Classification of outer actions of discrete amenable groupoids on injective factors

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Abstract

We classify outer actions (or $\mathcal{G}$-kernels) of discrete amenable groupoids on injective factors. Our method based on unified approach for classification of discrete amenable groups actions, and cohomology reduction theorem of discrete amenable equivalence relations. We do not use Katayama-Takesaki type resolution group approach.

1 Introduction

In the theory of operator algebras, research of automorphism groups and group actions is one of the central subject. In particular, classification of automorphisms and group actions has been developed by many hands since seminal works of A. Connes. Here we focus on classification of G-kernels and groupoid actions.

Classification of group actions and G-kernels was first taken place by A. Connes [4], [3]. Since then, G-kernels on injective, semifinite factors has been classified by V. F. R. Jones [8] for finite groups, and A. Ocneanu [20] for general discrete amenable groups. As the generalization of these works, Katayama-Takesaki classified G-kernels of discrete amenable groups on injective factors of type III. (In what follows, we use the term "outer actions" instead of "G-kernels" according to Katayama-Takesaki.) Their method based on clarification of cohomological aspect of outer actions, and they reduced classification to that of genuine group actions, which have been completely classified in [15].

On the other hand, classification of discrete amenable measurable groupoids on injective factors has been also developed, motivated by classification of compact abelian group actions. In [9], V. F. R. Jones and M. Takesaki classified actions of compact abelian groups on semifinite injective factors. By the Pontrjagin duality theorem, they first reduced classification to that of discrete abelian group actions on semifinite von Neumann algebras. Then they reduced to classification of groupoid actions on semifinite factors. This method has been further extended by Sutherland-Takesaki [24], and Kawahigashi-Takesaki [17].

In this paper, we further generalize these result and classify outer action of an amenable discrete groupoid on injective factors. We also realize actions with given invariant.

Our main idea is similar to [12], [13], [14], Namely, we split groupoid actions to isotropy part and equivalence relation part. We apply [18] to classify isotropy part, and [23] to classify equivalence relation part. However there are two big difference between our argument and proceeding ones.

The first point is that we do not use the idea of resolution groups in classification of outer actions. Indeed, Katayama and Takesaki's method is the following. They clarified cohomological aspect of outer actions, and reduced all problems to those of genuine actions of discrete amenable groups by means of resolution groups. One must note that the construction of resolution groups is not canonical. (We explain more detail in the beginning of §4.)

The second point is that we do not use model action type argument in our proof. In [9], [24], [17], model actions with special property were used to erase some obstruction arising in classification. As explained above, we split classification to those of isotropy part and equivalence relation part. Thus one must combine these two parts to obtain complete classification. In this procedure, some obstruction may arise to obtain cocycle conjugacy classification. In our approach, however, such obstruction does not arise, and we can avoid use of model actions, since our classification of group actions in [18] is slightly stronger than [15] and we apply [23] more carefully.

This paper is organized as follows. In §2, we give basic definition of outer actions of groupoids, introduce characteristic invariants of outer actions, and state main classification theorem. In §3, we show necessary facts on outer actions of a discrete amenable group. In particular, we present the classification theorem of outer actions of discrete amenable groups on injective factors. In §4, we present the proof of main classification theorem. In §5, we briefly review random walks of groupoids due to [11] [2], which is necessary to construct model actions. In §6, we construct model actions which realize given invariants. Main part is the construction of free actions on the injective factor of type II_1, where we use results in §5. In Appendix, we discuss relationship between our invariants and Katayama-Takesaki's invariants for outer actions of discrete groups.

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2 Preliminaries on outer actions of groupoids

Our standard references for operator algebras are [26], and for amenable groupoids is [1].

Let $\mathcal{M}$ be a von Neumann algebra, and $(\tilde{\mathcal{M}}, \theta, \tau)$ be its continuous core covariant system. Put $\mathcal{C} = Z(\tilde{\mathcal{M}})$, and $\mathrm{Aut}_\theta(\mathcal{C}) := \{\alpha \in \mathrm{Aut}(\mathcal{C}) \mid \alpha \circ \theta_t = \theta_t \circ \alpha, t \in \mathbb{R}\}$. Let $\tilde{\alpha} \in \mathrm{Aut}(\tilde{\mathcal{M}})$ be the canonical extension of $\alpha \in \mathrm{Aut}(\mathcal{M})$, the Connes-Takesaki module $\mathrm{mod}(\alpha) \in \mathrm{Aut}_\theta(\mathcal{C})$ the restriction of $\tilde{\alpha}$ on $\mathcal{C}$, and $\mathrm{Cnt_r}(\mathcal{M}) = \{\alpha \in \mathrm{Aut}(\mathcal{M}) \mid \tilde{\alpha} \in \mathrm{Int}(\tilde{\mathcal{M}})\}$.

For a groupoid $\mathcal{G}$, we use the following notation. Its source map, range map and unit space are denoted by $s(g)$, $r(g)$ and $\mathcal{G}^{(0)}$, respectively. For $S \subset \mathcal{G}$, $x \in \mathcal{G}^{(0)}$, $S^x := \{g \in S \mid r(g) = x\}$, $S_x := \{g \in S \mid s(g) = x\}$, $S_y^x = S_y \cap S^x$. We denote $x \sim y$, $x, y \in \mathcal{G}^{(0)}$, if there is $g \in \mathcal{G}$ with $s(g) = x$, $r(g) = y$, which is an equivalence relation. Let

$$\mathcal{G}^{(n)} := \{(t_1, t_2, \cdots t_n) \mid t_i \in \mathcal{G}, s(t_i) = r(t_{i+1}), 1 \leq i \leq n-1\}.$$

Let $(\mathcal{G}, \lambda, \nu)$ be a discrete measured groupoid. Namely, $\mathcal{G}^x$ is at most countable, ν is a quasi-invariant measure on $\mathcal{G}^{(0)}$, and $\lambda = \{\lambda^x\}_{x \in \mathcal{G}^{(0)}}$ is the Haar system, where λ^x is the countable measure on $\mathcal{G}^x$. In the following, we simply write $(\mathcal{G}, \lambda, \nu)$ by $\mathcal{G}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be its isotropy part and equivalence relation part, i.e., $\mathcal{H} = \{g \in \mathcal{G} \mid s(g) = r(g)\}$, and $\mathcal{K} = \{(y, x) \in G^{(0)} \times G^{(0)} \mid y \sim x\}$.

In this paper, we mainly treat amenable groupoids. Among many characterization of amenability, we adopt the following definition. (See [1, Definition 3.2.8]).

Definition 2.1 *A discrete measured groupoid $(\mathcal{G}, \lambda, \nu)$ is said to be amenable if there exists a mean $m : L^\infty(\mathcal{G}) \to \ell^\infty(\mathcal{G}^{(0)})$ such that $m(f * \varphi) = f * m(\varphi)$ for all $\varphi \in L^\infty(\mathcal{G})$ and $f \in \mathcal{G}^{(0)}$.*

In this paper, we do not use the above definition explicitly. What we need is the fact that amenability of $\mathcal{G}$ implies that amenability of $\mathcal{H}_x$, and AF property of $\mathcal{K}$.

Hereafter we assume that $\mathcal{G}$ is amenable, and ergodic. Since $\mathcal{K}$ is generated by a single ergodic transformation [5], we can embed $\mathcal{K}$ in $\mathcal{G}$, and express $\mathcal{G}$ as a semidirect product $\mathcal{G} = \mathcal{H} \rtimes \mathcal{K}$ (see [24, p.1091]).

For a measurable family of groups $\{G(x)\}_{x\in\mathcal{G}^0}$, let $C^n(\mathcal{G}, \{G(x)\})$ be a set of maps on $\mathcal{G}^{(n)}$ such that $f(g_1, \cdots g_n) \in G(r(g_1))$, and $f(g_1, \cdots, g_n) = e$ if any of g_i is in $\mathcal{G}^{(0)}$.

Definition 2.2 (1) *An outer action of $\mathcal{G}$ (or $\mathcal{G}$-kernel) on a measurable family of factors $\{\mathcal{M}(x)\}_{x\in\mathcal{G}^{(0)}}$ is a triple (α, w_α, c) consisting of a map*

$$\alpha : g \in \mathcal{G} \to \alpha_g \in \mathrm{Iso}\big(\mathcal{M}(s(g)), \mathcal{M}(r(g))\big),$$

$w_\alpha \in C^2(\mathcal{G}, \{\mathcal{U}(\mathcal{M}(x))\})$ and $c \in C^3(\mathcal{G}, \mathbb{T})$ satisfying

$$\begin{aligned} &\alpha_g \circ \alpha_h = \mathrm{Ad}\, w_\alpha(g, h) \circ \alpha_{gh}, \\ &\alpha_g(w_\alpha(h, k))w_\alpha(g, hk) = c(g, h, k)w_\alpha(g, h)w_\alpha(gh, k). \end{aligned}$$

Here $\mathrm{Iso}(\mathcal{M}_1, \mathcal{M}_2)$ is a set of all isomorphisms from $\mathcal{M}_1$ onto $\mathcal{M}_2$. Note that $c(g, h, k)$ satisfies the 3-cocycle identity

$$c(g, h, k)\overline{c(g, h, kl)}c(g, hk, l)\overline{c(gh, k, l)}c(h, k, l) = 1, \ \ (g, h, k, l) \in \mathcal{G}^{(4)}.$$

(2) *Two outer actions (α, w_α, c) and (β, w_β, c) of $\mathcal{G}$ with the same 3-cocycle c are said to be cocycle conjugate if there exist a measurable family $\{\theta_x\}_{x\in\mathcal{G}^{(0)}}$, $\theta_x \in \mathrm{Aut}(\mathcal{M}(x))$, and $u \in C^1(\mathcal{G}, \{\mathcal{U}(\mathcal{M}(x))\})$ such that*

$$\mathrm{Ad}\, u(g) \circ \alpha_g = \theta_{r(g)} \circ \beta_g \circ \theta^{-1}_{s(g)}, \ \ u(g)\alpha_g(u(h))w_\alpha(g, h)u(gh)^* = \theta_{r(g)}(w_\beta(g, h)).$$

If we can take $\theta_x \in \overline{\mathrm{Int}}(\mathcal{M}(x))$, we say that they are strongly cocycle conjugate.

We introduce invariants of α in a similar way as in the usual group action case. Let $\{\tilde{\mathcal{M}}(x), \theta^x\}$ and $\{\mathcal{C}(x), \theta^x\}$ be the continuous core of $\mathcal{M}(x)$, and the flow of weights of $\mathcal{M}(x)$, respectively. The first invariant is the Connes-Takesaki module $\mathrm{mod}(\alpha_g)$.

We next introduce the characteristic invariant of (α, w_α, c). Let $(\alpha^x, w^x_\alpha, c)$ be the restriction of (α, w_α, c) on $\mathcal{H}_x$. Of course $(\alpha^x, w^x_\alpha, c)$ is an outer action of $\mathcal{H}_x$ on $\mathcal{M}(x)$. Let $\mathcal{N}_{\alpha,x} \subset \mathcal{H}_x$ be a normal subgroup defined by

$$\mathcal{N}_{\alpha,x} := \{g \in \mathcal{H}_x \mid \alpha^x_g \in \mathrm{Cnt_r}(\mathcal{M}(x))\}.$$

Let $V_\alpha(n,g) := w_\alpha(g, g^{-1}ng)w_\alpha(n,g)^*$ for $n, g \in \mathcal{G}$. Then we have

$$\alpha_g \circ \alpha_{g^{-1}ng} \circ \alpha_g^{-1} = \mathrm{Ad}(V_\alpha(n,g)) \circ \alpha_n.$$

Fix $\tilde{u}_\alpha^x \in C^1\left(\mathcal{N}_{\alpha,x}, \mathcal{U}(\tilde{\mathcal{M}}(x))\right)$ such that $\tilde{\alpha}_n^x = \mathrm{Ad}\,\tilde{u}_\alpha^x(n)$. Define $\lambda(n,g), \mu(m,n), d(n,t) \in \mathcal{U}(\mathcal{C}(x))$ by

$$\alpha_g(\tilde{u}^{s(g)}(g^{-1}ng)) = V_\alpha(n,g)\lambda(n,g)\tilde{u}^{r(g)}(n),$$
$$\tilde{u}_\alpha^x(m)\tilde{u}_\alpha^x(n) = \mu(m,n)w_\alpha^x(m,n)\tilde{u}_\alpha^x(mn),$$
$$\theta_t^x(\tilde{u}_\alpha^x(n)) = d(n,t)\tilde{u}_\alpha^x(n).$$

These unitaries enjoy the following relations:

(CC1) $\lambda(n,g)^*\theta_t(\lambda(n,g)) = d(n,t)^*\tilde{\alpha}_g(d(g^{-1}ng,t))$,

(CC2) $d(m,t)d(n,t)d(mn,t)^* = \mu(m,n)^*\theta_t(\mu(m,n))$,

(CC3) $\mu(l,m)\mu(lm,n) = c(l,m,n)\mu(m,n)\mu(l,mn)$,

(CC4) $\lambda(n,gh) = \tilde{\alpha}_g(\lambda(g^{-1}ng,h))\lambda(n,g) \times \overline{c(g,g^{-1}ng,h)}c(n,g,h)c(g,h,h^{-1}g^{-1}ngh)$,

(CC5) $\lambda(mn,g)\lambda(m,g)^*\lambda(n,g)^* = \mu(m,n)\alpha_g(\mu(g^{-1}mg,g^{-1}ng)^*) \times c(m,g,g^{-1}ng)\overline{c(g,g^{-1}mg,g^{-1}ng)c(m,n,g)}$,

(CC6) $\lambda(n,m) = \mu(m,m^{-1}nm)\mu(n,m)^*$,

(CC7) $d(n,t+s) = \theta_t(d(n,s))d(n,t)$,

for $g,h \in \mathcal{G}$, $m,n \in \mathcal{N}$, $t,s \in \mathbb{R}$.

This (λ,μ,d) is called the characteristic cocycle of α. The triple (λ,μ,d) depends on the choice of $\tilde{u}^x(n)$. If we replace $\tilde{u}^x(n)$ by $z^x(n)\tilde{u}^x(n)$ for $z^x(n) \in \mathcal{U}(\mathcal{C}(x))$, then the characteristic cocycle changes into

$$\left(\alpha_g(z^{s(g)}(g^{-1}ng))\lambda(n,g)z^{r(g)}(n)^*, z^x(m)z^x(n)\mu(m,n)z^x(mn)^*, \theta_t(z^x(n))d(n,t)z^x(n)^*\right).$$

The above relation is an equivalence relation on the set of characteristic cocycle for (α, w_α, c). Thus the equivalence class, say $\chi(\alpha) = [\lambda,\mu,d]$, is a true invariant of α. Now we define the invariant of α by $\mathrm{Inv}(\alpha) = (\mathcal{N}_\alpha, \mathrm{mod}(\alpha), \chi(\alpha))$.

The main classification theorem is as follows.

Theorem 2.3 *Let $\mathcal{G}$ an ergodic amenable discrete groupoid, $\{\mathcal{M}(x)\}_{x\in\mathcal{G}^{(0)}}$ be a measurable family of injective factors, and (α, w_α, c) and (β, w_β, c) be two free outer actions of $\mathcal{G}$ on $\{\mathcal{M}(x)\}_{x\in\mathcal{G}^{(0)}}$ with the same 3-cocycle c. Then α and β are strongly cocycle conjugate if and only if* $\mathrm{Inv}(\alpha) = \mathrm{Inv}(\beta)$.

The proof of Theorem 2.3 will be presented in §4.

3 Preparation on outer actions of a group

In this section, we mainly treat group actions.

Let G be a discrete group, and $\mathcal{M}$ be a factor. We gift the topology on $C^1(G, \mathcal{U}(\mathcal{M}))$ by pointwise convergence in the strong topology of $\mathcal{U}(\mathcal{M})$.

In a similar way as in [18], we can classify outer actions of a discrete amenable group on an injective factor.

Theorem 3.1 *Let $\mathcal{M}$ be an injective factor, G a discrete amenable group. Let (α, w_α, c), and (β, w_β, c) be two outer actions of G on $\mathcal{M}$ with the same 3-cocycle c. Then α and β are strongly cocycle conjugate if and only if $(N_\alpha, \mathrm{mod}(\alpha), \chi(\alpha)) = (N_\beta, \mathrm{mod}(\beta), \chi(\beta))$ holds. Namely, there exist $v \in C^1(G, \mathcal{U}(\mathcal{M}))$, and $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$ such that*

$$\mathrm{Ad}\, v(g) \circ \alpha_g = \theta \circ \beta_g \circ \theta^{-1}, v(g)\alpha_g(v(h))w_\alpha(g,h)v(gh)^* = \theta(w_\beta(g,h)).$$

Since the proof is parallel to that of [18], we omit the proof.

Let (α, w_α, c) and (β, w_β, c) be as in Theorem 3.1, $N := N_\alpha = N_\beta$, $[\lambda, \mu, d] := \chi(\alpha) = \chi(\beta)$, and fix $\tilde{u}_\alpha, \tilde{u}_\beta \in C^1(N, \mathcal{U}(\tilde{\mathcal{M}}))$ so that

$$\begin{gathered}\tilde{\alpha}_g(\tilde{u}_\alpha(g^{-1}ng)) = \lambda(n,g)V_\alpha(n,g)\tilde{u}_\alpha(n),\ \tilde{\beta}_g(\tilde{u}_\beta(g^{-1}ng)) = \lambda(n,g)V_\beta(n,g)\tilde{u}_\beta(n), \\ \tilde{u}_\alpha(m)\tilde{u}_\alpha(n) = \mu(m,n)w_\alpha(m,n)\tilde{u}_\alpha(mn),\ \tilde{u}_\beta(m)\tilde{u}_\beta(n) = \mu(m,n)w_\beta(m,n)\tilde{u}_\beta(mn), \\ \theta_t(\tilde{u}_\alpha(n)) = d(n,t)\tilde{u}_\alpha(n),\ \theta_t(\tilde{u}_\beta(n)) = d(n,t)\tilde{u}_\beta(n).\end{gathered}$$

If we carefully examine the proof in [18], it turns out that we can choose $v \in C^1(N, \mathcal{U}(\mathcal{M}))$ so that

$$v(n)\tilde{u}_\alpha(n) = \theta(\tilde{u}_\beta(n)).$$

This choice is crucial in our argument.

In what follows, we assume that $\mathcal{M}$, α are as in Theorem 3.1, and fix the choice of $\tilde{u}_\alpha$.

Let Γ_α be a set of $(\theta, v) \in \mathrm{Ker(mod)} \times C^1(G, \mathcal{U}(\mathcal{M}))$ satisfying

$$\begin{aligned}&(\Gamma 1)\ \ \mathrm{Ad}\, v(g) \circ \alpha_g = \theta \circ \alpha_g \circ \theta^{-1},\ \ g \in G, \\ &(\Gamma 2)\ \ v(g)\alpha_g(v(h))w_\alpha(g,h)v(gh)^* = \theta(w_\alpha(g,h)),\ \ g,h \in G, \\ &(\Gamma 3)\ \ v(n)\tilde{u}_\alpha(n) = \theta(\tilde{u}_\alpha(n)),\ \ n \in N.\end{aligned}$$

We follow [9], [24], [17] for the definition of Γ_α. However, there is one different point comparing with their definition, that is, we further assume the third condition. (Strictly speaking, our definition coincides with one in [24], and differ from ones in [9], [17].) It turns out that the condition $(\Gamma 3)$ avoids us to use model actions in the proof of Theorem 2.3.

Define a group structure of Γ_α by

$$(\theta_1, v_1)(\theta_2, v_2) = (\theta_1\theta_2, \theta_1(v_2(\cdot))v_1(\cdot)),\ \ (\theta, v)^{-1} = (\theta^{-1}, \theta^{-1}(v(\cdot)^*)).$$

By the product topology, we regard Γ_α as a Polish group.

Let $\Gamma_\alpha^0 := \{(\mathrm{Ad}\, w, w\alpha_g(w^*)) \mid w \in \mathcal{U}(\mathcal{M})\}$. Then Γ_α^0 is a normal subgroup of Γ_α.

Remark. When α is a genuine action of G, Γ_α is identified with

$$\{\theta \in \mathrm{Aut}(\mathcal{M} \rtimes_\alpha G) \mid \hat{\alpha} \circ \theta = (\theta \otimes \mathrm{id}) \circ \hat{\alpha}, \tilde{\theta} = \mathrm{id} \text{ on } \tilde{\mathcal{M}}' \cap (\tilde{\mathcal{M}} \rtimes_{\tilde{\alpha}} G)\}$$

as topological groups, where $\hat{\alpha}$ is the dual coaction, and Γ_α^0 is $\{\mathrm{Ad}\, w \mid w \in \mathcal{U}(\mathcal{M})\}$ via this identification.

Lemma 3.2 *We have $\overline{\Gamma_\alpha^0} = \Gamma_\alpha$.*

Proof. The proof is similar to that of [9, Lemma 2.5.6]. Fix a free ultrafilter ω over $\mathbb{N}$. Take $(\theta, v) \in \Gamma_\alpha$. By [16, Theorem 1], $\theta \in \overline{\mathrm{Int}}(\mathcal{M})$. Let $\{w_n\} \subset \mathcal{U}(\mathcal{M})$ such that $\theta = \lim_{n\to\infty} \mathrm{Ad}\, w_n$, and set $W = (w_n) \in \mathcal{M}^\omega$. Since $\theta \circ \alpha_g \circ \theta^{-1} = \mathrm{Ad}\, v(g) \circ \alpha_g$, $V(g) := W^* v(g)\alpha_g(W) \in \mathcal{M}_\omega$. We also have $v(g)\alpha_g(v(h))w_\alpha(g,h)v(gh)^* = \theta(w_\alpha(g,h)) = Ww_\alpha(g,h)W^*$. Thus

$$\begin{aligned}
V(gh)\alpha_g(V(h))V(gh)^* &= W^*v(g)\alpha_g(W)\alpha_g(W^*v(h)\alpha_h(W))V(gh)^* \\
&= W^*v(g)\alpha_g(v(h))w_\alpha(g,h)\alpha_{gh}(W)w_\alpha(g,h)^*V(gh)^* \\
&= W^*v(g)\alpha_g(v(h))w_\alpha(g,h)\alpha_{gh}(W)\alpha_{gh}(W^*)v(gh)^*Ww_\alpha(g,h)^* \\
&= W^*v(g)\alpha_g(v(h))w_\alpha(g,h)v(gh)^*Ww_\alpha(g,h)^* \\
&= 1.
\end{aligned}$$

Hence $V(g)$ is a 1-cocycle for α_g.

We next show $V(n) = 1$ for $n \in N_\alpha$. By the definition of Γ_α and the fact $\mathcal{M}^\omega \subset \tilde{\mathcal{M}}^\omega$ [19], we have

$$v(n)\tilde{u}_\alpha(n) = \theta(\tilde{u}_\alpha(n)) = W\tilde{u}_\alpha(n)W^*,$$

and

$$V(n) = W^*v(n)\alpha_n(W) = W^*v(n)\tilde{u}_\alpha(n)W\tilde{u}_\alpha(n)^* = 1.$$

Thus we can regard $V(\cdot)$ as a G/N_α-cocycle on $\mathcal{M}_\omega$. Again by [16, Theorem 1], α is centrally free as an action of G/N_α. By the 1-cohomology vanishing [20], we can take $Z \in \mathcal{U}(\mathcal{M}_\omega)$ with $Z\alpha_g(Z^*) = V(g)$, and choose a representing sequence $WZ = (w'_n)$, $w'_n \in \mathcal{U}(\mathcal{M})$. Then $(\mathrm{Ad}\, w'_n, w'_n\alpha_g(w'^*_n)) \to (\theta, v)$ as $n \to \omega$. □

4 Classification

In this section, we present a proof of Theorem 2.3 by proving several lemmas.

Before the proof, we briefly explain Katayama-Takesaki's method in [12]. Assume a discrete amenable group G and its 3-cocycle $c \in Z^3(G, \mathbb{T})$ is given. At first, they construct a discrete amenable group $G(c)$ (called the resolution group for c) and its normal subgroup H such that $G(c)/H = G$ and $[\pi_*(c)] \in H^3(G(c), \mathbb{T})$ is trivial, where $\pi : G(c) \to G$ is a quotient map. Then they reduce the problem to construction of genuine actions of $G(c)$. Let α^c be an action of $G(c)$ on a given injective factor $\mathcal{R}$ such that $H \subset \{g \in G(c) \mid \alpha_g \in \mathrm{Int}(\mathcal{R})\}$. Fix a section $\mathfrak{s} : G = G(c)/H \to G(c)$. Then $\alpha_p := \alpha^c_{\mathfrak{s}(p)}$ is an outer action of G on $\mathcal{R}$. The modified HJR-exact sequence [12, Theorem 2.7] describes relation between $\mathrm{Inv}(\alpha)$ and $\mathrm{Inv}(\alpha^c)$, and it is shown that any outer action of G arises as above. In this way, they reduced all problems of outer actions to those of genuine actions, and obtained desired results by applying [15].

Here we remark that $G(c)$ is not the canonical object, and heavily depends on the choice of representative of a 3-cocycle c as remarked in [12, Remark 2.15]. This is an unsatisfactory point of Katayama-Takesaki's method. Therefore, we do not take this approach, and classify and construct outer actions directly without use of resolution groups.

In this section, we mainly denote elements of $\mathcal{H}_x$ by roman alphabet $g, h, \cdots$, and those of $\mathcal{K}$ by Greek letter $\gamma, \delta, \cdots$. We can assume $w_\alpha(\gamma, \delta) = w_\beta(\gamma, \delta) = 1$ for $\gamma, \delta \in \mathcal{K}$, and hence $\alpha|_\mathcal{K}$ and $\beta|_\mathcal{K}$ are genuine actions, since $\mathcal{K}$ is generated by a single transformation.

By Theorem 3.1, α^x and β^x are strongly cocycle conjugate for a.e. $x \in \mathcal{G}^{(0)}$ with conjugating automorphism θ_x. By replacing $\theta_{r(g)} \circ \beta_g \circ \theta_{s(g)}^{-1}$ with β_g and $\theta_x(\tilde{u}_\beta^x(n))$ with $\tilde{u}_\beta^x(n)$ respectively, we can assume that

$$\begin{aligned}
&\operatorname{Ad} v^x(g) \circ \alpha_g^x = \beta_g^x, \\
&\operatorname{mod}(\alpha_\gamma) = \operatorname{mod}(\beta_\gamma), \\
&v^x(g)\alpha_g^x(v^x(h))w_\alpha^x(g,h)v^x(gh)^* = w_\beta^x(g,h), \\
&\alpha_\gamma(\tilde{u}_\alpha^x(\gamma^{-1}n\gamma)) = V_\alpha(n,\gamma)\lambda(n,\gamma)\tilde{u}_\alpha^y(n), \\
&\beta_\gamma(\tilde{u}_\beta^x(\gamma^{-1}n\gamma)) = V_\beta(n,\gamma)\lambda(n,\gamma)\tilde{u}_\beta^y(n), \\
&v^x(m)\tilde{u}_\alpha^x(m) = \tilde{u}_\beta^x(m)
\end{aligned}$$

for $x \in \mathcal{G}^{(0)}$, $g,h \in \mathcal{H}_x$, $m \in \mathcal{N}_x$, $n \in \mathcal{N}_y$, $\gamma = (y,x) \in \mathcal{K}$, and some $v^x \in C^1(\mathcal{H}_x, \mathcal{U}(\mathcal{M}(x)))$.

Let $\Gamma(x) := \Gamma_{\alpha^x}$, $\Gamma^0(x) := \Gamma^0_{\alpha^x}$. As in [9], we can see that $\{\Gamma(x)\}$ is a Borel family of Polish groups.

For our purpose, it is convenient to extend the definition of $\Gamma(x)$ as follows. For $\gamma = (y,x) \in \mathcal{K}$, $\xi \in C^2(\mathcal{H}_y, \mathbb{T})$, and $a \in C^1(\mathcal{N}_y, \mathcal{U}(\mathcal{C}(y)))$, let $\Gamma(\gamma,\xi,a)$ be the set of $(\sigma,u) \in \operatorname{Iso}(\mathcal{M}(x),\mathcal{M}(y)) \times C^1(\mathcal{H}_y, \mathcal{U}(\mathcal{M}(y)))$ such that

$(\Gamma 0)'$ $\operatorname{mod}(\sigma) = \operatorname{mod}(\alpha_\gamma)$,

$(\Gamma 1)'$ $\sigma \circ \alpha^x_{\gamma^{-1}g\gamma} \circ \sigma^{-1} = \operatorname{Ad} u(g) \circ \alpha_g^y$, $g \in \mathcal{H}_y$,

$(\Gamma 2)'$ $u(g)\alpha_g^y(u(h))w_\alpha^y(g,h)u(gh)^* = \xi(g,h)\sigma(w_\alpha^x(\gamma^{-1}g\gamma, \gamma^{-1}h\gamma))$, $g,h \in \mathcal{H}_y$

$(\Gamma 3)'$ $a(n)u(n)\tilde{u}_\alpha^y(n) = \sigma(\tilde{u}_\alpha^x(\gamma^{-1}n\gamma))$, $n \in \mathcal{N}_y$.

Of course, we have $\Gamma((x,x),1,1) = \Gamma(x)$ in this notation.

Define product $(\sigma_1,u_1)(\sigma_2,u_2)$, and inverse $(\sigma,u)^{-1}$ by

$$\left(\sigma_1\sigma_2, \sigma_1(u_2(\gamma_1^{-1}g\gamma_1))u_1(g)\right), \quad (\sigma,u)^{-1} = \left(\sigma^{-1}, \sigma^{-1}(u(\gamma g\gamma^{-1}))^*\right)$$

for $(\sigma,u) \in \Gamma(\gamma,\xi,a)$, $(\gamma_1,\gamma_2) \in \mathcal{K}^{(2)}$, $(\sigma_1,u_1) \in \Gamma(\gamma_1,\xi_1,a_1)$, and $(\sigma_2,u_2) \in \Gamma(\gamma_2,\xi_2,a_2)$.

For $\xi \in C^2(\mathcal{H}_x,\mathbb{T})$, $a \in C^1(\mathcal{N}_x, \mathcal{U}(\mathcal{C}(x)))$, and $\gamma = (y,x) \in \mathcal{K}$, define $\gamma_*(\xi) \in C^2(\mathcal{H}_y,\mathbb{T})$, and $\gamma_*(a) \in C^1(\mathcal{N}_y, \mathcal{U}(\mathcal{C}(y)))$ by

$$\gamma_*(\xi)(g,h) = \xi(\gamma^{-1}g\gamma, \gamma^{-1}hg) \quad \gamma_*(a)(n) = \operatorname{mod}(\alpha_\gamma)(a(\gamma^{-1}n\gamma)).$$

Lemma 4.1 *We have following.*
(1) *Let* $(\gamma_1,\gamma_2) \in \mathcal{K}^{(2)}$. *Then* $(\sigma_1,u_1)(\sigma_2,u_2) \in \Gamma(\gamma_1\gamma_2, \xi_1\gamma_{1*}(\xi_2), a_1\gamma_{1*}(a_2))$ *holds for* $(\sigma_1,u_1) \in \Gamma(\gamma_1,\xi_1,a_1)$, *and* $(\sigma_2,u_2) \in \Gamma(\gamma_2,\xi_2,a_2)$.
(2) *For* $(\sigma,u) \in \Gamma(\gamma,\xi,a)$, $(\sigma,u)^{-1} \in \Gamma(\gamma^{-1}, \gamma_*^{-1}(\xi)^*, \gamma_*^{-1}(a)^*)$.
(3) *We have* $(\sigma_1,u_1)(\theta,v)(\sigma_2,u_2)^{-1} \in \Gamma(y)$ *for* $\gamma = (y,x)$, $(\sigma_1,u_1),(\sigma_2,u_2) \in \Gamma(\gamma,\xi,a)$, *and* $(\theta,v) \in \Gamma(x)$,

Proof. It is routine to show (1) and (2). Then (3) follows from (1) and (2). (Note $\operatorname{mod}(\sigma_1) = \operatorname{mod}(\sigma_2) = \operatorname{mod}(\alpha_\gamma)$.) □

Lemma 4.2 *Put*

$$\zeta_\gamma(g,h) := \overline{c(g,h,\gamma)}c(g,\gamma,\gamma^{-1}h\gamma)\overline{c(\gamma,\gamma^{-1}g\gamma,\gamma^{-1}h\gamma)},\ \ g,h\in\mathcal{H},\gamma\in\mathcal{K},$$
$$\eta_{\gamma_1,\gamma_2}(g) := \overline{c(\gamma_1,\gamma_1^{-1}g\gamma_1,\gamma_2)}c(\gamma_1,\gamma_2,\gamma_2^{-1}\gamma_1^{-1}g\gamma_1\gamma_2)\overline{c(g,\gamma_1,\gamma_2)},\ \ g\in\mathcal{H},\gamma_1,\gamma_2\in\mathcal{K}.$$

We have

(1) $V_\alpha(g,\gamma)\alpha_g^y(V_\alpha(h,\gamma))w_\alpha^y(g,h)V_\alpha(gh,\gamma)^* = \zeta_\gamma(g,h)\alpha_\gamma(w_\alpha^x(\gamma^{-1}g\gamma,\gamma^{-1}h\gamma))$.

(2) $\alpha_{\gamma_1}(V_\alpha(\gamma_1^{-1}g\gamma_1,\gamma_2))V_\alpha(g,\gamma_1) = \eta_{\gamma_1,\gamma_2}(g)V_\alpha(g,\gamma_1\gamma_2)$.

Proof. We only present the proof of (1);

$$\begin{aligned}
&V_\alpha(g,\gamma)\alpha_g^y(V_\alpha(h,\gamma))w_\alpha^y(g,h)V_\alpha(gh,\gamma)^*\\
&= w_\alpha^y(\gamma,\gamma^{-1}g\gamma)w_\alpha^y(g,\gamma)^*\alpha_g^y(w_\alpha^y(\gamma,\gamma^{-1}h\gamma)w_\alpha^y(h,\gamma)^*)w^y(g,h)\\
&\times w_\alpha^y(gh,\gamma)w_\alpha^y(\gamma,\gamma^{-1}gh\gamma)^*\\
&= \overline{c(g,h,\gamma)}w_\alpha^y(\gamma,\gamma^{-1}g\gamma)w_\alpha^y(g,\gamma)^*\alpha_g^y(w_\alpha^y(\gamma,\gamma^{-1}h\gamma))w_\alpha^y(g,h\gamma)w_\alpha^y(\gamma,\gamma^{-1}gh\gamma)^*\\
&= \overline{c(g,h,\gamma)}c(g,\gamma,\gamma^{-1}h\gamma)w_\alpha^y(\gamma,\gamma^{-1}g\gamma)w_\alpha^y(g\gamma,\gamma^{-1}h\gamma)w_\alpha^y(\gamma,\gamma^{-1}gh\gamma)^*\\
&= \overline{c(g,h,\gamma)}c(g,\gamma,\gamma^{-1}h\gamma)\overline{c(\gamma,\gamma^{-1}g\gamma,\gamma^{-1}h\gamma)}\alpha_\gamma(w_\alpha^x(\gamma^{-1}g\gamma,\gamma^{-1}h\gamma)).
\end{aligned}$$

The second equation can be verified in a similar way. □

Lemma 4.3 *For $\gamma=(y,x)\in\mathcal{K}$, set $W(g,\gamma)=\beta_\gamma(v^x(\gamma^{-1}g\gamma)^*)V_\beta(g,\gamma)v^y(g)$. Let $\pi_1(\gamma)=(\alpha_\gamma,V_\alpha(g,\gamma))$, and $\pi_2(\gamma)=(\beta_\gamma,W(g,\gamma))$.*
(1) $\pi_1(\gamma),\pi_2(\gamma)\in\Gamma(\gamma,\zeta_\gamma,\lambda(\cdot,\gamma))$.
(2) $\pi_i(\gamma_1)\pi_i(\gamma_2)=(\mathrm{id},\eta_{\gamma_1,\gamma_2})\pi_i(\gamma_1\gamma_2)$ for $(\gamma_1,\gamma_2)\in\mathcal{K}^{(2)}$ and $i=1,2$.

Proof. (1) By the definition of $V_\alpha(g,\gamma)$, $\lambda(n,\gamma)$, ζ_γ and Lemma 4.2(1), the fact $\pi_1(\gamma)\in\Gamma(\gamma,\zeta_\gamma,\lambda(\cdot,\gamma))$ follows.

For $\pi_2(\gamma)$, by the definition of $W(g,\gamma)$, $\lambda(n,\gamma)$, ζ_γ and Lemma 4.2(1), it is clear that $\pi_2(\gamma)$ satisfies $(\Gamma0)'$, $(\Gamma1)'$, $(\Gamma2)'$. We verify $(\Gamma3)'$. By the choice of $\tilde{u}_\alpha^x(n)$, $\tilde{u}_\beta^x(n)$, and $v^x(n)$, we have

$$\begin{aligned}
\beta_\gamma(u_\alpha^x(\gamma^{-1}n\gamma)) &= \beta_\gamma(v^x(\gamma^{-1}n\gamma)^*)\beta_\gamma(u_\beta^x(\gamma^{-1}n\gamma))\\
&= \lambda(n,\gamma)\beta_\gamma(v^x(\gamma^{-1}n\gamma)^*)V_\beta(n,\gamma)u_\beta^y(n)\\
&= \lambda(n,\gamma)W(n,\gamma)u_\alpha^y(n).
\end{aligned}$$

The statement (2) follows from Lemma 4.2(2). □

Lemma 4.4 *For $\gamma=(y,x)\in\mathcal{K}$, define $F_\gamma:\Gamma(x)\to\Gamma(y)$ by*

$$F_\gamma(\theta,v) := \pi_1(\gamma)(\theta,v)\pi_1(\gamma)^{-1}$$

Then $(\Gamma(x),F_\gamma)$ gives a Borel functor in the sense of [23, Definition 4.1] *with $F_\gamma(\Gamma_x^0)=\Gamma_y^0$.*

Proof. By Lemma 4.1 and Lemma 4.3, $F_\gamma:\Gamma(x)\to\Gamma(y)$ is well-defined. It is clear that $F_\gamma(\Gamma_x^0)=\Gamma_y^0$. Since $(\mathrm{id},\eta_{\gamma_1,\gamma_2})$ commutes with all elements of $\Gamma(r(\gamma_1))$, $F_{\gamma_1}F_{\gamma_2}=F_{\gamma_1\gamma_2}$ by Lemma 4.3(2). □

Lemma 4.5 *Define $\rho(\gamma) \in \Gamma(y)$, $\gamma = (y, x)$, by $\rho(\gamma) = \pi_2(\gamma)\pi_1(\gamma)^{-1}$. Then ρ is a cocycle in the sense $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)F_{\gamma_1}(\rho(\gamma_2))$.*

Proof. By Lemma 4.1 and Lemma 4.3, $\rho(\gamma) \in \Gamma(y)$. Since $(\mathrm{id}, \eta_{\gamma_1,\gamma_2})$ commutes with all element of $\Gamma(y)$,

$$\begin{aligned}\rho(\gamma_1)F_{\gamma_1}(\rho(\gamma_2)) &= \pi_2(\gamma_1)\pi_1(\gamma_1)^{-1}\pi_1(\gamma_1)\pi_2(\gamma_2)\pi_1(\gamma_2)^{-1}\pi_1(\gamma_1)^{-1} \\ &= \pi_2(\gamma_1)\pi_2(\gamma_2)\pi_1(\gamma_2)^{-1}\pi_1(\gamma_1)^{-1} \\ &= (\mathrm{id}, \eta_{\gamma_1,\gamma_2})\pi_2(\gamma_1\gamma_2)\pi_1(\gamma_1\gamma_2)^{-1}(\mathrm{id}, \eta_{\gamma_1,\gamma_2})^{-1} \\ &= \pi_2(\gamma_1\gamma_2)\pi_1(\gamma_1\gamma_2)^{-1} \\ &= \rho(\gamma_1\gamma_2).\end{aligned}$$

□

Remark. We can not show $\pi_2(\gamma) \in \Gamma(\gamma, \zeta_\gamma, \lambda(\cdot, n))$, and $\rho(\gamma) \in \Gamma(y)$ without the condition $v^x(n)\tilde{u}^x_\alpha(n) = \tilde{u}^x_\beta(n)$, which is missing in the proof of [24, Lemma 4.2].

We will explain this point in detail. In our choice, $v(n)$ satisfies the following three equation simultaneously:

$$v^x(n)\tilde{u}^x_\alpha(n) = \tilde{u}^x_\beta(n),$$

$$\alpha_\gamma(\tilde{u}^x_\alpha(\gamma^{-1}n\gamma)) = \lambda(n, \gamma)\tilde{u}^y_\alpha(n), \ \beta_\gamma(\tilde{u}^x_\beta(\gamma^{-1}n\gamma)) = \lambda(n, \gamma)\tilde{u}^y_\beta(n).$$

By the traditional model action splitting argument, we can not get the condition $v^x(n)\tilde{u}^x_\alpha(n) = \tilde{u}^x_\beta(n)$, and only have $v^x(n)\tilde{u}^x_\alpha(n) = c^x(n)\tilde{u}^x_\beta(n)$ for some $c^x(n) \in \mathcal{C}(x)$.

In the rest of this remark, we use notation in [24]. (Actions α_k, m_k in [24] correspond to β_γ and α_γ in our notation.) In [24, p.1109], they define a unitary $a(n) = v(n)u(n)$, claim $\lambda_\alpha = \lambda_m$, and conclude $\alpha_k(a(k^{-1}nk)) = \lambda(n, k)a(n)$. (Unitaries $a(n)$, $v(n)$, $u(n)$ correspond to $\tilde{u}^x_\beta(n)$, $v^x(n)$, $\tilde{u}^x_\alpha(n)$ respectively, in our notation.) However in their definition, $\lambda_\alpha = \lambda_m$ is not clear, and in fact we can only say that $\lambda_\alpha(n, k) = c(k^{-1}nk)c(n)^*\lambda_m(n, k)$ for some $c(n) \in \mathbb{T}$, and the proof of [24, Lemma 4.2(2)] fails. (Of course, this problem occurs only if $\mathcal{N}_\alpha$ is non-trivial.)

We also remark that such problem does not arise in [9], [17], because they treat abelian groups, use the fact that any character of a subgroup of an abelian group can be extended to whole group, and apply the cohomology lemma [23, Theorem 5.5] to groups which are slightly different from our $\Gamma(x)$. Anyway some model action type argument can not be avoided.

We continue the proof of Theorem 2.3. It is obvious that $\gamma = (y, x) \to (\mathrm{id}, 1) \in \Gamma(y)$ is a cocycle. By [23, Theorem 5.5], there exists $P(x) \in \Gamma(x)$ such that

$$(\mathrm{id}, 1) \equiv P(y)\rho(\gamma)F_\gamma(P(x))^{-1} \mod \Gamma^0(y),$$

i.e., for each $\gamma = (y, x) \in \mathcal{K}$, there exists $U(\gamma) \in \mathcal{U}(\mathcal{M}(y))$ such that

$$(\mathrm{Ad}\, U(\gamma), U(\gamma)\alpha^y_g(U(\gamma)^*)) = P(y)\rho(\gamma)F_\gamma(P(x))^{-1}.$$

Lemma 4.6 *Let $P(x) = (\theta_x, z^x(g))$, and $U(g) = \theta_x(v^x(g))z^x(g)$ for $g \in \mathcal{H}_x$. Then*

$$\begin{aligned}\mathrm{Ad}\, U(g) \circ \alpha^x_g &= \theta_x \circ \beta^x_g \circ \theta_x^{-1}, \ \ g \in \mathcal{H}_x, \\ \mathrm{Ad}\, U(\gamma) \circ \alpha_\gamma &= \theta_y \circ \beta_\gamma \circ \theta_x^{-1}, \ \ \gamma = (y, x) \in \mathcal{K},\end{aligned}$$

and

$$\begin{aligned}&\theta_y(w_\beta(g,\gamma)^*)U(g)\alpha_g^y(U(\gamma))w_\alpha(g,\gamma)\\&=\theta_y(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)U(\gamma)\alpha_\gamma(U(\gamma^{-1}g\gamma))w_\alpha(\gamma,\gamma^{-1}g\gamma),\ \ g\in\mathcal{H}_y,\ \ \gamma=(y,x)\in\mathcal{K}.\end{aligned}$$

hold.

Proof. Since $(\theta_x, z^x(g)) \in \Gamma(x)$,

$$\operatorname{Ad}U(g)\circ\alpha_g=\operatorname{Ad}(\theta_x(v^x(g))z^x(g))\circ\alpha_g^x=\operatorname{Ad}\theta_x(v^x(g))\circ\theta_x\circ\alpha_g^x\circ\theta_x^{-1}=\theta_x\circ\beta_g^x\circ\theta_x^{-1}.$$

By

$$\begin{aligned}(\operatorname{Ad}U(\gamma),U(\gamma)\alpha_g^y(U(\gamma)^*))&=P(y)\rho(\gamma)F_\gamma(P(x))^{-1}\\&=P(y)\pi_2(\gamma)\pi_1(\gamma)^{-1}\pi_1(\gamma)P(x)^{-1}\pi_1(\gamma)^{-1}.\end{aligned}$$

we have

$$\left(\operatorname{Ad}U(\gamma),U(\gamma)\alpha_g^y(U(\gamma)^*)\right)\pi_1(\gamma)P(x)=P(y)\pi_2(\gamma).$$

The left hand side is

$$\begin{aligned}&\left(\operatorname{Ad}U(\gamma),U(\gamma)\alpha_g^y(U(\gamma)^*)\right)\pi_1(\gamma)P(x)\\&=\left(\operatorname{Ad}U(\gamma),U(\gamma)\alpha_g^y(U(\gamma)^*)\right)(\alpha_\gamma,V_\alpha(g,\gamma))(\theta_x,z^x(g))\\&=\left(\operatorname{Ad}U(\gamma),U(\gamma)\alpha_g^y(U(\gamma)^*)\right)\left(\alpha_\gamma\theta_x,\alpha_\gamma(z^x(\gamma^{-1}g\gamma))V_\alpha(g,\gamma)\right)\\&=\left(\operatorname{Ad}U(\gamma)\alpha_\gamma\theta_x,U(\gamma)\alpha_g(z^x(\gamma^{-1}g\gamma))V_\alpha(g,\gamma)\alpha_g^y(U(\gamma))^*\right).\end{aligned}$$

The right hand side is

$$\begin{aligned}P(y)\pi_2(\gamma)&=(\theta_y,z^y(g))(\beta_\gamma,W(g,\gamma))\\&=(\theta_y\beta_\gamma,\theta_y(W(g,\gamma))z^y(g)).\end{aligned}$$

By comparing the first component, we have $\operatorname{Ad}U(\gamma)\circ\alpha_\gamma=\theta_y\circ\beta_\gamma\circ\theta_x^{-1}$.

We next compare the second component;

$$\begin{aligned}U(\gamma)\alpha_g(z^x(\gamma^{-1}g\gamma))V_\alpha(g,\gamma)\alpha_g^y(U(\gamma))^*&=\theta_y(W(g,\gamma))z^y(g)\\&=\theta_y\beta_\gamma(v^x(\gamma^{-1}g\gamma))^*\theta_y(V_\beta(g,\gamma))\theta_y(v^y(g))z^y(g)\\&=U(\gamma)\alpha_\gamma(v^x(\gamma^{-1}g\gamma))^*U(\gamma)^*\theta_y(V_\beta(g,\gamma))U(g).\end{aligned}$$

In the last equality, we used $\operatorname{Ad}U(\gamma)\circ\alpha_\gamma=\theta_y\circ\beta_\gamma\circ\theta_x^{-1}$. Hence we obtain

$$U(\gamma)\alpha_\gamma(U(\gamma^{-1}g\gamma))V_\alpha(g,\gamma)\alpha_g^y(U(\gamma))^*=\theta_y(V_\beta(g,\gamma))U(g).$$

Hence we get the conclusion by $V_\alpha(g,\gamma)=w_\alpha(\gamma,\gamma^{-1}g\gamma)w_\alpha(g,\gamma)^*$. □

We extend the definition of $U(\cdot)$ for whole $\mathcal{G}$ by

$$U(g\gamma):=\theta_y(w_\beta(g,\gamma)^*)U(g)\alpha_g^y(U(\gamma))w_\alpha(g,\gamma),\ \ g\in\mathcal{H}_y,\ \ \gamma=(y,x)\in\mathcal{K}.$$

By the above result, we also have

$$U(g\gamma)=\theta_y(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)U(\gamma)\alpha_\gamma(U(\gamma^{-1}g\gamma))w_\alpha(\gamma,\gamma^{-1}g\gamma).$$

We can easily see $\mathrm{Ad}\, U(g)\circ\alpha_g=\theta_{r(g)}\circ\beta_g\circ\theta_{s(g)}^{-1}$, $g\in\mathcal{G}$.

Remark. Here we emphasize that the condition $v^x(n)\tilde{u}^x_\alpha(n)=\tilde{u}^x_\beta(n)$ is essential to deduce Lemma 4.6. Without this condition, we can say only weaker statement, i.e.,

$$U(g\gamma)=\varphi(g,\gamma)\theta_y(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)U(\gamma)\alpha_\gamma(U(\gamma^{-1}g\gamma))w_\alpha(\gamma,\gamma^{-1}g\gamma)$$

for some $\varphi(g,\gamma)\in\mathbb{T}$, which is an obstruction for cocycle conjugacy of actions. In [9], [24], [17], such obstruction appears, and model actions with special property are used to get rid of $\varphi(g,\gamma)$.

By verifying the following lemma, we finish the proof of Theorem 2.3.

Lemma 4.7 *We have*

$$U(g)\alpha_g(U(h))w_\alpha(g,h)U(gh)^*=\theta_{r(g)}(w_\beta(g,h)),\ \ g,h\in\mathcal{G}.$$

Proof. We denote $\theta_{r(g)}$ by θ for simplicity. We first show lemma for $w_\alpha(g\gamma,h)$ and $w_\alpha(g\gamma,\delta)$ for $g\in\mathcal{H}_y$, $h\in\mathcal{H}_x$, $\gamma,\delta\in\mathcal{K}$. First note that

$$w_\alpha(g\gamma,\delta)=\overline{c(g,\gamma,\delta)}w_\alpha(g,\gamma)^*w_\alpha(g,\gamma\delta).$$

Then we have

$$\begin{aligned}
&U(g\gamma)\alpha_{g\gamma}(U(\delta))w_\alpha(g\gamma,\delta)U(g\gamma\delta)^*\\
&=\overline{c(g,\gamma,\delta)}\theta_y(w_\beta(g,\gamma)^*)U(g)\alpha_g(U(\gamma))w_\alpha(g,\gamma)\alpha_{g\gamma}(U(\delta))w_\alpha(g,\gamma)^*w_\alpha(g,\gamma\delta)\\
&\times w_\alpha(g,\gamma\delta)^*\alpha_g(U(\gamma\delta)^*)U(g)^*\theta(w_\beta(g,\gamma\delta))\\
&=\overline{c(g,\gamma,\delta)}\theta_y(w_\beta(g,\gamma)^*)U(g)\alpha_g(U(\gamma))\alpha_g\alpha_\gamma(U(\delta))\alpha_g(U(\gamma\delta)^*)U(g)^*\theta(w_\beta(g,\gamma\delta))\\
&=\overline{c(g,\gamma,\delta)}\theta_y(w_\beta(g,\gamma)^*)U(g)\alpha_g(U(\gamma\delta))\alpha_g(U(\gamma\delta)^*)U(g)^*\theta(w_\beta(g,\gamma\delta))\\
&=\overline{c(g,\gamma,\delta)}\theta_y(w_\beta(g,\gamma)^*w_\beta(g,\gamma\delta))\\
&=\theta(w_\beta(g\gamma,\delta)).
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
w_\alpha(g\gamma,h)&=w_\alpha(\gamma\gamma^{-1}g\gamma,h)\\
&=\overline{c(\gamma,\gamma^{-1}g\gamma,h)}w_\alpha(\gamma,\gamma^{-1}g\gamma)^*\alpha_\gamma(w_\alpha(\gamma^{-1}g\gamma,h))w_\alpha(\gamma,\gamma^{-1}g\gamma h).
\end{aligned}$$

Thus

$$\begin{aligned}
&U(g\gamma)\alpha_{g\gamma}(U(h))w_\alpha(g\gamma,h)U(g\gamma h)\\
&=\theta(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)U(\gamma)\alpha_\gamma(U(\gamma^{-1}g\gamma))w_\alpha(\gamma,\gamma^{-1}g\gamma)\alpha_{g\gamma}(U(h))\\
&\quad\times\overline{c(\gamma,\gamma^{-1}g\gamma,h)}w_\alpha(\gamma,\gamma^{-1}g\gamma)^*\alpha_\gamma(w_\alpha(\gamma^{-1}g\gamma,h))w_\alpha(\gamma,\gamma^{-1}g\gamma h)\\
&\quad\times w_\alpha(\gamma,\gamma^{-1}g\gamma h)^*\alpha_\gamma(U(\gamma^{-1}g\gamma h))^*U(\gamma)^*\theta(w_\beta(\gamma,\gamma^{-1}g\gamma h))\\
&=\overline{c(\gamma,\gamma^{-1}g\gamma,h)}\theta(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)U(\gamma)\alpha_\gamma(U(\gamma^{-1}g\gamma))\alpha_\gamma\alpha_{\gamma^{-1}g\gamma}(U(h))\\
&\quad\times\alpha_\gamma\left(w_\alpha(\gamma^{-1}g\gamma,h)\right)\alpha_\gamma\left(U(\gamma^{-1}g\gamma h)\right)^*U(\gamma)^*\theta(w_\beta(\gamma,\gamma^{-1}g\gamma h))\\
&=\overline{c(\gamma,\gamma^{-1}g\gamma,h)}\theta(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)\theta\beta_\gamma\theta^{-1}\left(U(\gamma^{-1}g\gamma)\alpha_{\gamma^{-1}g\gamma}(U(h))w_\alpha(\gamma^{-1}g\gamma,h)u^*_{\gamma^{-1}g\gamma h}\right)\\
&\quad\times\theta(w_\beta(\gamma,\gamma^{-1}g\gamma h))\\
&=\overline{c(\gamma,\gamma^{-1}g\gamma,h)}\theta(w_\beta(\gamma,\gamma^{-1}g\gamma)^*)\theta\beta_\gamma\left(w_\beta(\gamma^{-1}g\gamma,h)\right)\theta(w_\beta(\gamma,\gamma^{-1}g\gamma h))\\
&=\theta(w_\beta(g\gamma,h)),
\end{aligned}$$

and hence the statement holds for any $w_\alpha(g,h)$, $g\in\mathcal{G}$, $h\in\mathcal{H}$, or $h\in\mathcal{K}$.

Finally, by using the equality

$$w_\alpha(g,h\gamma)=c(g,h,\gamma)\alpha_g(w_\alpha(h,\gamma))^*w_\alpha(g,h)w_\alpha(gh,\gamma),\ \ g\in\mathcal{G},h\in\mathcal{H},\gamma\in\mathcal{K},$$

we can show the statement in a similar way as follows;

$$\begin{aligned}
&U(g)\alpha_g(U(h\gamma))w_\alpha(g,h\gamma)U(gh\gamma)^*\\
&=U(g)\alpha_g\left(\theta(w_\beta(h,\gamma)^*)U(h)\alpha_h(U(\gamma))w_\alpha(h,\gamma)\right)\\
&\quad\times c(g,h,\gamma)\alpha_g(w_\alpha(h,\gamma))^*w_\alpha(g,h)w_\alpha(gh,\gamma)U(gh\gamma)^*\\
&=c(g,h,\gamma)U(g)\alpha_g\theta(w_\beta(h,\gamma)^*)\alpha_g(U(h))\alpha_g\alpha_h(U(\gamma))w_\alpha(g,h)w_\alpha(gh,\gamma)U(gh\gamma)^*\\
&=c(g,h,\gamma)\theta\beta_g(w_\beta(h,\gamma)^*)U(g)\alpha_g(U(h))w_\alpha(g,h)\alpha_{gh}(U(\gamma))w_\alpha(gh,\gamma)U(gh\gamma)^*\\
&=c(g,h,\gamma)\theta\beta_g(w_\beta(h,\gamma)^*)\theta(w_\beta(g,h)w_\beta(gh,\gamma))\\
&=\theta\left(w_\beta(g,h\gamma)\right).
\end{aligned}$$

□

Thus we have shown Theorem 2.3, which is a generalization of [24] to outer action case.

5 Random walks on groupoids

In this section, we make preparation for construction of model actions. For this purpose, we collect necessary facts on random walks on groupoids. Contents of this section is mainly based on [11], [2].

Let f be a function on $\mathcal{G}$. We often denote its restriction on $\mathcal{G}^x$ by f^x. Let f be a function on $\mathcal{G}^x$, and $g\in\mathcal{G}^y_x$. Then $g\cdot f(a):=f(g^{-1}a)$ is a function on $\mathcal{G}^y$.

Definition 5.1 *Let μ be a measurable function on $\mathcal{G}$. We say μ is a family of probability measure if each μ^x is a probability measure on $\mathcal{G}^x$, and often denote it by $\mu=\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$.*

Let $f\in\ell^1(\mathcal{G}^x)$, and $\mu=\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$ be a family of probability measures on $\mathcal{G}$ such that the support of μ^x is included in $\mathcal{G}^x$. Define $f*\mu\in\ell^1(\mathcal{G}^x)$ by

$$\begin{aligned}
f*\mu(a)&=\sum_{h\in\mathcal{G}_{s(a)}}f(ah^{-1})\mu^{r(h)}(h)=\sum_{y\sim x}\sum_{h\in\mathcal{G}^y_{s(a)}}f(ah^{-1})\mu^y(h)\\
&=\sum_{y\sim x}\sum_{h\in\mathcal{G}^x_y}f(h)\mu^y(h^{-1}a)=\sum_{y\sim x}\sum_{h\in\mathcal{G}^x_y}f(h)h\cdot\mu^y(a).
\end{aligned}$$

Lemma 5.2 *We have $\|f*\mu\|_1\le\|f\|_1$ for $f\in\ell^1(\mathcal{G}^x)$, where equality holds when f is positive.*

Proof.

$$
\begin{aligned}
\|f*\mu\|_1 &= \sum_{a\in\mathcal{G}^x} |f*\mu(a)| = \sum_{z\sim x}\sum_{a\in\mathcal{G}^x_z} |f*\mu(a)| \\
&= \sum_{z\sim x}\sum_{a\in\mathcal{G}^x_z} \left| \sum_{(y,x)\in\mathcal{K}}\sum_{h\in\mathcal{G}^y_z} f(ah^{-1})\mu^y(h) \right| \\
&\leq \sum_{z\sim x}\sum_{a\in\mathcal{G}^x_z}\sum_{(y,x)\in\mathcal{K}}\sum_{h\in\mathcal{G}^y_z} \left|f(ah^{-1})\right| \mu^y(h) \\
&= \sum_{z\sim x}\sum_{(y,x)\in\mathcal{K}}\sum_{h\in\mathcal{G}^y_z}\sum_{a\in\mathcal{G}^x_y} |f(a)|\,\mu^y(h) \\
&= \sum_{(y,x)\in\mathcal{K}}\sum_{h\in\mathcal{G}^y}\sum_{a\in\mathcal{G}^x_y} |f(a)|\,\mu^y(h) = \sum_{y\sim x}\sum_{a\in\mathcal{G}^x_y} |f(a)| \\
&= \|f\|_1
\end{aligned}
$$

If f is positive, the forth inequality becomes an equality. □

Let $\mu = \{\mu^x\}$, and $\nu = \{\nu^x\}$ be families of probability measures. By Lemma 5.2, $\{\mu^x*\nu\}$ is also a family of probability measures. Hence we can define a family of probability measure $\mu*\nu$ by $(\mu*\nu)^x := \mu^x*\nu$.

Let μ be a family of probability measures on $\mathcal{G}$. For each $x\in\mathcal{G}^{(0)}$, define a positive operator P^x_μ on $\ell^\infty(\mathcal{G}^x)$ by

$$
P^x_\mu(f)(g) = \sum_{h\in\mathcal{G}^x} \mu^{s(g)}(g^{-1}h)f(h) = \sum_{h\in\mathcal{G}^{s(g)}} f(gh)\mu^{s(g)}(h).
$$

Lemma 5.3 *P^x_μ is a unital positive operator, and $P^x_\mu P^x_\nu = P^x_{\mu*\nu}$.*

Proof. Positivity of P^x_μ is clear. We first show $P^x_\mu(1) = 1$;

$$
P^x_\mu(1)(g) = \sum_{h\in\mathcal{G}^x} \mu^{s(g)}(g^{-1}h) = \sum_{h\in\mathcal{G}^{s(g)}} \mu^{s(g)}(h) = 1.
$$

We next show $P^x_\mu P^x_\nu = P^x_{\mu*\nu}$. Let $y = s(g)$. Then

$$
\begin{aligned}
P^x_\mu P^x_\nu(f)(g) &= \sum_{z\sim x}\sum_{h\in\mathcal{G}^x_z} \mu^y(g^{-1}h)P_\nu(f)(h) \\
&= \sum_{z\sim x}\sum_{h\in\mathcal{G}^x_z}\sum_{k\in\mathcal{G}^x} \mu^y(g^{-1}h)\nu^z(h^{-1}k)f(k) \\
&= \sum_{k\in\mathcal{G}^x}\left(\sum_{z\sim x}\sum_{h\in\mathcal{G}^x_z} \mu^y(g^{-1}h)\nu^z(h^{-1}k)\right) f(k) \\
&= \sum_{k\in\mathcal{G}^x}\left(\sum_{z\sim x}\sum_{l\in\mathcal{G}^z_{s(k)}} \mu^y(g^{-1}kl^{-1})\nu^z(l)\right) f(k) \\
&= \sum_{k\in\mathcal{G}^x} \mu^y*\nu(g^{-1}k)f(k) \\
&= P^x_{\mu*\nu}(f)(g).
\end{aligned}
$$

□

Lemma 5.4 *For $\theta \in \ell^1(\mathcal{G}^x)$, define $\theta P_\mu^x \in \ell^1(\mathcal{G}^x)$ by*

$$\langle \theta P_\mu^x, f\rangle = \langle \theta, P_\mu^x f\rangle,\ f \in \ell^\infty(\mathcal{G}^x).$$

Then θP_μ^x is given by $\theta * \mu$.

Proof.

$$\begin{aligned}
\langle \theta, P_\mu^x f\rangle &= \sum_{y\sim x}\sum_{g\in \mathcal{G}_y^x} \theta(g) P_\mu^x(f)(g) \\
&= \sum_{y\sim x}\sum_{g\in \mathcal{G}_y^x}\sum_{h\in\mathcal{G}^x} \theta(g)\mu^y(g^{-1}h) f(h) \\
&= \sum_{h\in\mathcal{G}^x}\left(\sum_{y\sim x}\sum_{g\in \mathcal{G}_y^x} \theta(g)\mu^y(g^{-1}h)\right) f(h) \\
&= \sum_{h\in\mathcal{G}_z^x} \theta * \mu(h) f(h) \\
&= \langle \theta * \mu, f\rangle
\end{aligned}$$

□

In [2, Proposition 5.4], C- H. Chu and X. Li proved the following result by using the Reiter condition, which is equivalent to the amenability of $\mathcal{G}$ [1, Proposition 3.2.14], and concluded that an amenable groupoid is Liouville, i.e., the Poisson boundary of P_μ^x is trivial.

Theorem 5.5 *Let $\mathcal{G}$ be an amenable discrete measured groupoid. Then there exists a family of probability measures $\mu = \{\mu^x\}$ such that $\lim_n \|g\cdot \mu^{*n,s(g)} - \mu^{*n,r(g)}\|_1 = 0$ for a.e. $x \in \mathcal{G}^{(0)}$ and every $g \in \mathcal{G}^x$, where $\mu^{*n} = \underbrace{\mu * \cdots * \mu}_{n\text{-times}}$.*

If we look their proof, we can see that μ can be chosen so that $\mathrm{supp}(\mu^x) = \mathcal{G}^x$. Thus we always assume $\mathrm{supp}(\mu^x) = \mathcal{G}^x$, when we apply Theorem 5.5.

For our purpose, we need stronger result than the Liouville property.

Theorem 5.6 *Let $\mathcal{G}$ be an amenable, discrete measured groupoid, and take $\mu = \{\mu^x\}$ as in Theorem 5.5. Then the tail boundary of P_μ^x is trivial for a.e. $x \in \mathcal{G}^{(0)}$.*

Proof. We denote P_μ^x by P for simplicity. Let $\{u_n\} \subset \ell^\infty(\mathcal{G}^x)$ be a bounded harmonic sequence, i.e., $P(u_{n+1}) = u_n$, $n \in \mathbb{N}$, and $\sup_n \|u_n\|_\infty = M < \infty$. Since the tail boundary of P is identified with the space of all bounded harmonic sequences [10], we only have to show that u_n is a constant function for any $n \in \mathbb{N}$. Let $e \in \mathcal{G}^{(0),x}$ a unit element. On one hand, $P^k(u_{n+k}) = u_n$, and hence we have

$$\delta_g P^k(u_{n+k}) - \delta_e P^k(u_{n+k}) = \langle \delta_g - \delta_e, u_n\rangle = u_n(g) - u_n(e).$$

On the other hand, we have

$$\begin{aligned}\left|\delta_g P^k(u_{n+k}) - \delta_e P^k(u_{n+k})\right| &= \left|\langle \delta_g * \mu^{*k} - \delta_e * \mu^{*k}, u_{n+k}\rangle\right| \\ &\leq \|\delta_g * \mu^{*k} - \delta_e * \mu^{*k}\|_1 \|u_{n+k}\|_\infty \\ &\leq M\|g\cdot \mu^{*k,s(g)} - \mu^{*k,x}\|_1 \\ &\to 0\end{aligned}$$

as $k \to \infty$ for all $g \in \mathcal{G}^x$ by Lemma 5.3, Lemma 5.4 and the choice of μ. Hence u_n is a constant function, and $u_n = P(u_{n+1}) = u_{n+1}$. □

Remark. (1) By [10, Theorem 2.1], the tail boundary of $P = P^x_\mu$ is trivial if and only if $\lim_{n\to\infty} \|\delta_e P^{n+d} - \nu P^n\|_1 = 0$ for any $d \in \mathbb{N}$, and probability measure ν on $\mathcal{G}^x$ with $\nu \prec \delta_e P^d$. We can prove Theorem 5.6 by verifying this condition.
(2) If we assume $\mathcal{G}$ is an etale groupoid, then $\mathcal{G}$ is topologically amenable if and only if measurewise amenable, i.e., $(\mathcal{G}, \nu)$ is amenable for every quasi-invariant measure ν on $\mathcal{G}^{(0)}$ by [1, Theorem 3.3.7]. Then $\mathcal{G}$ has a Reiter condition by [1, Corollary 3.3.8], and we can get rid of "a.e." in Theorem 5.6.

6 Construction of model actions

In this section, we construct an outer action of $\mathcal{G}$ with given invariant. As stated in §4, we do not use resolution group method of Katayama-Takesaki [12].

We first construct a free outer action on the injective factor of type II_1 in §6.1. This construction is inspired by ones in subfactor theory [21], [6].

Then we realize outer actions with given invariant in §6.2 by the same method used in [18].

6.1 Existence of free actions

Let $\mathcal{G}$ be a discrete measurable groupoid. (For instance, we do not assume the amenability of $\mathcal{G}$.) We use the following notations;

$$\mathcal{G}^{x,(n)} := \{(t_1, t_2, \cdots t_n) \in \mathcal{G}^{(n)} \mid t_1 \in \mathcal{G}^x\},$$

$$\mathcal{G}^{(n)}_y := \{(t_1, t_2, \cdots t_n) \in \mathcal{G}^{(n)} \mid t_n \in \mathcal{G}_y\}$$

and $\mathcal{G}^{x,(n)}_y = \mathcal{G}^{x,(n)} \cap \mathcal{G}^{(n)}_y$.

Define $B_n(k)$ and $A_n(x)$, $k \in \mathcal{G}$, $x \in \mathcal{G}^{(0)}$, $n \geq 1$, by

$$B_n(k) := B\left(\ell^2\left(\mathcal{G}^{r(k),(n)}_{s(k)}\right)\right), \quad A_n(x) := \bigoplus_{k\in\mathcal{G}^x} B_n(k).$$

Let us denote a matrix unit of $B_n(k)$ by $e_{\boldsymbol{s},\boldsymbol{t}}$, $\boldsymbol{s}, \boldsymbol{t} \in \mathcal{G}^{r(k),(n)}_{s(k)}$. For $g \in \mathcal{G}$, and $\boldsymbol{t} = (t_1, t_2, \cdots t_n) \in \mathcal{G}^{s(g),(n)}$, define $g\boldsymbol{t} = (gt_1, t_2, \cdots t_n) \in \mathcal{G}^{r(g),(n)}$.

Fix a 3-cocycle c of $\mathcal{G}$. Define $u^n_g(k)$, $w^n(g,h)(k)$, $I_n(k,l)$ by

$$u^n_g(k) := \sum_{\boldsymbol{t}\in\mathcal{G}^{s(g),(n)}_{s(k)}} c(g, t_1, t_1^{-1}g^{-1}k) e_{g\boldsymbol{t},\boldsymbol{t}} \in B\left(l^2(\mathcal{G}^{s(g),(n)}_{s(k)}), l^2(\mathcal{G}^{r(g),(n)}_{s(k)})\right), \; n \geq 1, \; s(g) = r(k),$$

$$w^n(g,h)(k) := \sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(h),(n)}} \overline{c(g,h,t_1)} e_{gh\boldsymbol{t},gh\boldsymbol{t}} \in B_n(k),\ n\geq 1,\ (g,h)\in\mathcal{G}^{(2)}$$

$$I_n(k,l) := \sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{r(k),(n-1)}} \overline{c(t_1,t_1^{-1}k,l)} e_{\boldsymbol{t},\boldsymbol{t}}\otimes e_{ll} \in B_n(kl),\ n\geq 2,\ (k,l)\in\mathcal{G}^{(2)}$$

Of course, $u_g^n(k)$ and $w^n(g,h)(k)$ are unitaries, and $I_n(k,l)I_n(k,l)^* = I_n(k,l)^* I_n(k,l)^* = 1\otimes e_{ll}$.

Let Define two injective homomorphisms $\alpha_g^n : A_n(s(g))\to A_n(r(g))$, and $\phi_n^x : A_n(x)\to A_{n+1}(x)$ by

$$\alpha_g^n(a)(k) := \mathrm{Ad}\, u_g^n(k)\left(a(g^{-1}k)\right),$$

and

$$\phi_n^x(b)(k) := \sum_{l\in\mathcal{G}_{s(k)}} I^{n+1}(kl^{-1},l)\left(b(kl^{-1})\otimes 1\right) I^{n+1}(kl^{-1},l)^*$$

respectively for $a=\bigoplus_{k\in\mathcal{G}^{s(g)}} a(k)\in A_n(s(g))$, and $b=\bigoplus_{k\in\mathcal{G}^x} b(k)\in A_n(x)$.

Lemma 6.1 *We have following relations.*
(1) $u_g^n(k)u_h^n(g^{-1}k) = c(g,h,h^{-1}g^{-1}k)w^n(g,h)(k)u_{gh}^n(k)$, $(h^{-1},g^{-1},k)\in\mathcal{G}^{(3)}$.
(2) $c(g,g^{-1}k,l)I_{n+1}(k,l)(u_g^n(k)\otimes 1) = u_g^{n+1}(kl)I_{n+1}(g^{-1}k,l)$, $(g^{-1},k,l)\in\mathcal{G}^{(3)}$.

Proof. (1) It is shown as follows;

$$\begin{aligned}
u_g^n(k)u_h^n(g^{-1}k) &= \left(\sum_{\boldsymbol{s}\in\mathcal{G}_{s(k)}^{s(g),(n)}} c(g,s_1,s_1^{-1}g^{-1}k)e_{g\boldsymbol{s},\boldsymbol{s}}\right)\left(\sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(h),(n)}} c(h,t_1,t_1^{-1}h^{-1}g^{-1}k)e_{h\boldsymbol{t},\boldsymbol{t}}\right)\\
&= \sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(h),(n)}} c(g,ht_1,t_1^{-1}h^{-1}g^{-1}k)c(h,t_1,t_1^{-1}h^{-1}g^{-1}k)e_{gh\boldsymbol{t},\boldsymbol{t}}\\
&= \sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(h),(n)}} \overline{c(g,h,t_1)}c(g,h,h^{-1}g^{-1}k)c(gh,t_1,t_1^{-1}h^{-1}g^{-1}k)e_{gh\boldsymbol{t},\boldsymbol{t}}\\
&= c(g,h,h^{-1}g^{-1}k)w^n(g,h)(k)u_{gh}^n(k).
\end{aligned}$$

Here we apply the 3-cocycle identity

$$\begin{aligned}
&c(g,h,t_1)\overline{c(g,h,h^{-1}g^{-1}k)}c(g,ht_1,t_1^{-1}h^{-1}g^{-1}k)\overline{c(gh,t_1,t_1^{-1}h^{-1}g^{-1}k)}c(h,t_1,t_1^{-1}h^{-1}g^{-1}k)\\
&=1
\end{aligned}$$

for $(g,h,t_1,t_1^{-1}h^{-1}g^{-1}k)$ at the third equality.
(2) The left hand side becomes as follows;

$$\begin{aligned}
&I_{n+1}(k,l)(u_g^n(k)\otimes 1)\\
&= \left(\sum_{\boldsymbol{s}\in\mathcal{G}_{s(k)}^{r(k),(n)}} \overline{c(s_1,s_1^{-1}k,l)}e_{\boldsymbol{s},\boldsymbol{s}}\otimes e_{ll}\right)\left(\sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(g),(n)}} c(g,t_1,t_1^{-1}gk)e_{g\boldsymbol{t},\boldsymbol{t}}\otimes 1\right)\\
&= \sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(g),(n)}} \overline{c(gt_1,t_1^{-1}g^{-1}k,l)}c(g,t_1,t_1^{-1}gk)e_{g\boldsymbol{t},\boldsymbol{t}}\otimes e_{ll}.
\end{aligned}$$

The right hand side is as follows;

$$
\begin{aligned}
&u_g^{n+1}(kl)I_{n+1}(g^{-1}k,l)\\
&=\left(\sum_{\boldsymbol{s}\in\mathcal{G}_{s(l)}^{s(g),(n+1)}} c(g,s_1,s_1^{-1}g^{-1}kl)e_{g\boldsymbol{s},\boldsymbol{s}}\right)\left(\sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(g),(n)}} \overline{c(t_1,t_1^{-1}g^{-1}k,l)}e_{\boldsymbol{t},\boldsymbol{t}}\otimes e_{ll}\right)\\
&=\sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{s(g),(n)}} c(g,t_1,t_1^{-1}g^{-1}kl)\overline{c(t_1,t_1^{-1}g^{-1}k,l)}e_{g\boldsymbol{t},\boldsymbol{t}}\otimes e_{ll}.
\end{aligned}
$$

By the 3-cocycle identity

$$
c(g,t_1,t_1^{-1}g^{-1}k)\overline{c(g,t_1,t_1^{-1}g^{-1}kl)}c(g,g^{-1}k,l)\overline{c(gt_1,t_1^{-1}g^{-1}k,l)}c(t_1,t_1^{-1}g^{-1}k,l)=1
$$

for $(g,t_1,t_1^{-1}g^{-1}k,l)$, we have

$$
c(g,g^{-1}k,l)I_{n+1}(k,l)(u_g(k)\otimes 1)=u_g^{n+1}(kl)I_{n+1}(g^{-1}k).
$$

□

Lemma 6.2 *We have following.*
(1) $w^{n+1}(g,h)=\phi_n^{r(g)}(w^n(g,h))$, $(g,h)\in\mathcal{G}^{(2)}$.
(2) $\phi_n^{r(g)}\circ\alpha_g^n=\alpha_g^{n+1}\circ\phi_n^{s(g)}$.
(3) $\alpha_g^n\circ\alpha_h^n=\operatorname{Ad}w^n(g,h)\circ\alpha_{gh}$, $(g,h)\in\mathcal{G}^{(2)}$.
(4) $\alpha_g^{(n)}(w^1(h,k))w^n(g,hk)=c(g,h,k)w^n(g,h)w^n(gh,k)$, $(g,h,k)\in\mathcal{G}^{(3)}$.

Proof. It is easy to see (1).
(2) Take arbitrary $\bigoplus_{l\in\mathcal{G}^{s(g)}}a(l)\in A_n(s(g))$, and we will show

$$
\phi_n^{r(g)}\alpha_g^n(a)(k)=\alpha_g^{n+1}\phi_n^{s(g)}(a)(k),\ \ k\in\mathcal{G}^{r(g)}.
$$

On one hand, we have

$$
\begin{aligned}
&\phi_n^{r(g)}\alpha_g^n(a)(k)\\
&=\sum_{l\in\mathcal{G}_{s(k)}} I^{n+1}(kl^{-1},l)\left(\alpha_g^n(a)(kl^{-1})\otimes 1\right)I^{n+1}(kl^{-1},l)^*\\
&=\sum_{l\in\mathcal{G}_{s(k)}} I^{n+1}(kl^{-1},l)\left(u_g^n(kl^{-1})a(g^{-1}kl^{-1})u_g^n(kl^{-1})^*\otimes 1\right)I^{n+1}(kl^{-1},l)^*\\
&=\sum_{l\in\mathcal{G}_{s(k)}} I^{n+1}(kl^{-1},l)\left(u_g^n(kl^{-1})\otimes 1\right)\left(a(g^{-1}kl^{-1})\otimes 1\right)\left(u_g^n(kl^{-1})^*\otimes 1\right)I^{n+1}(kl^{-1},l)^*.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\alpha_g^{n+1}\phi_n^{s(g)}(a)(k)=u_g^{n+1}(k)\phi_n^{s(g)}(a)(g^{-1}k)u_g^{n+1}(k)^*\\
&=\sum_{l\in\mathcal{G}_{s(k)}} u_g^{n+1}(k)I^{n+1}(g^{-1}kl^{-1},l)\left(a(g^{-1}kl^{-1})\otimes 1\right)I^{n+1}(g^{-1}kl^{-1},l)^*u_g^{n+1}(k)^*.
\end{aligned}
$$

By Lemma 6.1, $\phi_n^{r(g)}\alpha_g^n(a) = \alpha_g^{n+1}\phi_n^{s(g)}(a)$ holds.

(3) By Lemma 6.1, we have

$$\begin{aligned}\alpha_g^n\alpha_h^n(a)(k) &= \operatorname{Ad} u_g(k)\left(\alpha_h(a)(g^{-1}k)\right)\\ &= \operatorname{Ad}\left(u_g(k)u_h(g^{-1}k)\right)\left(a(h^{-1}g^{-1}k)\right)\\ &= \operatorname{Ad}\left(w^{(n)}(g,h)u_{gh}^n(k)\right)\left(a(h^{-1}g^{-1}k)\right)\\ &= \operatorname{Ad}\left(w^{(n)}(g,h)\right)\alpha_{gh}^n(a)(k)\end{aligned}$$

for $k \in \mathcal{G}^{r(g)}$, and the statement (3) holds.

(4) At first, we consider the case $n = 1$. Let $l \in \mathcal{G}$. Then we have

$$\begin{aligned}&\alpha_g^1(w^1(h,k))(l)w^1(g,hk)(l)\\ &= \left(\sum_{t\in\mathcal{G}_{s(l)}^{s(k)}}\overline{c(h,k,t)}e_{ghkt,ghkt}\right)\left(\sum_{s\in\mathcal{G}_{s(l)}^{s(k)}}\overline{c(g,hk,s)}e_{ghks,ghks}\right)\\ &= \sum_{t\in\mathcal{G}_{s(l)}^{s(k)}}\overline{c(h,k,t)c(g,hk,t)}e_{ghkt,ghkt}\\ &= \sum_{t\in\mathcal{G}_{s(l)}^{s(k)}} c(g,h,k)\overline{c(g,h,kt)c(gh,k,t)}e_{ghkt,ghkt} \quad \text{(by the 3-cocycle identity)}\\ &= c(g,h,k)\left(\sum_{s\in\mathcal{G}_{s(l)}^{s(k)}}\overline{c(g,h,s)}e_{s,s}\right)\left(\sum_{t\in\mathcal{G}_{s(l)}^{s(k)}}\overline{c(gh,k,t)}e_{ghkt,ghkt}\right)\\ &= c(g,h,k)w^1(g,h)(l)w^1(gh,k)(l).\end{aligned}$$

By (1) and (2), we have $\alpha_g^n(w^n(h,k))w^n(g,hk) = c(g,h,k)w^n(g,h)w^n(gh,k)$ for all $n \in \mathbb{N}$.
□

By Lemma 6.2, we get the model action in C*-level.

Theorem 6.3 *Let $A(x) := \lim_n(A_n(x), \phi_n^x)$ be an inductive limit C*-algebra. Then we can define an isomorphism $\alpha_g : A(x) \to A(y)$, $g \in \mathcal{G}_x^y$, by $\alpha_g(a) = \alpha_g^n(a)$, $a \in A_n(x)$, and $(\alpha_g, w(g,h))$ is an outer action of $\mathcal{G}$ with 3-cocycle $c(\cdot,\cdot,\cdot)$.*

We will construct a suitable state ψ_n^x on $A(x)$ for W*-completion. Let $\mu = \{\mu^x\}$ be a family of probability measures on $\mathcal{G}$ with $\operatorname{supp}(\mu^x) = \mathcal{G}^x$. Denote the non-normalized trace on $B_n(k)\left(= B\left(\ell^2\left(\mathcal{G}_{s(k)}^{r(k),(n)}\right)\right)\right)$ by $\operatorname{Tr}_{n,k}$.

Let

$$\rho_n(k) := \sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{r(k),(n)}}\sum_{y\sim r(k)}\mu^{r(k)}(kt_n^{-1}t_{n-1}^{-1}\cdots t_1^{-1}(r(k),y))\mu^y((y,r(k))t_1)\mu^{x_2}(t_2)\cdots\mu^{x_n}(t_n)e_{\boldsymbol{t},\boldsymbol{t}},$$

$$\psi_n^x(a) := \sum_{k\in\mathcal{G}^x}\operatorname{Tr}_{n,k}(\rho_n(k)a(k)),\ a = \bigoplus_{l\in\mathcal{G}^x}a(l) \in A_n(x),$$

$$E_n^x(a)(k) := \sum_{l\in\mathcal{G}^{s(k)}}\mu^{r(l)}(l)\left(\operatorname{id}\otimes\operatorname{Tr}_{1,l}\right)\left(I^{n+1}(k,l)^*a(kl)I^{n+1}(k,l)\right), a = \bigoplus_{l\in\mathcal{G}^x} \in A_{n+1}(x).$$

(Note that we insert $(y, r(k))$ before t_1 in the definition of $\rho_n(k)$.)

Lemma 6.4 (1) *ψ_n^x is a faithful state on $A_n(x)$.*
(2) *$\psi_{n+1}^x(\phi_n^x(a)b) = \psi_n^x(aE_n^x(b))$, $a \in A_n(x)$, $b \in A_{n+1}(x)$. Thus E_n^x is a ψ_{n+1}^x-preserving conditional expectation via identification $A_n(x)$ with $\phi_n^x(A_n(x)) \subset A_{n+1}(x)$.*
(3) *$\alpha_g^n \circ E_n^{s(g)} = E_n^{r(g)} \circ \alpha_g^{n+1}$.*

Proof. (1) Let $k \in \mathcal{G}^x$. Since we insert $(y, r(k))$ before t_1 in the definition of $\rho_n(k)$, we have

$$
\begin{aligned}
&\mathrm{Tr}_{n,k}(\rho_n(k)) \\
&= \sum_{y \sim x} \sum_{\boldsymbol{t} \in \mathcal{G}_{s(k)}^{r(k),(n)}} \mu^{r(k)}(kt_n^{-1}t_{n-1}^{-1} \cdots t_1^{-1}(r(k), y))\mu^y((y, r(k))t_1)\mu^{r(t_2)}(t_2) \cdots \mu^{r(t_n)}(t_n) \\
&= \sum_{y \sim x} \sum_{\boldsymbol{t} \in \mathcal{G}_{s(k)}^{y,(n)}} \mu^{r(k)}(kt_n^{-1}t_{n-1}^{-1} \cdots t_1^{-1})\mu^y(t_1)\mu^{r(t_2)}(t_2) \cdots \mu^{r(t_n)}(t_n) \\
&= \mu^{*(n+1),x}(k).
\end{aligned}
$$

Hence $\psi_n^x(1) = \sum_{k \in \mathcal{G}^x} \mu^{*(n+1),x}(k) = 1$. Since each $\rho_n(k)$ is a non-singular positive element, ψ_n is a faithful normal state.

(2) At first note the following relation for $(k, l) \in \mathcal{G}^{(2)}$;

$$
\begin{aligned}
&\rho_{n+1}(kl)(1 \otimes e_{ll}) \\
&= \sum_{\boldsymbol{t} \in \mathcal{G}_{s(l)}^{r(k),(n+1)}} \sum_{y \sim r(k)} \mu^x(klt_{n+1}^{-1}t_n^{-1} \cdots t_1^{-1}(r(k), y))\mu^y((y, r(k))t_1)\mu^{x_2}(t_2) \cdots \mu^{x_{n+1}}(t_{n+1})e_{\boldsymbol{t},\boldsymbol{t}} \\
&\quad \times (1 \otimes e_{ll}) \\
&= \sum_{\boldsymbol{t} \in \mathcal{G}_{s(k)}^{r(k),(n)}} \sum_{y \sim r(k)} \mu^x(kt_n^{-1} \cdots t_1^{-1}k^{-1})\mu^{r(k)}(kt_1)\mu^{x_2}(t_2) \cdots \mu^{x_n}(t_n)\mu^{r(l)}(l)e_{\boldsymbol{t},\boldsymbol{t}} \otimes e_{ll} \\
&= \mu^{r(l)}(l)\rho_n(k) \otimes e_{ll}.
\end{aligned}
$$

On one hand, we have

$$
\begin{aligned}
&\psi_{n+1}^x(\phi_n^x(a)b) \\
&= \sum_{k \in \mathcal{G}^x} \mathrm{Tr}_{n+1,k}\left(\rho_{n+1}(k)\phi_n^x(a)(k)b(k)\right) \\
&= \sum_{k \in \mathcal{G}^x} \sum_{l \in \mathcal{G}_{s(k)}} \mathrm{Tr}_{n+1,l}\left(\rho_{n+1}(k)I_{n+1}(kl^{-1}, l)\left(a(kl^{-1}) \otimes e_{ll}\right) I_{n+1}(kl^{-1}, l)^*b(k)\right) \\
&= \sum_{k \in \mathcal{G}^x} \sum_{l \in \mathcal{G}^{s(k)}} \mathrm{Tr}_{n+1,kl}\left(\rho_{n+1}(kl)(1 \otimes e_{ll})I_{n+1}(k, l)\left(a(k) \otimes 1\right) I_{n+1}(k, l)^*b(kl)\right) \\
&= \sum_{k \in \mathcal{G}^x} \sum_{l \in \mathcal{G}^{s(k)}} \mu^{r(l)}(l)\, \mathrm{Tr}_{n+1,kl}\left((\rho_n(kl) \otimes e_{ll})I_{n+1}(k, l)\left(a(k) \otimes 1\right) I_{n+1}(k, l)^*b(kl)\right)
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
\psi_n^x(aE_n^x(b)) &= \sum_{k\in\mathcal{G}^x} \mathrm{Tr}_{n,k}\left(\rho_n(k)a(k)E_n^x(b)(k)\right) \\
&= \sum_{k\in\mathcal{G}^x}\sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(k)}(l)\,\mathrm{Tr}_{n,k}\left(\rho_n(k)a(k)(\mathrm{id}\otimes\mathrm{Tr}_{1,l})\left(I_{n+1}(k,l)^*b(kl)I_{n+1}(k,l)\right)\right) \\
&= \sum_{k\in\mathcal{G}^x}\sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(l)}(l)\,\mathrm{Tr}_{n+1,kl}\left(\left(\rho_n(k)a(k)\otimes 1\right)I_{n+1}(k,l)^*b(kl)I_{n+1}(k,l)\right) \\
&= \sum_{k\in\mathcal{G}^x}\sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(l)}(l)\,\mathrm{Tr}_{n+1,kl}\left(I_{n+1}(k,l)\left(\rho_n(k)a(k)\otimes 1\right)I_{n+1}(k,l)^*b(kl)\right) \\
&= \sum_{k\in\mathcal{G}^x}\sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(l)}(l)\,\mathrm{Tr}_{n+1,kl}\left((\rho_n(k)\otimes e_{ll})I_{n+1}(k,l)\left(a(k)\otimes 1\right)I_{n+1}(k,l)^*b(kl)\right).
\end{aligned}$$

Here note $I_{n+1}(k,l)$ and $\rho_n(k)\otimes e_{ll}$ commute. Hence we have $\psi_{n+1}^x(\phi_n^x(a)b) = \psi_n^x(aE_n^x(b))$, $a\in A_n(x)$, $b\in A_{n+1}(x)$, and E_n^x is a ψ_{n+1}^x-preserving conditional expectation.

(3) Take $a=\bigoplus_{l\in\mathcal{G}(s(g))} a(l)\in A_n(s(g))$. Then the statement is shown as follows;

$$\begin{aligned}
&E_n^{r(g)}\alpha_g^{n+1}(a)(k) \\
&= \sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(l)}(l)\,(\mathrm{id}\otimes\mathrm{Tr}_{1,l})\left(I_{n+1}(k,l)^*\alpha_g^{n+1}(a)(kl)I_{n+1}(k,l)\right) \\
&= \sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(l)}(l)\,(\mathrm{id}\otimes\mathrm{Tr}_{1,l})\left(I_{n+1}(k,l)^*u_g^{n+1}(kl)a(g^{-1}kl)u_g^{n+1}(kl)^*I_{n+1}(k,l)\right) \\
&= \sum_{l\in\mathcal{G}^{s(k)}} \mu^{r(l)}(l)\,(\mathrm{id}\otimes\mathrm{Tr}_{1,l})\big((u_g^n(k)\otimes 1)I_{n+1}(g^{-1}kl)a(g^{-1}kl)I_{n+1}(g^{-1}kl)(u_g^n(k)^*\otimes 1)\big) \\
&= (\mathrm{id}\otimes\mathrm{Tr}_{1,l})\big((u_g^n(k)\otimes 1)\phi_n^{s(g)}(a)(g^{-1}k)(u_g^n(k)^*\otimes 1)\big) \\
&= \alpha_g^n E_n^{s(g)}(a)(k).
\end{aligned}$$

In the third equality, we used Lemma 6.1. □

By Lemma 6.4, we can define a state ψ^x on $A(x)$ by $\psi^x(a)=\psi_n^x(a)$, $a\in A_n(x)$. Let (π_x, H_x, ξ_x) be the GNS triple of $A(x)$ by ψ_x, and $\mathcal{M}(x) := \pi_x(A(x))''$.

Let σ^{ψ^x} be the modular automorphism group. By Lemma 6.4, the restriction of σ^{ψ^x} on $A_n(x)$ is $\sigma^{\psi_n^x}$, and there exists the ψ^x-preserving conditional expectation $F_n^x:\mathcal{M}(x)\to A_n(x)$.

Theorem 6.5 *The center $Z(\mathcal{M}(x))$ is identified with the tail boundary of P_μ^x. Hence if μ satisfies the condition in Theorem 5.5, then $\mathcal{M}(x)$ is a factor for a.e. x.*

Proof. We simply denote F_n^x and E_n^x by F_n and E_n, respectively. Take $z\in Z(\mathcal{M}(x))$, and set $w_n=F_n(z)$. Then $w_n\in Z(A_n(x))=\ell^\infty(\mathcal{G}^x)$, and $\|w_n\|_\infty\le\|z\|$. Since $F_n=E_nF_{n+1}$,

$E_n(w_{n+1}) = w_n$, and

$$\begin{aligned}E_n(w_{n+1})(k) &= \sum_{l\in\mathcal{G}^{s(k)}} \mu^l(l)(\mathrm{id}\otimes \mathrm{Tr}_{1,l})(I(k,l)(w_{n+1}(kl)\otimes e_{ll})I_{n+1}(k,l)^*)\\ &= \sum_{l\in\mathcal{G}^{s(k)}} w_{n+1}(kl)\mu^{r(l)}(l)(\mathrm{id}\otimes \mathrm{Tr}_{1,l})(1\otimes e_{ll}))\\ &= \sum_{l\in\mathcal{G}^{s(k)}} w_{n+1}(kl)\mu^{r(l)}(l)\\ &= P^x_\mu(w_{n+1})(k)\end{aligned}$$

holds. Hence $\{w_n\}_n$ is a bounded harmonic sequence for P^x_μ. Note that $\lim_n w_n = z$ in the σ-weak topology by the martingale convergence theorem. This implies that $z \in Z(\mathcal{M}(x)) \mapsto \{w_n\}$ is an injective map.

Suppose that a bounded harmonic sequence $\{w_n\}$ is given, and $M = \sup_n \|w_n\|_\infty$. We regard $w_n \in Z(A_n(x))$. Let $C_n := \{a \in \mathcal{M}(x) \mid F_n(a) = w_n, \|a\| \le M\}$. Then $\{C_n\}$ is a decreasing sequence of non-empty σ-weakly compact sets, since $w_m \in C_n$ for $m \ge n$. Hence $\bigcap_n C_n \neq \emptyset$. Take $a \in \bigcap_n C_n \neq \emptyset$. By the martingale convergence, $a = \lim_n w_n$ in σ-weak topology. Let $u \in A_m(x)$ be an arbitrary unitary. Then $uau^* = \lim_n uw_nu^*$, and $uw_nu^* = w_n$ for $n \ge m$. Thus $uau^* = a$, and hence $a \in Z(\mathcal{M}(x))$ holds. These results implies $z \in Z(\mathcal{M}(x)) \mapsto \{w_n\}$ is a bijection map.

Take μ as in Theorem 5.5. Then $w^x_n(\cdot)$ is a constant function on $\mathcal{G}^x$ by Theorem 5.6, and hence $\mathcal{M}(x)$ is a factor. □

In the following, we fix a μ as in Theorem 5.5. We next consider the lift of α to $\mathcal{M}(x)$.

Lemma 6.6 *Define a positive operator $d^n_g = \bigoplus_{k\in\mathcal{G}^x} d^n_g(k)$ affiliated with $A_n(x)$ by*

$$d^n_g(k) = \sum_{\boldsymbol{t}\in\mathcal{G}^{r(k),n}_{s(k)}} \sqrt{\frac{\sum_{y\sim r(g)} \mu(gkt_n^{-1}\cdots t_1^{-1}g^{-1}(r(g),y))\mu((y,r(g))gt_1)}{\sum_{z\sim r(k)} \mu(kt_n^{-1}\cdots t_1^{-1}(r(k),z))\mu((z,r(k))t_1)}}\, e_{\boldsymbol{t},\boldsymbol{t}}.$$

(1) *We have $\psi^{r(g)}_n(\alpha_g(a)) = \psi^{s(g)}_n(d^n_g a d^n_g)$ for $a \in A_n(s(g))$.*
(2) *$\phi^x_n(d^n_g) = d^{n+1}_g$.*

Proof. (1) The left hand side is as follows;

$$\begin{aligned}\psi^{r(g)}(\alpha_g(a)) &= \sum_{k\in\mathcal{G}^{s(g)}} \mathrm{Tr}(\rho_n(gk)\alpha_g(a)(k))\\ &= \sum_{k\in\mathcal{G}^{s(g)}} \mathrm{Tr}(u_g(k)^*\rho_n(gk)u_g(gk)a(k))\\ &= \sum_{k\in\mathcal{G}^{s(g)}} \psi^{s(g)}_n(u_g(k)^*\rho_n(gk)u_g(gk)\rho_n(k)^{-1}a(k)).\end{aligned}$$

Here

$$
\begin{aligned}
&u_g(gk)^*\rho_n(gk)u_g(gk)\rho_n(k)^{-1}\\
&=\sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{r(k),(n)}}\sum_{y\sim r(k)}\mu^{r(k)}(gkt_n^{-1}t_{n-1}^{-1}\cdots t_1^{-1}g^{-1}(r(k),y))\mu^y((y,r(k))gt_1)\mu^{x_2}(t_2)\cdots\mu^{x_n}(t_n)e_{\boldsymbol{t},\boldsymbol{t}}\\
&\times\left(\sum_{z\sim r(k)}\mu^{r(k)}(kt_n^{-1}t_{n-1}^{-1}\cdots t_1^{-1}(r(k),z))\mu^z((z,r(k))t_1)\mu^{x_2}(t_2)\cdots\mu^{x_n}(t_n)e_{\boldsymbol{t},\boldsymbol{t}}\right)^{-1}\\
&=d_g^n(k)^2.
\end{aligned}
$$

Thus $\psi_n^{r(g)}(\alpha_g(a))=\psi_n^{s(g)}(d_g^n a d_g^n)$ holds. (Note that $\rho_n(k)$ and $d_g^n(k)$ commute.)
(2)

$$
\begin{aligned}
&\phi_n(d_g^n)(k)\\
&=\sum_{l\in\mathcal{G}_{s(k)}} I_{n+1}(kl^{-1},l)(d_g^n(kl^{-1})\otimes 1)I_{n+1}(kl^{-1},l)^*\\
&=\sum_{l\in\mathcal{G}_{s(k)}}\left(\sum_{\boldsymbol{t}\in\mathcal{G}_{r(l)}^{r(k),(n)}}\sqrt{\frac{\sum_{y\in[r(g)]}\mu(gkl^{-1}t_n^{-1}\cdots t_1^{-1}g^{-1}(r(g),y))\mu((y,r(g))gt_1)}{\sum_{z\in[r(k)]}\mu(kl^{-1}t_n^{-1}\cdots t_1^{-1}(r(k),z))\mu((z,r(k))t_1)}}e_{\boldsymbol{t},\boldsymbol{t}}\otimes e_{ll}\right)\\
&=\sum_{\boldsymbol{t}\in\mathcal{G}_{s(k)}^{r(k),(n+1)}}\sqrt{\frac{\sum_{y\in[r(g)]}\mu(gkt_{n+1}^{-1}t_n^{-1}\cdots t_1^{-1}g^{-1}(r(g),y))\mu((y,r(g))gt_1)}{\sum_{z\in[r(k)]}\mu(kt_{n+1}^{-1}t_n^{-1}\cdots t_1^{-1}(r(k),z))\mu((z,r(k))t_1)}}e_{\boldsymbol{t},\boldsymbol{t}}\\
&=d_g^{n+1}(k)
\end{aligned}
$$

□

By Lemma 6.6, there exists a positive operator d_g^x affiliated with $\mathcal{M}(x)$ such that $\psi^{r(g)}(\alpha_g(a))=\psi^{s(g)}(d_g^{s(g)}ad_g^{s(g)})$.

Lemma 6.7 *We can extend α_g to an isomorphism in* $\mathrm{Iso}\big(\mathcal{M}(s(g)),\mathcal{M}(r(g))\big)$.

Proof. Let $x=s(g)$, $y=r(g)$. Define an operator $U\in B(H_x,H_y)$ by $U(a\xi_x)=\alpha_g(a(d_g^x)^{-1})\xi_y$. Then

$$\|U(a\xi_\psi)\|^2=\|\alpha_g(a(d_g^x)^{-1})\xi_\psi\|^2=\psi^y\left(\alpha_g((d_g^x)^{-1}a^*a(d_g^x)^{-1})\right)=\psi^x(a^*a)=\|a\xi_x\|^2.$$

An adjoint U^* is given by $U^*(a\xi_y)=\alpha_g^{-1}(a)d_g\xi_x$. Then it is easy to see $U^*U=UU^*=1$. Hence U is a unitary. We can verify $U\pi_x(a)U^*=\pi_y(\alpha_g(a))$ as follows:

$$
\begin{aligned}
U\pi_x(a)U^*b\xi_y&=U\pi_x(a)\alpha_g^{-1}(b)d_g^x\xi_x=Ua\alpha_g^{-1}(b)d_g^x\xi_x\\
&=\alpha_g(a\alpha_g^{-1}(b)d_g^x(d_g^x)^{-1})\xi_y=\pi_y(\alpha_g(a))b\xi_y.
\end{aligned}
$$

Similarly, we have $U^*\pi_y(a)U=\pi_x(\alpha_g^{-1}(a))$. Thus $U\mathcal{M}(x)U^*=\mathcal{M}(y)$ holds, and $\mathrm{Ad}\,U$ is an extension of α_g. □

Theorem 6.8 *Let $\mathcal{G}$ be an amenable measurable discrete groupoid. Then there exists an outer free action of $\mathcal{G}$ on the injective factor of type II_1 $\mathcal{R}_0$ with given 3-cocycle c.*

Proof. Let (α, c) be an outer action on $\{\mathcal{M}(x)\}_{x\in\mathcal{G}^{(0)}}$ constructed in Lemma 6.7. Let $\mathcal{R}_\infty$ be the injective factor of type III_1, and consider $\alpha\otimes\mathrm{id}$ on $\mathcal{M}(x)\otimes\mathcal{R}_\infty$, which is an injective factor of type III_1. Then the canonical extension $\widetilde{\alpha\otimes\mathrm{id}}$ is a trace preserving outer action on an injective factor of type II_∞. By the standard argument, we get an outer action (α^0, w^0, c) of $\mathcal{G}$ with 3-cocycle c on an injective factor of type II_1 $\mathcal{R}_0$.

If we apply the above construction to a trivial 3-cocycle, then we get a genuine faithful action of $\mathcal{G}$ on $\mathcal{R}_0$. By taking an infinite tensor product, we get a free genuine action σ of $\mathcal{G}$ on $\mathcal{R}_0$. (Such an action is also constructed in [24].) Then $(\alpha^0\otimes\sigma, w^0\otimes 1, c)$ is a desired one. □

6.2 Realization of actions with given invariants

Once we have a free outer action of $\mathcal{G}$ on $\mathcal{R}_0$, we can construct model actions with given invariants as in [18]. Let (α^0, w^0, c) be a free outer action of $\mathcal{G}$ with 3-cocycle c on $\mathcal{R}_0$.

Let us $(\mathcal{N}, \beta, \chi)$ be given, where $\mathcal{N}\subset\mathcal{G}$ is a normal subgroupoid, β is a genuine action of $\mathcal{G}$ on a family of ergodic flows $\{\mathcal{C}(x), \theta^x\}$ such that $\beta_n = \mathrm{id}$ for $n\in\mathcal{N}$, and $\chi = (\lambda, \mu, d)$ is a characteristic invariant. Thus we are given a family of ergodic flows $\{\mathcal{C}(x), \theta^x\}$, and $\chi = (\lambda, \mu, d)$ for some $\lambda(n, g), \mu(m, n), d(n, t) \in \mathcal{U}(\mathcal{C}(x))$. We will construct an outer action α with $\mathrm{Inv}(\alpha) = (\mathcal{N}, \beta, \chi)$. Let $\mathcal{P}(x)$ be an injective factor, whose flow of weights is $\{\mathcal{C}(x), \theta^x\}$.

At first we recall the following fact [25].

Theorem 6.9 *Let $\mathcal{R}$ be an injective factor, and $\{\mathcal{C}, \theta\}$ the flow of weights of $\mathcal{R}$. Then the following exact sequence splits.*

$$1 \longrightarrow \overline{\mathrm{Int}}(\mathcal{R}) \longrightarrow \mathrm{Aut}(\mathcal{R}) \stackrel{\mathrm{mod}}{\longrightarrow} \mathrm{Aut}_\theta(\mathcal{C}) \longrightarrow 1.$$

In a similar way, we can show an action β of $\mathcal{G}$ on $\{\mathcal{C}(x), \theta^x\}_{x\in\mathcal{G}^{(0)}}$ can be lift to that on $\{\mathcal{P}(x)\}_{x\in\mathcal{G}^{(0)}}$, which we denote by β.

Let φ_x be a dominant weight on $\mathcal{P}(x)$, $\mathcal{Q}(x) := \mathcal{P}(x)_{\varphi_x}$, $\mathcal{P}(x) = \mathcal{Q}(x)\rtimes_{\theta^x}\mathbb{R}$ the continuous decomposition, and $u^x(s)\in\mathcal{P}(x)$ be the implementing unitary. We may assume that $\varphi_{s(g)} = \varphi_{r(g)}\circ\beta_g$ for all $g\in\mathcal{G}$.

Let $\sigma_n^x := \sigma_{d(n)^*}^{\varphi_x}$ be an extended modular automorphism for 1-cocycle $d(n,\cdot)^*$. Then $(\sigma_n^x, \mu(\cdot,\cdot))$ is an outer action of $\mathcal{N}_x$ with 3-cocycle $\overline{c(\cdot,\cdot,\cdot)}$ by the relation (CC2) and (CC3). Hence $(\sigma_m^x\otimes\alpha_m^0, \mu(m,n)\otimes w^0(m,n))$ is a free cocycle crossed action of $\mathcal{N}_x$ on $\mathcal{P}(x)\otimes\mathcal{R}_0$.

Let $\mathcal{M}(x) := (\mathcal{P}(x)\otimes\mathcal{R}_0)\rtimes\mathcal{N}_x$, and define $\alpha_g\in\mathrm{Iso}(\mathcal{M}(x), \mathcal{M}(y))$, $x = s(g)$, $y = r(g)$ by

$$\alpha_g(a) = \beta_g\otimes\alpha_g^0(a), \ \ a\in\mathcal{P}(x)\otimes\mathcal{R}_0,$$
$$\alpha_g(v^x(g^{-1}ng)) = (\lambda^y(n,g)\otimes d^0(n,g))v^y(n),$$

where $v^x(n)$ is the implementing unitary in $\mathcal{M}(x)$, and $d^0(n,g) = w^0(g, g^{-1}ng)w^0(n,g)^*$.

Theorem 6.10 *Let α be an outer action of $\mathcal{G}$ constructed above. Then we have* $\mathrm{Inv}(\alpha) = (\beta, \mathcal{N}, \chi)$.

Proof is similar to that of [18], so we omit the detail.

A Relation with Katayama-Takesaki's classification

Let (α, w_α, c) be an outer action of a discrete group G on a factor $\mathcal{M}$. We will describe outline of correspondence between $\mathrm{Inv}(\alpha) = (N, \mathrm{mod}(\alpha), \chi)$ and Katayama-Takesaki's invariant, and explain their classification theorem follows from Theorem 3.1.

Let $Q := G/N$, and $\pi : G \to Q$ the quotient map. In this section, we use letters g, h, k for general elements in G, p, q, r for those in Q, and l, m, n for those in N.

Fix a section $p \in Q \to \tilde{p} \in G$, and let $\mathfrak{n}(p, q) = \tilde{p}\tilde{q}\widetilde{pq}^{-1} \in N$.

We recall the definition of modular obstruction Ob_m defined by Katayama-Takesaki [12]. At first note that $\tilde{\alpha}_{\tilde{p}}$ is a Q-kernel on $\tilde{\mathcal{M}}$. Fix $z \in C^2(Q, \mathcal{U}(\tilde{\mathcal{M}}))$ such that $\tilde{\alpha}_{\tilde{p}} \circ \tilde{\alpha}_{\tilde{q}} = \mathrm{Ad}\, z(p, q) \circ \tilde{\alpha}_{\widetilde{pq}}$. Then we get $d_1 \in Z^3(Q, \mathcal{U}(\mathcal{C}))$ and $d_2(s; p, q) \in \mathcal{U}(\mathcal{C})$ by

$$\tilde{\alpha}_{\tilde{p}}(z(q, r))z(p, qr) = d_1(p, q, r)z(p, q)z(pq, r), \;\; \theta_s(z(p, q)) = d_2(p, q; s)z(p, q)$$

We can see

$$d_1 d_2((p, s), (q, t), (r, u)) := d_1(p, q, r)d_2(s; q, r) \in Z^3(Q \times \mathbb{R}, \mathcal{C}(\mathcal{U}))$$

and hence get an element in $H^3(Q \times \mathbb{R}, \mathcal{C}(\mathcal{U}))$. We have another invariant

$$\nu : N \to H^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$$

by $\nu(n) = [d(n, \cdot)]$. The invariant ν relates with d_2 by $\nu(\mathfrak{n}(p, q)) = [d_2(\cdot; \mathfrak{n}(p, q))]$ in $H^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C}))$. By definition, $\mathrm{Ob}_{\mathrm{m}}(\alpha) = ([d_1 d_2], \nu)$ is the modular obstruction for α.

In what follows, we denote the canonical extension of $\alpha_{\tilde{p}}$ by the same symbol $\alpha_{\tilde{p}}$.

To describe relation between $\mathrm{Inv}(\alpha) = (N, \mathrm{mod}(\alpha), \chi)$ and $\mathrm{Ob}_{\mathrm{m}}(\alpha)$, it is convenient to replace (α, w_α) with suitable form.

Note that any element of G can be expressed as $l\tilde{p}$, $l \in N$, $p \in Q$, uniquely . Define $\hat{\alpha}_{l\tilde{p}} = \alpha_l \circ \alpha_{\tilde{p}}$. Let

$$\begin{aligned}
\widehat{w_\alpha}(m, n) &= w_\alpha(m, n), \\
\widehat{w_\alpha}(\tilde{p}, n) &= w_\alpha(\tilde{p}, n) w_\alpha(\tilde{p} n \tilde{p}^{-1}, \tilde{p})^*, \\
\widehat{w_\alpha}(\tilde{p}, \tilde{q}) &= w_\alpha(\tilde{p}, \tilde{q}) w_\alpha(\mathfrak{n}(p, q), \widetilde{pq})^*
\end{aligned}$$

for $m, n \in N$, $p, q \in Q$. Then we have

$$\begin{aligned}
\hat{\alpha}_m \circ \hat{\alpha}_n &= \mathrm{Ad}\, \widehat{w_\alpha}(m, n) \circ \hat{\alpha}_{mn}, \\
\hat{\alpha}_{\tilde{p}} \circ \hat{\alpha}_n &= \mathrm{Ad}\, \widehat{w_\alpha}(\tilde{p}, n) \circ \alpha_{\tilde{p} n \tilde{p}^{-1}} \circ \alpha_{\tilde{p}} = \mathrm{Ad}\, \widehat{w_\alpha}(\tilde{p}, n) \circ \hat{\alpha}_{\tilde{p} n}, \\
\hat{\alpha}_{\tilde{p}} \circ \hat{\alpha}_{\tilde{q}} &= \mathrm{Ad}\, \widehat{w_\alpha}(\tilde{p}, \tilde{q}) \circ \alpha_{\mathfrak{n}(\mathrm{p},\mathrm{q})} \circ \alpha_{\widetilde{pq}} = \mathrm{Ad}\, \widehat{w_\alpha}(\tilde{p}, \tilde{q}) \circ \hat{\alpha}_{\tilde{p}\tilde{q}}.
\end{aligned}$$

Define

$$\widehat{w_\alpha}(m\tilde{p}, n\tilde{q}) = \alpha_m(\widehat{w_\alpha}(\tilde{p}, n))\alpha_m \alpha_{\tilde{p} n \tilde{p}^{-1}}(\widehat{w_\alpha}(\tilde{p}, \tilde{q}))\, \widehat{w_\alpha}(m, \tilde{p} n \tilde{p}^{-1}) \widehat{w_\alpha}(m \tilde{p} n \tilde{p}^{-1}, \mathfrak{n}(p, q)).$$

We can see that $\hat{\alpha}_g \circ \hat{\alpha}_h = \mathrm{Ad}\, \widehat{w_\alpha}(g, h) \circ \hat{\alpha}_{gh}$. Let $\hat{c} \in Z^3(G, \mathbb{T})$ be a 3-cocycle associated with $(\hat{\alpha}, \widehat{w_\alpha})$. Then we have $[\hat{c}] = [c]$ in $H^3(G, \mathbb{T})$.

In what follows, we replace (α, w_α) with $(\hat{\alpha}, \widehat{w_\alpha})$. In particular, we have $w_\alpha(n, \tilde{p}) = 1$ and hence $\alpha_{n\tilde{p}} = \alpha_n \alpha_{\tilde{p}}$. We also fix $z(p, q)$ as $z(p, q) = w_\alpha(\tilde{p}, \tilde{q})\tilde{u}_{\mathfrak{n}(p,q)}$. Then we have $\alpha_{\tilde{p}} \circ \alpha_{\tilde{q}} = \mathrm{Ad}\, z(p, q) \circ \alpha_{\widetilde{pq}}$, and $d(\mathfrak{n}(p, q), s) = d_2(\mathfrak{n}(p, q; s))$.

Definition A.1 *Let $\chi = [\lambda, \mu, d]$ be a characteristic invariant for (α, w_α). Define $\delta[\lambda, \mu] \in C^3(Q, \mathcal{U}(\mathcal{C}))$ by*

$$\delta[\lambda,\mu](p,q,r) = \lambda(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}, \tilde{p})\mu(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}, \mathfrak{n}(p,qr))\mu(\mathfrak{n}(p,q), \mathfrak{n}(pq,r))^*.$$

The map δ is an analogue of Huebschmann-Jones-Ratcliffe map, [7], [8], [22].

At first we express d_1 in terms of λ, μ and c.

Lemma A.2 *We have*

$$d_1(p,q,r) = c(\tilde{p}, \tilde{q}, \tilde{r})\delta[\lambda,\mu](p,q,r).$$

Proof. By

$$\alpha_{\tilde{p}}(w_\alpha(\tilde{p},\tilde{q}))w_\alpha(\tilde{p},\widetilde{qr}) = c(\tilde{p},\tilde{q},\tilde{r})w_\alpha(\tilde{p},\tilde{q})w_\alpha(\widetilde{pq},\tilde{r}),$$

we have

$$\begin{aligned}
&\alpha_{\tilde{p}}(w_\alpha(\tilde{p},\tilde{q}))w_\alpha(\tilde{p},\mathfrak{n}(q,r))\alpha_{\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}}(w_\alpha(\tilde{p},\widetilde{qr}))w_\alpha(\tilde{p}\mathfrak{n}(p,q)\tilde{p}^{-1}, \mathfrak{n}(p,qr)) \\
&= c(\tilde{p},\tilde{q},\tilde{r})w_\alpha(\tilde{p},\tilde{q})\alpha_{\mathfrak{n}(p,q)}(w_\alpha(\widetilde{pq},\tilde{r}))w_\alpha(\mathfrak{n}(p,q),\mathfrak{n}(pq,r)).
\end{aligned}$$

Recall that we have fixed $z \in C^2(Q, \mathcal{U}(\mathcal{C}))$ as $z(p,q) = w_\alpha(\tilde{p},\tilde{q})\tilde{u}_{\mathfrak{n}(p,q)}$. Then

$$\begin{aligned}
d_1(p,q,r) &= \alpha_{\tilde{p}}(z(q,r))z(p,qr)z(pq,r)^*z(p,q)^* \\
&= \alpha_{\tilde{p}}\left(w_\alpha(\tilde{q},\tilde{r})\tilde{u}_{\mathfrak{n}(q,r)}\right)w_\alpha(\tilde{p},\widetilde{qr})\tilde{u}_{\mathfrak{n}(p,qr)}\tilde{u}^*_{\mathfrak{n}(pq,r)}w_\alpha(\widetilde{pq},\tilde{r})^*\tilde{u}^*_{\mathfrak{n}(p,q)}w_\alpha(\tilde{p},\tilde{q})^* \\
&= \lambda(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1},\tilde{p})\alpha_{\tilde{p}}\left(w_\alpha(\tilde{q},\tilde{r})\right)w_\alpha(\tilde{p},\mathfrak{n}(q,r))\tilde{u}_{\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}}w_\alpha(\tilde{p},\widetilde{qr})\tilde{u}_{\mathfrak{n}(p,qr)} \\
&\times \tilde{u}^*_{\mathfrak{n}(pq,r)}w_\alpha(\widetilde{pq},\tilde{r})^*\tilde{u}^*_{\mathfrak{n}(p,q)}w_\alpha(\tilde{p},\tilde{q})^* \\
&= \lambda(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1},\tilde{p})\alpha_{\tilde{p}}\left(w_\alpha(\tilde{q},\tilde{r})\right)w_\alpha(\tilde{p},\mathfrak{n}(q,r))\alpha_{\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}}\left(w_\alpha(\tilde{p},\widetilde{qr})\right) \\
&\times \tilde{u}_{\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}}\tilde{u}_{\mathfrak{n}(p,qr)}\tilde{u}^*_{\mathfrak{n}(pq,r)}\tilde{u}^*_{\mathfrak{n}(p,q)}\alpha_{\mathfrak{n}(p,q)}\left(w_\alpha(\widetilde{pq},\tilde{r})^*\right)w_\alpha(\tilde{p},\tilde{q})^* \\
&= \lambda(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1},\tilde{p})\mu(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1},\mathfrak{n}(p,qr))\mu(\mathfrak{n}(p,q),\mathfrak{n}(pq,r))^* \\
&\times \alpha_{\tilde{p}}\left(w_\alpha(\tilde{q},\tilde{r})\right)w_\alpha(\tilde{p},\mathfrak{n}(q,r))\alpha_{\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1}}\left(w_\alpha(\tilde{p},\widetilde{qr})\right)w_\alpha(\tilde{p}\mathfrak{n}(q,r)\tilde{p}^{-1},\mathfrak{n}(p,qr)) \\
&\times w_\alpha(\mathfrak{n}(p,q),\mathfrak{n}(pq,r))^*\alpha_{\mathfrak{n}(p,q)}\left(w_\alpha(\widetilde{pq},\tilde{r})^*\right)w_\alpha(\tilde{p},\tilde{q})^* \\
&= c(\tilde{p},\tilde{q},\tilde{r})\delta[\lambda,\mu](p,q,r).
\end{aligned}$$

□

Conversely, we can recover the 3-cocycle c from d_1, λ, μ as following lemma.

Lemma A.3 *Put*

$$a(m\tilde{p}, n\tilde{q}) = \lambda(\tilde{p}n\tilde{p}^{-1},\tilde{p})^*\mu(m,\tilde{p}n\tilde{p}^{-1})^*\mu(m\tilde{p}n\tilde{p}^{-1},\mathfrak{n}(p,q))^*.$$

Then

$$c(g,h,k) = \partial(a)(g,h,k)d_1(\pi(g),\pi(h),\pi(k)).$$

Here

$$\partial(a)(g,h,k) = \alpha_g(a(h,k))a(g,hk)a(gh,k)^*a(g,h)^*.$$

Proof. At first we express $w_\alpha(g,h)$ by $z, \lambda, \mu, \tilde{u}_n$. Let $g = m\tilde{p}$, $h = n\tilde{q}$.

$$\begin{aligned}
&w_\alpha(m\tilde{p}, n\tilde{q})\\
&= \alpha_m(w_\alpha(\tilde{p}, n))\alpha_m\alpha_{\tilde{p}n\tilde{p}^{-1}}\left(w_\alpha(\tilde{p},\tilde{q})\right) w_\alpha(m, \tilde{p}n\tilde{p}^{-1})w_\alpha(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))\\
&= \alpha_m(w_\alpha(\tilde{p}, n))w_\alpha(m, \tilde{p}n\tilde{p}^{-1})\alpha_{m\tilde{p}n\tilde{p}^{-1}}\left(w_\alpha(\tilde{p},\tilde{q})\right) w_\alpha(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))\\
&= \alpha_m(w_\alpha(\tilde{p}, n))w_\alpha(m, \tilde{p}n\tilde{p}^{-1})\tilde{u}_{m\tilde{p}n\tilde{p}^{-1}}z(p,q)\tilde{u}^*_{\mathfrak{n}(p,q)}\tilde{u}^*_{m\tilde{p}n\tilde{p}^{-1}}w_\alpha(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))\\
&= \mu(m, \tilde{p}n\tilde{p}^{-1})^*\mu(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))^*\alpha_m(w_\alpha(\tilde{p}, n))\tilde{u}_m\tilde{u}_{\tilde{p}n\tilde{p}^{-1}}z(p,q)\tilde{u}_{m\tilde{p}n\tilde{p}^{-1}\mathfrak{n}(p,q)}\\
&= \mu(m, \tilde{p}n\tilde{p}^{-1})^*\mu(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))^*\tilde{u}_m w_\alpha(\tilde{p}, n)\tilde{u}_{\tilde{p}n\tilde{p}^{-1}}z(p,q)\tilde{u}_{m\tilde{p}n\tilde{p}^{-1}\mathfrak{n}(p,q)}\\
&= \mu(m, \tilde{p}n\tilde{p}^{-1})^*\mu(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))^*\lambda(\tilde{p}n\tilde{p}^{-1}, \tilde{p})^*\tilde{u}_m\alpha_{\tilde{p}}(\tilde{u}_n)z(p,q)\tilde{u}_{m\tilde{p}n\tilde{p}^{-1}\mathfrak{n}(p,q)}
\end{aligned}$$

Put

$$a(m\tilde{p}, n\tilde{q}) = \lambda(\tilde{p}n\tilde{p}^{-1}, \tilde{p})^*\mu(m, \tilde{p}n\tilde{p}^{-1})^*\mu(m\tilde{p}n\tilde{p}^{-1}, \mathfrak{n}(p,q))^*.$$

For $g \in G$, let $p = \pi(g)$ and $n(g) = g\tilde{p}^{-1}$. Then we have

$$w_\alpha(g,h) = a(g,h)\tilde{u}_{n(g)}\alpha_{\tilde{p}}(\tilde{u}_{n(h)})z(\pi(g),\pi(h))\tilde{u}^*_{n(gh)}.$$

We compute the 3-cocycle c for (α, w_α). Let $p = \pi(g)$, $q = \pi(h)$, $r = \pi(k)$. Then

$$\begin{aligned}
&c(g,h,k)\\
&= \alpha_g(w_\alpha(h,k))w_\alpha(g,hk)w_\alpha(gh,k)^*w_\alpha(g,h)^*\\
&= \alpha_{n(g)}\alpha_{\tilde{p}}\left(a(h,k)\tilde{u}_{n(h)}\alpha_{\tilde{q}}(\tilde{u}_{n(k)})z(q,r)\tilde{u}^*_{n(hk)}\right) \times a(g,hk)\tilde{u}_{n(g)}\alpha_{\tilde{p}}(\tilde{u}_{n(hk)})z(p,qr)\tilde{u}^*_{n(ghk)}\\
&\quad \times \left(a(gh,k)\tilde{u}_{n(gh)}\alpha_{\widetilde{pq}}(\tilde{u}_{n(k)})z(pq,r)\tilde{u}^*_{n(ghk)}\right)^* \times \left(a(g,h)\tilde{u}_{n(g)}\alpha_{\tilde{p}}(\tilde{u}_{n(h)})z(p,q)\tilde{u}^*_{n(gh)}\right)^*\\
&= \partial(a)(g,h,k)\alpha_{n(g)}\alpha_{\tilde{p}}\left(\tilde{u}_{n(h)}\alpha_{\tilde{q}}(\tilde{u}_{n(k)})z(q,r)\tilde{u}^*_{n(hk)}\right) \times \tilde{u}_{n(g)}\alpha_{\tilde{p}}(\tilde{u}_{n(hk)})z(p,qr)\tilde{u}^*_{n(ghk)}\\
&\quad \times \tilde{u}_{n(ghk)}z(pq,r)^*\alpha_{\widetilde{pq}}(\tilde{u}^*_{n(k)})\tilde{u}^*_{n(gh)} \times \tilde{u}_{n(gh)}z(p,q)^*\alpha_{\tilde{p}}(\tilde{u}^*_{n(h)})\tilde{u}^*_{n(g)}\\
&= \partial(a)(g,h,k)\tilde{u}_{n(g)}\alpha_{\tilde{p}}\left(\tilde{u}_{n(h)}\alpha_{\tilde{q}}(\tilde{u}_{n(k)})z(q,r)\tilde{u}^*_{n(hk)}\right)\\
&\quad \times \alpha_{\tilde{p}}(\tilde{u}_{n(hk)})z(p,qr)z(pq,r)^*\alpha_{\widetilde{pq}}(\tilde{u}^*_{n(k)})z(p,q)^*\alpha_{\tilde{p}}(\tilde{u}^*_{n(h)})\tilde{u}^*_{n(g)}\\
&= \partial(a)(g,h,k)\tilde{u}_{n(g)}\alpha_{\tilde{p}}\left(\tilde{u}_{n(h)}\alpha_{\tilde{q}}(\tilde{u}_{n(k)})\right)\\
&\quad \times \alpha_{\tilde{p}}(z(q,r))z(p,qr)z(pq,r)^*z(p,q)^*\alpha_{\tilde{p}}\alpha_{\tilde{q}}(\tilde{u}^*_{n(k)})\alpha_{\tilde{p}}(\tilde{u}^*_{n(h)})\tilde{u}^*_{n(g)}\\
&= \partial(a)(g,h,k)d_1(p,q,r).
\end{aligned}$$

Thus c is given by $c(g,h,k) = \partial(a)(g,h,k)d_1(p,q,r)$. □

Remark.
(1) The formula in Lemma A.3 is obtained by Katayama-Takesaki in [12, Lemma 2.11].
(2) In Katayama-Takesaki's formulation, λ and μ do not appear explicitly. However we can recover these by $\nu(mn) = \nu(m)\nu(n)$ and $\mathrm{mod}(\alpha_g)(\nu(g^{-1}ng)) = \nu(n)$, which are equivalent to the relation (CC1) and (CC2), respectively.

With a bit of effort, we can show that the above correspondence preserves equivalence classes of invariants. By Theorem 3.1, we can show Katayama-Takesaki's classification. Namely, we can conclude the following.

Theorem A.4 *Let α and β be outer actions on an injective factor $\mathcal{M}$. Then α and β are conjugate in* $\mathrm{Out}(\mathcal{M})$ *if and only if* $\sigma(\mathrm{mod}(\alpha), \mathrm{Ob_m}(\alpha)) = (\mathrm{mod}(\beta), \mathrm{Ob_m}(\beta))$ *for some* $\sigma \in \mathrm{Aut}_\theta(\mathcal{C})$.

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