O}_{\bar{Y}})_1 \in \mathrm{Pic}(X)$ is trivial, then $1 \in H^0(\bar{Y}, \mathcal{O}_{\bar{Y}})$ would have a nontrivial square root and $\bar{Y}$ would be non-reduced, which is absurd. Therefore $\mathrm{Pic}(X)$ has nontrivial torsion, hence X is classical, and $\bar{Y}$ is the canonical covering of X.

Suppose D is of additive type. Since $\mathrm{Fix}(D) = \varnothing$, we have $\mathrm{Im}(D) = \mathcal{O}_X$, and the extension

$$0 \to \mathcal{O}_X \to \bar{\pi}_* \mathcal{O}_{\bar{Y}} \xrightarrow{D} \mathcal{O}_X \to 0$$

is non-split (otherwise $\bar{Y}$ would be non-reduced). We obtain $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\bar{Y}})/2 = 1$, hence X is an Enriques surface. Since $\bar{Y} \to X$ is purely inseparable, the Frobenius image $F(e)$ of the nontrivial class $e \in H^1(X, \mathcal{O}_X)$ of this extension is zero. This shows that X is supersingular and that $\bar{Y}$ is the canonical covering of X. (cf. [BM76, Corollary in Section 3].)

(3) Assume $\operatorname{Sing}(\bar{Y})$ has only RDPs. Let n_i and m_j be the indices of the RDPs on $\bar{Y}$ and $\bar{X}$ respectively. Then we have $b_2(\bar{Y}) = b_2(Y) - \sum n_i = 22 - \sum n_i$ and $b_2(\bar{X}) = b_2(X) - \sum m_j = 10 - \sum m_j$, where $Y \to \bar{Y}$ and $X \to \bar{X}$ are the minimal resolutions. Since $\bar{\pi}$ is purely inseparable we have $b_2(\bar{Y}) = b_2(\bar{X})$. Hence $\sum n_i = 12 + \sum m_j \geq 12$ and the equality is equivalent to $\sum m_j = 0$. $\square$

We slightly generalize the results of Ekedahl–Hyland–Shepherd-Barron and Schröer on the tangent sheaf of the canonical covering and the fixed loci of global sections.

PROPOSITION 3.7 (cf. [EHSB12, Corollary 3.7(3)], [Sch21, Theorem 6.4]). *Suppose $\bar{Y}$ and D are as in Proposition 3.6. Then,*

(1) *The tangent sheaf $T_{\bar{Y}}$ is free (of rank 2).*

(2) *For each $w \in \operatorname{Sing}(\bar{Y})$ there exists a line $\mathfrak{l}(w) \subset H^0(\bar{Y}, T_{\bar{Y}})$ such that, for $D' \in H^0(\bar{Y}, T_{\bar{Y}})$, we have $w \in \operatorname{Fix}(D')$ if and only if $D' \in \mathfrak{l}(w)$.*

(3) *An element $D' \in H^0(\bar{Y}, T_{\bar{Y}})$ is fixed-point-free if and only if it belongs to the complement of the (finite) union of the lines $\mathfrak{l}(w)$.*

Again, if we assume moreover $X = Y^D$ is smooth, then under some assumptions on $\bar{Y}$ the assertions follow from [Sch21, Proposition 6.1 and Theorem 6.4], and the proofs of (2) and (3) are parallel.

PROOF. (1) $T_{\bar{Y}}$ is locally free by Lemma 3.4(1). Then we can apply the proof of [EHSB12, Corollary 3.7(3)] as follows (although it is stated for smooth Enriques surfaces). Since D is fixed-point-free, the quotient $L := T_{\bar{Y}}/\mathcal{O}_{\bar{Y}}D$ is an invertible sheaf. Since $K_{\bar{Y}^{\mathrm{sm}}} = 0$, comparing the Chern classes we obtain $L \cong \mathcal{O}_{\bar{Y}}$. Since $H^1(\bar{Y}, \mathcal{O}) = 0$, the extension is trivial.

(2), (3) We can apply the proof of [Sch21, Theorem 6.4] as follows (although it is stated for smooth Enriques surfaces). For each closed point $w \in \bar{Y}$, the composite $H^0(\bar{Y}, T_{\bar{Y}}) \to T_{\bar{Y},w} \to T_{\bar{Y},w} \otimes k(w)$ is an isomorphism of restricted Lie algebras. If w is a smooth point, then $w \in \operatorname{Fix}(D')$ if and only if $D' = 0$. If w is a singular point, then $w \in \operatorname{Fix}(D')$ if and only if $D' \in \mathfrak{l}(w)$, where $\mathfrak{l}(w) \subset H^0(\bar{Y}, T_{\bar{Y}})$ is the inverse image of the line of $T_{\bar{Y},w} \otimes k(w)$ mentioned in Lemma 3.4(2). Therefore $\operatorname{Fix}(D') = \varnothing$ if and only if $D' \notin \{0\} \cup \bigcup_{w \in \operatorname{Sing}(\bar{Y})} \mathfrak{l}(w)$. Since $\bar{Y}$ has at least one singular point (by Proposition 3.6(3)), we have $0 \in \bigcup \mathfrak{l}(w)$. $\square$

Following Schröer [Sch21, Section 2], we call this line $\mathfrak{l}(w) \subset \mathfrak{g} = H^0(\bar{Y}, T_{\bar{Y}})$ to be the *canonical line* attached to $w \in \operatorname{Sing}(\bar{Y})$.

COROLLARY 3.8. *Suppose $\bar{Y}$ and D are as in Proposition 3.6. If $w \in \operatorname{Sing}(\bar{Y})$ is an RDP of type A_{2n-1} for some $n \geq 1$ (resp. any other singularity), then the attached canonical line $\mathfrak{l}(w) \subset \mathfrak{g}$ is of multiplicative type (resp. of additive type).*

PROOF. The canonical line $\mathfrak{l}(w)$ is p-closed since $\mathrm{Fix}(D^p) \supset \mathrm{Fix}(D) \ni w$ for $D \in \mathfrak{l}(w)$. Hence it is either of multiplicative type or of additive type. Take a generator D'_1 of $\mathfrak{l}(w)$ and extend it to a basis D'_1, D'_2 of $\mathfrak{g}$. Then Lemma 3.4(3) excludes one possibility. $\square$

4. Proof of the main theorem

Hereafter we assume $p = 2$.

In this section we prove Theorem 1.4. Our proof is case-by-case: $\bar{Y}$ having an EDP are discussed in Section 4.1, those with only RDPs of type D_n and E_n (those with $\mathrm{Sing}(\bar{Y}) = 3D_4^0$, $D_4^0 + D_8^0$, $D_4^0 + E_8^0$, or D_{12}^0) in Section 4.2, and those having at least one RDP of type A_1 (those with $\mathrm{Sing}(\bar{Y}) = 12A_1$, $8A_1 + D_4^0$, $6A_1 + D_6^0$, or $5A_1 + E_7^0$) in Section 4.3. By Theorems 1.1 and 1.2, this covers all cases to be considered.

Before splitting into cases, we note the following.

PROPOSITION 4.1 (cf. [Mat21, Theorem 9.1]). *Let $\bar{Y}$ be a normal surface that is the canonical covering of a supersingular Enriques surface X. Then $\mathrm{Sing}(\bar{Y})$ is one of*

$$12A_1,\ 3D_4^0,\ D_4^0 + D_8^0,\ D_4^0 + E_8^0,\ D_{12}^0,\ \textit{or}\ E_{12}.$$

PROOF. We have $K_X = 0$ since X is supersingular, and $K_{\bar{Y}} = 0$ and $\bar{Y}$ is normal by assumption. Then the "dual" morphism $\pi' \colon X^{(1/2)} \to \bar{Y}$ is, by the argument in [Mat21, proof of Theorem 4.3], the quotient morphism by either a μ_2- or α_2-action with only isolated fixed points, and the fixed locus $\langle D' \rangle$ of the corresponding derivation D' on $X^{(1/2)}$ has degree 12 by the Katsura–Takeda formula. We use the classification ([Mat21, Lemma 3.6 and Corollary 3.9]) of μ_2- and α_2-quotient singularities with degree ≤ 12. If it is a μ_2-quotient, then each singular point of $\bar{Y}$ is an RDP of A_1. If it is an α_2-quotient, then each singular point of $\bar{Y}$ is an RDP of type D_{4n}^0 or E_8^0 or an EDP of type E_{12}. In each case, the degree of $\langle D' \rangle$ at each point is equal to the index of the quotient singularity. $\square$

4.1. Case of $\bar{Y}$ with an EDP

REMARK 4.2. Suppose $\bar{Y}$ has a non-RDP. It is claimed in the proof of [CD89, Proposition 1.3.1] that then $\bar{Y}$ has exactly one non-RDP singularity and it is an EDP. The proof is however incomplete where they use the Leray spectral sequence. This is fixed in the new version of the book [CDL]. Schröer [Sch21, proof of Proposition 5.4] also gives an argument. We can also use the classification ([Mat21, Lemma 3.6 and Corollary 3.9]) of 2-closed derivation quotient singularities with small degree, saying that the singularity is an RDP if degree ≤ 10 and that the singularity is either an RDP or an EDP if degree ≤ 12.

The essential part of the proof of this case is:

PROPOSITION 4.3. *Suppose X is a classical Enriques surface whose canonical covering $\bar{Y}$ is normal. Then $\mathrm{Sing}(\bar{Y})$ does not contain an EDP.*

DEFINITION 4.4. Following Schröer [Sch21, Section 8], we say that an integral curve $A \subset X$ is a *radical two-section* of an elliptic fibration $X \to \mathbb{P}^1$ if the composite $A \to X \to \mathbb{P}^1$ is surjective and inseparable of degree 2.

Following arguments of [Sch21, proof of Proposition 8.9], we can prove the following assertion on Enriques surfaces having no elliptic fibrations admitting a radical two-section.

Lemma 4.5 (cf. [Sch21, Proposition 8.9]). *Suppose X is a classical or supersingular Enriques surface whose canonical covering $\bar{Y}$ is normal. Assume that no elliptic fibration on X admits a radical two-section. Then either X is supersingular or $\#\operatorname{Sing}(\bar{Y}) \geq 5$.*

Proof. Since $\bar{Y}$ is normal, any genus one fibration on X is elliptic ([Sch21, Theorem 5.6(i)]). Suppose no elliptic fibration on X admits a radical two-section. Then X does not admit a smooth rational curve nor a non-movable cuspidal rational curve ([Sch21, Proposition 8.8]) nor a non-movable nodal rational curve (same proof as in the cuspidal case). Let $\phi\colon X \to \mathbb{P}^1$ be an elliptic fibration and $\phi'\colon J \to \mathbb{P}^1$ its Jacobian fibration. By above, any half-fiber of ϕ is smooth, and any singular fiber of ϕ is of Kodaira type I_1 or II. (We call $(X_a)_{\mathrm{red}}$ a half-fiber if X_a is a multiple fiber of multiplicity 2.) By [LLR04, Theorem 6.6], if a fiber of an elliptic fibration is of type mT, where $m \in \mathbb{Z}_{>0}$ is the multiplicity and $T \in \{\mathrm{I}_n, \mathrm{I}_n^*, \mathrm{II}, \mathrm{II}^*, \mathrm{III}, \mathrm{III}^*, \mathrm{IV}, \mathrm{IV}^*\}$ is the symbol denoting the Kodaira type, then the corresponding fiber of its Jacobian fibration is of type T. Hence ϕ' has the same types of singular fibers as ϕ (up to multiplicity).

Suppose ϕ' has no fibers of type I_1. Then, by Lang's classification of configurations of singular fibers of rational elliptic surfaces ([Lan00, Section 2 or 4]), the relative j-invariant for ϕ' is 0. This shows that any smooth fiber of ϕ' is a supersingular elliptic curve. Let $(X_a)_{\mathrm{red}}$ be a half-fiber of ϕ. Then it is smooth by above, and isogenous to the corresponding fiber J_a of ϕ' (consider the base change to a finite cover $C \to \mathbb{P}^1$ over which ϕ acquires a section), hence supersingular. Then X cannot be classical by Proposition 2.1.

Now suppose there is at least one fiber of type I_1 (and no singular fiber of type other than I_1 and II). Again by Lang's classification ([Lan00, Sections 2–3 or 4]), we observe that the singular fibers of ϕ', and hence those of ϕ, are $12\mathrm{I}_1$, $8\mathrm{I}_1+\mathrm{II}$, $6\mathrm{I}_1+\mathrm{II}$, or $5\mathrm{I}_1+\mathrm{II}$. By [Sch21, Proposition 4.7], the point above the node of each fiber of type I_1 is a singular point of $\bar{Y}$. Hence $\bar{Y}$ has at least 5 singular points. □

Proof of Proposition 4.3. Since $\bar{Y}$ is normal, the "dual" morphism $X \to \bar{Y}^{(2)}$ is the quotient by a rational derivation D' on X. We have $\operatorname{Sing}(\bar{Y}) = \bar{\pi}^{-1}(\operatorname{Supp}\langle D'\rangle)$. By the Rudakov–Shafarevich formula we have $(D') \sim -K_X \equiv 0$, hence by the Katsura–Takeda formula we have $\deg\langle D'\rangle = 12$. Here $\equiv$ is the numerical equivalence. By [Mat21, Corollary 3.9], if the quotient singularity on $\bar{Y}^{(2)}$ is an EDP then $\langle D'\rangle$ has degree at least 11 at the corresponding point of X. Hence if X has an EDP then $\#\operatorname{Sing}(\bar{Y}) \leq 2$.

Since X is classical, we may assume by Lemma 4.5 that X admits an elliptic fibration $\phi\colon X \to \mathbb{P}^1$ with a radical two-section. Then by [Sch21, Propositions 8.1 and 8.5], $\phi \circ \bar{\pi}\colon \bar{Y} \to \mathbb{P}^1$ factors as $\bar{Y} \xrightarrow{\psi} \mathbb{P}^1 \xrightarrow{F} \mathbb{P}^1$ and this ψ admits a section (e.g. $\bar{\pi}^{-1}(A)_{\mathrm{red}}$ for any radical two-section A of ϕ). Let $\phi'\colon J \to \mathbb{P}^1$ be the Jacobian fibration of ϕ, and $\psi'\colon J^{(2/\mathbb{P}^1)} := J \times_{\mathbb{P}^1} \mathbb{P}^1 \to \mathbb{P}^1$ be the Frobenius base change of ϕ'. Then the existence of a section of ψ implies that the generic fiber of $\psi\colon \bar{Y} \to \mathbb{P}^1$ is isomorphic to the generic fiber of $\psi'\colon J^{(2/\mathbb{P}^1)} \to \mathbb{P}^1$ by [Sch21, Proposition 8.4]. In particular $\bar{Y}$ and $J^{(2/\mathbb{P}^1)}$ are birational. As above, if X_a is of type mT then J_a is of type T. Since $\bar{Y}$ is normal, we have $T \in \{\mathrm{I}_n, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ by [Sch21, Theorem 5.6(ii)], in particular J_a is reduced for all $a \in \mathbb{P}^1$. By [Sch21, Proposition 11.1], $J^{(2/\mathbb{P}^1)}$ also has trivial dualizing sheaf. By [Sch21, Proposition 11.2], $\mathrm{Sing}(J^{(2/\mathbb{P}^1)})$ is precisely the points over the non-smooth locus of $J \to \mathbb{P}^1$, and then it is isolated since J has only finitely many singular fibers and all of them are reduced.

Suppose $\bar{Y}$ has an EDP. Then $\bar{Y}$ and hence $J^{(2/\mathbb{P}^1)}$ are rational surfaces. Since $J^{(2/\mathbb{P}^1)}$ has trivial dualizing sheaf and $\mathrm{Sing}(J^{(2/\mathbb{P}^1)})$ is isolated, $J^{(2/\mathbb{P}^1)}$ also has a non-RDP singularity. By [Sch21, Theorem 12.1], based on Lang's classification [Lan94, Section 2A] of local Weierstrass equations in characteristic 2, this can happen only if the corresponding fiber of J is of Lang type 9C (i.e. J is of the form

$$y^2 + t^3\gamma_0 y = x^3 + t\gamma_1 x^2 + t\gamma_3 x + t\gamma_5,$$

with polynomials $\gamma_i \in k[t]$ of degree $\leq i$ satisfying $t \nmid \gamma_0$ and $t \nmid \gamma_5$) and moreover $t \mid \gamma_3$. In particular, $\phi' \colon J \to \mathbb{P}^1$ has only one singular fiber (at $t = 0$) and all remaining fibers are supersingular elliptic curves.

As in the previous lemma, $(X_a)_{\mathrm{red}}$ is smooth if and only if J_a is smooth, and in this case these elliptic curves are isogenous. Hence $\phi \colon X \to \mathbb{P}^1$ has, up to multiplicity, only one singular fiber and all remaining fibers are supersingular elliptic curves.

On the other hand, since X is classical, the elliptic fibration $\phi \colon X \to \mathbb{P}^1$ has two multiple fibers, and each multiple fiber is either a smooth ordinary elliptic curve or a singular fiber of additive type (Proposition 2.1). Contradiction. $\square$

The supersingular case remains.

Proof of Theorem 1.4 in the case $\bar{Y}$ has an EDP. Let D be a fixed-point-free derivation on $\bar{Y}$ with Enriques quotient $X := Y^D$. By Proposition 4.3, X is supersingular. By Proposition 4.1, $\mathrm{Sing}(\bar{Y})$ consists of one point, of type E_{12}. By Corollary 3.8, the canonical line $\mathfrak{l}$ attached to the singularity is of additive type. Since the 2 lines $[D]$ and $\mathfrak{l}$ of $\mathfrak{g}$ of additive type are distinct (Proposition 3.7(3)), it follows from Corollary 2.6(2) that all lines of $\mathfrak{g}$ are of additive type and that $\mathfrak{g}$ is abelian. $\square$

Remark 4.6. Combining Propositions 4.3 and 4.1, we obtain another proof of Schröer's result [Sch21, Theorem 14.1] that if $\bar{Y}$ has an EDP then it has no other singularities.

4.2. Case of $\bar{Y}$ with only RDPs of type D_n or E_n

The following lemma on RDP K3 surfaces follows from arguments in [Mat21].

Lemma 4.7. *Suppose $\bar{Y}$ is an RDP K3 surface with* $\mathrm{Sing}(\bar{Y}) \neq \varnothing$, *with* $\mathrm{Sing}(\bar{Y}) = \{w_i\}_{i=1}^N$, *and $(n_i)_{i=1}^N$ are positive integers such that for each i one of the following holds.*

- *w_i is an RDP of type $D^0_{4n_i}$.*
- *w_i is an RDP of type E^0_8 and $n_i = 2$.*

For each i, let $I_{w_i} \subset \mathcal{O}_{\bar{Y},w_i}$ be the ideal defined in [Mat21, Section 6.2], *and let* $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{\bar{Y}} \to \bigoplus_i \mathcal{O}_{\bar{Y},w_i}/I_{w_i})$. *Then,*

(1) *the Frobenius map* $F \colon \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O}) \to \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}^{(2)}, \mathcal{O})$ *is zero and we have* $\dim \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O}) = -1 + \sum n_i$.

(2) *There is a family* $(\bar{Z}'_e, D_e)$ *of* α_2*-coverings* $\pi'_e \colon \bar{Z}'_e \to \bar{Y}^{\mathrm{sm}}$ *and global derivations* $D_e \in H^0(\bar{Y}, T_{\bar{Y}})$ *of additive type, parametrized by* $e \in \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O})$*, such that*

- $\mathrm{Sing}(\bar{Z}'_e) = \pi'_e(\mathrm{Fix}(D_e|_{\bar{Y}^{\mathrm{sm}}}))$,
- *The sequence* $0 \to \mathcal{O}_{\bar{Y}^{\mathrm{sm}}} \to \mathcal{O}_{\bar{Z}'_e} \xrightarrow{\delta} \mathcal{O}_{\bar{Y}^{\mathrm{sm}}} \to 0$*, where* δ *is the derivation corresponding to the* α_2*-action, is exact and represents the restriction of* e *to* $\bar{Y}^{\mathrm{sm}}$*, and*
- $\mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O}) \to H^0(\bar{Y}, T_{\bar{Y}}) \colon e \mapsto D_e$ *is an injective semilinear map.*

PROOF. (1) Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O}) \arrow[r] \arrow[d, "F"] & \bigoplus_i \mathrm{Ext}^2_{\bar{W}_i}(\mathcal{O}/I_{w_i}, \mathcal{O}) \arrow[r] \arrow[d, "F"] & H^2(Y, \mathcal{O}) \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}^{(2)}, \mathcal{O}) \arrow[r] & \bigoplus_i \mathrm{Ext}^2_{\bar{W}_i}(\mathcal{O}/{I_{w_i}}^{(2)}, \mathcal{O}) \arrow[r] & H^2(Y, \mathcal{O}) \arrow[r] & 0
\end{tikzcd}$$

constructed as in [Mat21, proof of Theorem 7.3(2)], where, for each i, $\bar{W}_i = \mathrm{Spec}\, \hat{\mathcal{O}}_{\bar{Y}, w_i}$ is the completion at the RDP w_i of $\bar{Y}$. As proved in [Mat21, Lemma 6.6(1)], the Frobenius map $F \colon \mathrm{Ext}^2_{\bar{W}_i}(\mathcal{O}/I_{w_i}, \mathcal{O}) \to \mathrm{Ext}^2_{\bar{W}_i}(\mathcal{O}/{I_{w_i}}^{(2)}, \mathcal{O})$ associated with the local ring $\bar{W}_i$ are zero. This implies the former assertion.

The latter equality follows from $\dim_k \mathrm{Ext}^2_{\bar{W}_i}(\mathcal{O}/I_{w_i}, \mathcal{O}) = \dim_k(\mathcal{O}/I_{w_i}) = n_i$ and $\dim H^2(Y, \mathcal{O}) = 1$.

(2) (This construction imitates Bombieri–Mumford's construction [BM76, Section 3] of the canonical α_2-covering of a supersingular Enriques surface X from a nontrivial class in $H^1(X, \mathcal{O})^{F=0} = H^1(X, \mathcal{O})$.)

We fix a nontrivial 2-form ω on Y^{sm}. Take a class $e \in \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O})$ and consider the corresponding extension

$$0 \to \mathcal{O} \to V \xrightarrow{\delta} \mathcal{I} \to 0.$$

Then we obtain, as in [Mat21, proof of Theorem 7.3(2)], an α_2-covering $\pi'_e \colon \bar{Z}'_e \to \bar{Y}^{\mathrm{sm}}$ with $V|_{\bar{Y}^{\mathrm{sm}}} = \mathcal{O}_{\bar{Z}'_e}$ and δ being the derivation corresponding to the α_2-action. As in [Mat21, Proposition 2.15] we define a 1-form η on $\bar{Y}^{\mathrm{sm}}$ and a p-closed derivation D on $\bar{Y}^{\mathrm{sm}}$ in the following way: let t a local section of V such that $\delta(t) = 1$, so that $\bar{Z}'_e$ is locally defined as $\mathcal{O}_{\bar{Z}'_e} = \mathcal{O}_{\bar{Y}^{\mathrm{sm}}}[t]/(t^2 - c)$ with $c \in \mathcal{O}_{\bar{Y}^{\mathrm{sm}}}$, let $\eta = dc$, and define D by $D(f)\omega = df \wedge \eta$. Then we have $\pi'_e(\mathrm{Sing}(\bar{Z}'_e)) = \mathrm{Fix}(D) = \mathrm{Zero}(\eta)$.

Since $\bar{Y}$ is normal, the derivation D on $\bar{Y}^{\mathrm{sm}}$ extends to one on $\bar{Y}$. This map $\mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O}) \to H^0(\bar{Y}, T_{\bar{Y}}) \colon e \mapsto D$ is F-semilinear by construction. We will show that this is injective. Suppose $D = 0$. Then $\eta = 0$. Then $c = b^2$ for some local sections b of $\mathcal{O}_{\bar{Y}^{\mathrm{sm}}}$. Then $t' := t - b$ glue to a global section $t' \in H^0(\bar{Y}^{\mathrm{sm}}, V)$ with $\delta(t') = 1$, and moreover to a global section on $\bar{Y}$, hence the extension is trivial and $e = 0$. □

Proof of Theorem 1.4 in the case $\mathrm{Sing}(\bar{Y})$ *is* $3D_4^0$*,* $D_4^0 + D_8^0$*,* $D_4^0 + E_8^0$*, or* D_{12}^0*.* By Lemma 4.7 we obtain a 2-dimensional family $\bar{Z}'_e$ of α_2-coverings of $\bar{Y}^{\mathrm{sm}}$ parametrized by $e \in \mathrm{Ext}^1_{\bar{Y}}(\mathcal{I}, \mathcal{O}) \cong H^0(\bar{Y}, T_{\bar{Y}})$. We show that if $e \neq 0$ then this extends to a family $\bar{Z}_e$

of α_2-coverings of $\bar{Y}$, and show that the family $(\bar{Z}_e^{(2)})_{e\in\mathrm{Ext}^1_{\bar{Y}}(\mathcal{I},\mathcal{O})\setminus\{0\}}$ exhaust nontrivial p-closed derivation quotients of Y.

Suppose $e \neq 0$ and let $D \neq 0$ be the corresponding derivation on $\bar{Y}$. Since $T_{\bar{Y}}$ is free, D has no fixed points on $\bar{Y}^{\mathrm{sm}}$. Then $\bar{Z}'_e$ is normal, since it is regular outside the codimension 2 subscheme ${\pi'_e}^{-1}(\mathrm{Fix}(D))$ and is Gorenstein everywhere. Let $\bar{Z}_e \to \bar{Y}$ be the normalization of $\bar{Y}$ in $k(\bar{Z}'_e)$. Then the derivation δ on $\bar{Z}'_e$ extends to a derivation of $\bar{Z}_e$, which defines an α_2-action with quotient $\bar{Y}$. Since $\bar{Z}_e$ is normal and $\mathcal{O}_{\bar{Z}_e^{(2)}}|_{\bar{Y}^{\mathrm{sm}}} = V^{(2)}|_{\bar{Y}^{\mathrm{sm}}} \subset \operatorname{Ker} D$, we obtain $\bar{Y}^D = \bar{Z}_e^{(2)}$. Since $[k(\bar{Y}) : k(\bar{Z}_e^{(2)})] = 2$, it follows that D is p-closed, and since $T_{\bar{Y}}$ is free, we have $D^2 = \lambda D$ with $\lambda \in k$. Note that replacing e with a nonzero multiple replaces D with a nonzero multiple, hence results in the same quotient.

Take any D that does not belong to any canonical line. Then D is fixed-point-free by Proposition 3.7(3), hence $\bar{Y}^D = \bar{Z}_e^{(2)}$ is an Enriques surface. It is supersingular and thus $D^2 = 0$, since a classical Enriques surface does not admit a regular p-closed derivation with K3-like quotient by the Rudakov–Shafarevich formula (cf. [Mat21, Proposition 4.5]). By Corollary 2.6(2), $\mathfrak{g}$ is abelian and all elements D satisfy $D^2 = 0$. In particular $\bar{Y}$ has no p-closed derivation quotient that is a classical Enriques surface. □

4.3. Case of $\bar{Y}$ having A_1

Proof of Theorem 1.4 in the case $\mathrm{Sing}(\bar{Y})$ *is* $8A_1 + D_4^0$, $6A_1 + D_6^0$, *or* $5A_1 + E_7^0$. Let w_1 be a singular point of type A_1 and w_2 a singular point not of type A_1. Then the attached canonical lines $\mathfrak{l}(w_1)$ and $\mathfrak{l}(w_2)$ of $\mathfrak{g} = H^0(\bar{Y}, T_{\bar{Y}})$ are respectively of multiplicative type and additive type by Corollary 3.8. The line generated by a fixed-point-free p-closed derivation (which exists by assumption) is different from $\mathfrak{l}(w_1)$ and $\mathfrak{l}(w_2)$ (Proposition 3.7(3)). By Corollary 2.6(1), all lines of $\mathfrak{g}$ are p-closed, and among them exactly one is of additive type, which should be $\mathfrak{l}(w_2)$. Hence all Enriques quotients of $\bar{Y}$ are classical. The assertion on the bracket follows from Corollary 2.6(1). □

Proof of Theorem 1.4 in the case $\mathrm{Sing}(\bar{Y})$ *is* $12A_1$. If all 12 canonical lines are equal, then a generator of the line extends to a derivation on the blow-up Y of $\bar{Y}$ at the 12 points, but since Y is a (smooth) K3 surface this is impossible by [RS76, Theorem 7]. Hence there are at least 2 distinct canonical lines, both of multiplicative type by Corollary 3.8.

By applying Proposition 2.4 to the rational derivation on Y induced by a fixed-point-free derivation D, where $Y \to \bar{Y}$ is the minimal resolution with exceptional curves $\{e_w\}_{w\in\mathrm{Sing}(\bar{Y})}$, we see that $\sum_{w\in\mathrm{Sing}(\bar{Y})} e_w \in 2\operatorname{Pic}(Y)$. This induces, as in [Mat21, Theorem 7.3(2)], a μ_2-covering $\bar{Z} \to \bar{Y}$ that is regular above a neighborhood of $\mathrm{Sing}(\bar{Y})$. Let $D' \neq 0$ be the resulting p-closed derivation on $\bar{Y}$ (cf. [Mat21, proof of Theorem 7.3(2)]). Then D' is fixed-point-free, since $\mathrm{Fix}(D')$ contains none of $\mathrm{Sing}(\bar{Y})$. Hence $\bar{Z}$ is an Enriques surface. As in the previous subsection, it is supersingular. Hence the line $[D'] \subset \mathfrak{g}$ is of additive type.

By Corollary 2.6(1), all lines of $\mathfrak{g} = H^0(\bar{Y}, T_{\bar{Y}})$ are p-closed and among them exactly one is of additive type, which is $[D']$, which is fixed-point-free. □

5. Examples

In this section we prove Theorem 1.5. If $\bar{Y}$ contains a non-RDP singularity, then $\bar{Y}$ has an EDP by Theorem 1.1, and we proved in Theorem 1.4 that $\bar{Y}$ has one EDP of type E_{12} and contains no other singularity. If $\operatorname{Sing}(\bar{Y})$ consists only of RDPs, then the configuration is one of the 8 given in Theorem 1.2. Hence it remains to show that each of the 9 configuration is indeed possible. We will give explicit examples.

5.1. Examples of canonical coverings that are RDP K3 surfaces

It turns out that all configurations of RDPs are realized by Enriques surfaces admitting elliptic fibrations admitting a radical two-section (Definition 4.4).

In each example, we give two elliptic RDP K3 surfaces $\bar{Y}' \to \mathbb{P}^1$ and $\bar{Y}'' \to \mathbb{P}^1$ satisfying the following properties.

- The generic fibers of $\bar{Y}'$ and $\bar{Y}''$ are isomorphic.
- $\bar{Y}'$ is isomorphic to the Frobenius base change $\bar{J} \times_{\mathbb{P}^1} \mathbb{P}^1$ of the Weierstrass form $\bar{J} \to \mathbb{P}^1$ of some rational elliptic surface $J \to \mathbb{P}^1$.
- We give a basis D_1, D_2 for $H^0(\bar{Y}'', T_{\bar{Y}''})$. A generic element $D = e_1D_1 + e_2D_2$ ($e_1, e_2 \in k$) has no fixed points, hence the quotient $\bar{X} := (\bar{Y}'')^D$ is an RDP Enriques surface, and $\bar{Y} := \bar{Y}'' \times_{\bar{X}} X \to X$ is the canonical covering of the Enriques surface X, where $X \to \bar{X}$ is the minimal resolution.

We do not give $\bar{J}$ explicitly since it will be clear from the equation defining $\bar{Y}'$. We will describe the type of particular fibers of J according to Lang's classification [Lan00] (for short, we call it the Lang type).

Example 5.1 ($12A_1$, $8A_1 + D_4^0$, $6A_1 + D_6^0$, $5A_1 + E_7^0$). The examples with $12A_1$ ((1) below) and $8A_1 + D_4^0$ ((2), $n = 0$) are the ones given by Katsura–Kondo [KK18, Section 3] and Kondo [Kon21, Section 3.3] respectively.

Let $A(t), B(t), C(t) \in k[t]$ be one of the following.

(1) $(A, B, C) = (t^3(t-1), t^3(t-1)^3, 0)$,

(2) $(A, B, C) = (0, t^{3-n}(t-1)^3, n(t-1)^4)$, $n \in \{0, 2, 3\}$.

We have equalities $d(A(t)B(t))/dt = 0$ and $C(t) = d(t(t-1)B(t))/dt$ in each case.

Let $\bar{Y}'$ be the elliptic RDP K3 surface defined by

$$y^2 + xy + t(t-1)A(t)y + x^3 + t(t-1)B(t)x = 0,$$
$$y'^2 + s^2x'y' + (1-s)\tilde{A}(s)y' + x'^3 + (1-s)\tilde{B}(s)x' = 0,$$

where $s = t^{-1}$, $x' = t^{-4}x$, $y' = t^{-6}y$, and

$$\tilde{A}(s) = s^4B(s^{-1}),\ \tilde{B}(s) = s^6B(s^{-1}),\ \tilde{C}(s) = s^4C(s^{-1}).$$

The RDPs of $\bar{Y}'$ and the corresponding singular fibers of the minimal resolution Y are

(1) $2A_9$ ($2\mathrm{I}_{10}$) at $t = 0, 1$ and $2A_1$ ($2\mathrm{I}_2$) at $t = \omega, \omega^2$, where ω and ω^2 are the roots of $t^2 + t + 1 = 0$,

(2) A_{7-2n} (I_{8-2n}) at $t=0$, A_7 (I_8) at $t=1$, and D_5^0 or D_7^0 or E_7^0 (I_1^* or I_3^* or III^*) at $s=0$ if $n=0$ or $n=2$ or $n=3$ respectively.

Let $\bar{Y}''$ be the elliptic RDP K3 surface which is birational to $\bar{Y}'$ and isomorphic outside the fibers $t=0,1$, defined by

$$\begin{aligned} y^2+xy+t(t-1)A(t)y+x^3+t(t-1)B(t)x&=0 \quad (t\neq 0,1),\\ y_1^2+x_1y_1+A(t)y_1+t(t-1)x_1^3+B(t)x_1&=0,\\ y_2^2+x_2y_2+A(t)x_2^2y_2+t(t-1)x_2+B(t)x_2^3&=0,\\ y'^2+s^2x'y'+(1-s)\tilde{A}(s)y'+x'^3+(1-s)\tilde{B}(s)x'&=0 \quad (s\neq 1),\end{aligned}$$

where the coordinates are given by

$$x_1=\frac{x}{t(t-1)},\ y_1=\frac{y}{t(t-1)},\qquad x_2=\frac{t(t-1)}{x},\ y_2=\frac{t(t-1)y}{x^2}.$$

The RDPs of $\bar{Y}''$ at the fibers $t=0,1$ are

(1) A_7+A_7 at $(x_1,y_1,t)=(0,0,0),(0,0,1)$ and A_1+A_1 at $(x_2,y_2,t)=(0,0,0),(0,0,1)$,

(2) A_{5-2n} at $(x_1,y_1,t)=(0,0,0)$ (if $n=0,2$), A_5 at $(x_1,y_1,t)=(0,0,1)$, and A_1+A_1 at $(x_2,y_2,t)=(0,0,0),(0,0,1)$.

The other fibers remain unchanged.

Let D_1 and D_2 be the derivations on $\bar{Y}''$ defined as follows, where A_t, B_t, and $\tilde{B}_s$ are the derivatives.

$-$	$D_1(-)$	$D_2(-)$
x	0	$(t(t-1))^{-1}(x+t(t-1)A(t))$
y	$t(t-1)C(t)$	$(t(t-1))^{-1}(y+x^2+t^2(t-1)^2B_t(t))$
t	$t(t-1)$	1
x_1	x_1	$A_t(t)$
y_1	$y_1+C(t)$	$x_1^2+B_t(t)$
t	$t(t-1)$	1
x_2	x_2	$A_t(t)x_2^2$
y_2	$y_2+C(t)x_2^2$	$1+B_t(t)x_2^2$
t	$t(t-1)$	1
x'	0	$(1-s)^{-1}s^2x'+\tilde{A}(s)$
y'	$(1-s)\tilde{C}(s)$	$(1-s)^{-1}(s^2y'+x'^2+(1-s)^2\tilde{B}_s(s))$
s	$1-s$	s^2

In case (1) (resp. case (2) with $n=0$), the derivations $D_{a,b}$ given by Katsura–Kondo [KK18, Section 3] (resp. Kondo [Kon21, Section 3.3]) are equal to abD_1+D_2 (resp. $D_1+(ab)^{-1}D_2$).

Consider the derivation $D=e_1D_1+e_2D_2$ ($e_1,e_2\in k$). We observe that $D^2=e_1D$ and that if (e_1,e_2) is generic (that is, (1) $e_1-e_2\neq 0$ and $e_2\neq 0$, and (2) $e_1\neq 0$ and $e_2\neq 0$) then $\mathrm{Fix}(D)=\varnothing$. Therefore, for such D, $\bar{X}=\bar{Y}''^D$ is an RDP Enriques surface with A_3, A_2, A_1, A_1 at the images of A_7, A_5, D_5^0, D_7^0 respectively and no other

RDPs. It is supersingular if $e_1 = 0$ in case (1), and classical in all other cases. Let $X \to \bar{X}$ be the minimal resolution and let $\bar{Y} = \bar{Y}'' \times_{\bar{X}} X$. Then $\bar{Y}$ is the canonical (μ_2- or α_2-) covering of the smooth Enriques surface X with (1) $\mathrm{Sing}(\bar{Y}) = 12A_1$ (2) $\mathrm{Sing}(\bar{Y}) = 8A_1 + D_4^0, 6A_1 + D_6^0, 5A_1 + E_7^0$ $(n = 0, 2, 3)$ respectively.

If $e_1 = 0$ in case (1), the multiple fiber of X corresponds to the fiber $s = 0$ of Y, which is a supersingular elliptic curve. In all other cases, the multiple fibers of X correspond to the fibers $t = \beta_i$ of Y, which are ordinary elliptic curves, where β_1, β_2 are the two (distinct) roots of $e_1 t(t-1) + e_2 = 0$ (equivalently, $\beta_1 + \beta_2 = 1$ and $\beta_1\beta_2 = e_2/e_1$).

In case (2), the singular fiber of additive type (at $s = 0$) of J is of type II and more precisely it is of Lang type 2A, 2B, 1C for $n = 0, 2, 3$ respectively.

Example 5.2 ($3D_4^0$, $D_4^0 + D_8^0$, $D_4^0 + E_8^0$). Let $A(t), B(t), C(t), G(t) \in k[t]$ be one of the following.

(1) $(A, B, C, G) = (t^2 + t + 1, t^2 + t + 1, t^2, 0)$,

(2) $(A, B, C, G) = (t + 1, (t + 1)^2, t^2, 0)$,

(3) $(A, B, C, G) = ((t + 1)^2, (t + 1)^2, (t + 1)^2, t + 1)$.

Note that $t^2C(t) = B(t)^2 + A_{\mathrm{ev}}(t)$, where $A_{\mathrm{ev}}(t)$ consists of the terms of $A(t)$ of even degree. Let $\bar{Y}'$ be the elliptic RDP K3 surface defined by

$$\begin{aligned} y^2 + t^2B(t)^2y + x^3 + tA(t)x^2 + t^{10}G(t)^2 &= 0, \\ y'^2 + \tilde{B}(s)^2y' + x'^3 + s\tilde{A}(s)x'^2 + \tilde{G}(s)^2 &= 0, \end{aligned}$$

where $s = t^{-1}$, $x' = t^{-4}x$, $y' = t^{-6}y$, and

$$\tilde{A}(s) = s^2A(s^{-1}),\ \tilde{B}(s) = s^2B(s^{-1}),\ \tilde{C}(s) = s^2C(s^{-1}),\ \tilde{G}(s) = sG(s^{-1}).$$

The RDPs of $\bar{Y}'$ and the corresponding singular fibers of the minimal resolution Y are

(1) $3D_5^0$ ($3\mathrm{I}_1^*$) at $t = 0, \omega, \omega^2$,

(2) D_5^0 (I_1^*) at $t = 0$ and D_9^0 (I_5^*) at $t = 1$,

(3) D_5^0 (I_1^*) at $t = 0$ and E_8^0 (II^*) at $t = 1$.

Here ω and ω^2 are the roots of $t^2 + t + 1 = 0$.

Let $\bar{Y}''$ be the elliptic RDP K3 surface which is birational to $\bar{Y}'$ and isomorphic outside the fiber $t = 0$, defined by

$$\begin{aligned} y^2 + t^2B(t)^2y + x^3 + tA(t)x^2 + t^{10}G(t)^2 &= 0 \quad (t \neq 0), \\ y_0^2 + B(t)^2x_0^2y_0 + t^2x_0 + tA(t)x_0^2 + t^6G(t)^2x_0^4 &= 0, \\ y'^2 + \tilde{B}(s)^2y' + x'^3 + s\tilde{A}(s)x'^2 + \tilde{G}(s)^2 &= 0, \end{aligned}$$

where the coordinates are given by

$$x_0 = \frac{t^2}{x},\ y_0 = \frac{t^2y}{x^2}.$$

Then $\bar{Y}''$ has D_4^0 on the fiber $t = 0$, and the RDPs of $\bar{Y}'$ on the other fibers remain unchanged.

Let D_1 and D_2 be the derivations on $\bar{Y}''$ defined as follows.

$-$	$D_1(-)$	$D_2(-)$
x	$t^2 A_{\mathrm{ev}}(t)$	$t^2 C(t)$
y	0	$t^{-2}x^2$
t	t^2	1
x_0	$A_{\mathrm{ev}}(t)x_0^2$	$x_0^2 C(t)$
y_0	0	1
t	t^2	1
x'	$\tilde{A}_{\mathrm{ev}}(s)$	$\tilde{C}(s)$
y'	0	x'^2
s	1	s^2

Consider the derivation $D = e_1 D_1 + e_2 D_2$ $(e_1, e_2 \in k)$. We observe that $D^2 = 0$ and that if (e_1, e_2) is generic (that is, if $e_2 \neq 0$ and $B(\sqrt{e_2/e_1}) \neq 0$) then $\mathrm{Fix}(D) = \varnothing$. Therefore, for such D, $\bar{X} = \bar{Y}''^D$ is a supersingular RDP Enriques surface with A_1 at the images of D_5^0 and D_9^0. Let $X \to \bar{X}$ be the minimal resolution and let $\bar{Y} = \bar{Y}'' \times_{\bar{X}} X$. Then $\bar{Y}$ is the canonical α_2-covering of the smooth supersingular Enriques surface X with (1) $\mathrm{Sing}(\bar{Y}) = 3D_4^0$, (2) $\mathrm{Sing}(\bar{Y}) = D_4^0 + D_8^0$, (3) $\mathrm{Sing}(\bar{Y}) = D_4^0 + E_8^0$.

The multiple fiber of X corresponds to the fiber $t = \sqrt{e_2/e_1}$ of Y, which is a supersingular elliptic curve.

The singular fiber at $t = 0$ of J is of type III and of Lang type 10A, and the remaining singular fibers are (1) both of type III and of Lang type 10A, (2) of type III and of Lang type 10B, (3) of type II and of Lang type 9B.

Example 5.3 (D_{12}^0). Let $\bar{Y}'$ be the elliptic RDP K3 surface defined by

$$y^2 + t^6 y + x^3 + (t^2 + t^6)x + t^7 = 0,$$
$$y'^2 + y' + x'^3 + (s^6 + s^2)x' + s^5 = 0,$$

where $s = t^{-1}$, $x' = t^{-4}x$, $y' = t^{-6}y$. The RDP of $\bar{Y}'$ and the corresponding singular fiber of the minimal resolution Y are D_{12}^0 (I_8^*) at $t = 0$.

Let $\bar{Y} = \bar{Y}''$ be the elliptic RDP K3 surface which is birational to $\bar{Y}'$ and isomorphic outside the fiber $t = 0$, defined by

$$y^2 + t^6 y + x^3 + (t^2 + t^6)x + t^7 = 0 \quad (t \neq 0),$$
$$y_0^2 + t^4 x_0^2 y_0 + x_0^3 + t^2 x_0 + t^4 x_0^3 + t^3 x_0^4 = 0,$$
$$y'^2 + y' + x'^3 + (s^6 + s^2)x' + s^5 = 0,$$

where the coordinates are given by

$$x_0 = \frac{t^2}{x},\ y_0 = \frac{t^2 y}{x^2}.$$

The RDP of $\bar{Y}$ is D_{12}^0 at $t = x_0 = y_0 = 0$.

Let D_1 and D_2 be the derivations on $\bar{Y}$ defined as follows.

$-$	$D_1(-)$	$D_2(-)$
x	0	t^4
y	t^2	$t^{-2}(x^2+t^6)$
t	t^2	1
x_0	0	$t^2x_0^2$
y_0	x_0^2	$t^2x_0^2+1$
t	t^2	1
x'	0	1
y'	s^4	$s^2+x'^2$
s	1	s^2

Consider the derivation $D = e_1D_1 + e_2D_2$ ($e_1, e_2 \in k$). We observe that $D^2 = 0$ and that if (e_1, e_2) is generic (that is, if $e_2 \neq 0$) then $\mathrm{Fix}(D) = \varnothing$. Therefore, for such D, $X = \bar{Y}^D$ is a supersingular smooth Enriques surface and $\bar{Y}$ is its canonical α_2-covering with $\mathrm{Sing}(\bar{Y}) = D_{12}^0$.

The multiple fiber of X corresponds to the fiber $t = \sqrt{e_2/e_1}$ of Y, which is a supersingular elliptic curve.

The singular fiber at $t = 0$ of J is of type II and of Lang type 9C.

EXAMPLE 5.4 ($D_4^0 + D_8^0$ on the same fiber). Let $\bar{Y}'$ be the elliptic RDP K3 surface defined by

$$y^2 + t^6y + x^3 + tx^2 = 0,$$
$$y'^2 + y' + x'^3 + s^3x'^2 = 0,$$

where $s = t^{-1}$, $x' = t^{-4}x$, $y' = t^{-6}y$. The RDP of $\bar{Y}'$ and the corresponding singular fiber of the minimal resolution Y are D_{13}^0 (I_9^*) at $t = 0$.

Let $\bar{Y}''$ be the elliptic RDP K3 surface which is birational to $\bar{Y}'$ and isomorphic outside the fiber $t = 0$, defined by

$$y^2 + t^6y + x^3 + tx^2 = 0 \quad (t \neq 0),$$
$$y_0^2 + t^4x_0^2y_0 + t^2x_0 + tx_0^2 = 0,$$
$$y_3^2 + t^4y_3 + t^2x_3^3 + tx_3^2 = 0,$$
$$y'^2 + y' + x'^3 + s^3x'^2 = 0,$$

where the coordinates are given by

$$x_0 = \frac{t^2}{x},\ y_0 = \frac{t^2y}{x^2}, \qquad x_3 = \frac{x}{t^2},\ y_3 = \frac{y}{t^2}.$$

Then $\bar{Y}''$ has D_4^0 at $t = x_0 = y_0 = 0$ and D_9^0 at $t = x_3 = y_3 = 0$.

Let D_1 and D_2 be the derivations on $\bar{Y}''$ defined as follows.

$-$	$D_1(-)$	$D_2(-)$
x	t^2	t^4
y	0	$t^{-2}x^2$
t	t^2	0
x_0	x_0^2	$t^2x_0^2$
y_0	0	1
t	t^2	0
x_3	1	t^2
y_3	0	x_3^2
t	t^2	0
x'	s^2	1
y'	0	x'^2
s	1	0

Consider the derivation $D = e_1D_1 + e_2D_2$ $(e_1, e_2 \in k)$. We observe that $D^2 = 0$ and that if (e_1, e_2) is generic (that is, if $e_1 \neq 0$ and $e_2 \neq 0$) then $\mathrm{Fix}(D) = \varnothing$. Therefore, for such D, $\bar{X} = \bar{Y}''^D$ is a supersingular RDP Enriques surface with A_1 at the image of D_9^0. Let $X \to \bar{X}$ be the minimal resolution and let $\bar{Y} = \bar{Y}'' \times_{\bar{X}} X$. Then $\bar{Y}$ is the canonical α_2-covering of the smooth supersingular Enriques surface X with $\mathrm{Sing}(\bar{Y}) = D_4^0 + D_8^0$.

The multiple fiber of X corresponds to the fiber $t = 0$ of Y. In this case this fiber does not move when D vary.

The singular fiber at $t = 0$ of J is of type III and of Lang type 10C.

We also note that in this example the natural morphism $H^0(\bar{Y}, T_{\bar{Y}}) \to H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$ is not injective.

5.2. An example of a canonical covering with an elliptic singularity

EXAMPLE 5.5 (E_{12}). This is the example the author gave in [Mat21, Example 9.4]. Let $\bar{Y} \subset \mathbb{P}^5$ be the intersection of three quadrics

$$\begin{aligned} x_1^2 + x_3^2 + y_1^2 + x_2y_3 + x_3y_2 &= 0,\\ x_2^2 + y_1^2 + y_3^2 + x_1y_3 + x_3y_1 &= 0,\\ y_2^2 + x_1y_2 + x_2y_1 &= 0. \end{aligned}$$

Then it has single singularity at $(x_1, x_2, x_3, y_1, y_2, y_3) = (1, 0, 1, 0, 0, 0)$, which is an EDP singularity of type E_{12}. Letting $s^{-1} = t := \frac{x_2}{y_2} = \frac{x_1+y_2}{y_1}$, $\bar{Y}$ admits a structure of an elliptic surface (without assuming the existence of a section) over $\mathbb{P}^1 = \operatorname{Spec} k[s] \cup \operatorname{Spec} k[t]$. It can be written as the intersection of two quadrics in a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$ as follows:

$$\begin{aligned} (1 + s^2)x_1^2 + s^4x_2^2 + x_3^2 + x_2(sx_3 + y_3) &= 0,\\ s^2x_1^2 + (1 + s^4)x_2^2 + s^2x_2x_3 + x_1(sx_3 + y_3) + y_3^2 &= 0 \end{aligned}$$

over $\operatorname{Spec} k[s]$, and

$$\begin{aligned} (t^2 + 1)y_1^2 + y_2^2 + x_3^2 + x_3y_2 + ty_2y_3 &= 0,\\ y_1^2 + t^2y_2^2 + y_3^2 + x_3y_1 + ty_1y_3 + y_2y_3 &= 0 \end{aligned}$$

over $\operatorname{Spec} k[t]$, glued by

$$y_1 = s(x_1 + y_2),\ y_2 = sx_2, \qquad x_1 = ty_1 + y_2,\ x_2 = ty_2.$$

The (EDP) singularity is at $s = 0$, $(x_1 : x_2 : x_3 : y_3) = (1 : 0 : 1 : 0)$.

Let D_1 and D_2 be the derivations on $\bar{Y}$ defined by

$$D_1(x_i) = 0,\ D_1(y_i) = x_i, \qquad D_2(x_i) = y_i,\ D_2(y_i) = 0.$$

(To be precise, we consider the derivations taking $\frac{y_j}{x_i}$ to $\frac{D_h(y_j)}{x_i} - y_j\frac{D_h(x_i)}{x_i^2}$, etc.) Under the elliptic surface coordinate these derivations are expressed as follows.

$-$	$D_1(-)$	$D_2(-)$
x_1	0	$sx_1 + s^2x_2$
x_2	0	sx_2
x_3	0	y_3
y_3	x_3	0
s	1	s^2
y_1	$ty_1 + y_2$	0
y_2	ty_2	0
y_3	x_3	0
x_3	0	y_3
t	t^2	1

Consider the derivation $D = e_1D_1 + e_2D_2$ $(e_1, e_2 \in k)$. We observe that $D^2 = 0$ and that if (e_1, e_2) is generic (that is, if $e_1 \neq 0$) then $\operatorname{Fix}(D) = \varnothing$. For such D, $X = \bar{Y}^D$ is a supersingular smooth Enriques surface and $\bar{Y}$ is its canonical α_2-covering with $\operatorname{Sing}(\bar{Y}) = E_{12}$.

The multiple fiber of X corresponds to the fiber $t = \sqrt{e_2/e_1}$ of Y, which is a supersingular elliptic curve.

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