A combinatorial perspective on shape theory

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Abstract

We give a description of shape theory using finite topological T_0-spaces (finite partially ordered sets). This description serves as a first step towards developing computational methods in shape theory for future works. Additionally, we introduce the notion of core for inverse sequences of finite topological spaces and prove some properties.

1 Introduction

The theory of dynamical systems is a very active area of researching. This is due to its wide variety of applications to other areas of science such as physics, biology or engineering [25, 38, 42]. Due to the need for computational methods to study dynamical systems, the theory of finite topological spaces [4, 29] has recently developed in this direction. Classical topological methods in dynamical systems, such as the Conley index [13, 35, 43] or the Lefschetz fixed point theorem [26], have been adapted to this framework [27, 5, 11] (ordered chronologically). This combinatorial approach has led to the development of persistence algorithms that can be used to analyze data collected from dynamical systems [16, 17, 18]. Classical Morse theory [31] has also been adapted for finite spaces in [20, 32] and for simplicial complexes in [21]. This reformulation of the Morse theory for simplicial complexes has found interesting applications in computational aspects in a fruitful manner; see, for example [15] or [24]. Therefore, it is natural to think that by adapting classical notions, we could develop computational applications for the general study of dynamical systems, among other areas.

Moreover, shape theory appears as a generalization of homotopy theory. It provides a weaker classification of compact metric spaces than homotopy theory. Originally, it was developed to study global properties of compact metric spaces that do not necessarily have good local properties. Since some dynamical objects, such as attractors, do not behave well locally, shape theory has several applications in this area (for a recent account of this treatment, we refer the reader to [40] and its references). For example, the Conley index can also be defined using this theory [39]. There are different approaches to shape theory: the original approach of K. Borsuk [7], the categorical approach of S. Mardešić and J. Segal [28], the categorical approach of J.M. Cordier and T. Porter [14], and the intrinsic description using multivalued maps by J.M.R. Sanjurjo [41]. In general, the main idea is to think about a topological space

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X as an inverse system of "easier" spaces approximating X, where the morphisms are given in terms of these systems.

Recently, methods to reconstruct compact metric spaces using finite topological spaces have been developed. For every compact metric space, an inverse sequence of finite spaces and continuous maps between them was associated in [2]. Later, in [12], E. Clader constructed an inverse sequence of finite spaces for every compact simplicial complex X, such that its inverse limit contains a homeomorphic copy of X which is a strong deformation retract. A generalization of this result to compact metric spaces was given in [9] and [33], where computational aspects are studied and a method to reconstruct algebraic invariants is proposed. These methods are somewhat motivated by the recent theory of topological data analysis [8, 19], which has grown very rapidly in recent years (e.g., see [37] and the references given there). It is also worth noting that there is a more general method to reconstruct locally compact, paracompact and Hausdorff spaces using Alexandroff spaces (partially ordered sets) [6], but it is not suitable from a computational viewpoint because it generally uses non-finite spaces.

Therefore, it is reasonable to ask whether shape theory can be described in terms of these inverse sequences. This description can be considered a first step towards developing computational aspects within the context of shape theory in future works, as has been done in other areas discussed earlier. Previous results have highlighted various relationships between finiteness and shape theory (see [10, 22, 23]). In [23], an intrinsic description of shape theory is given using sequences of continuous maps defined on open dense subsets of the compact metric spaces and having finite images. In [22], an intrinsic representation of Čech homology of compacta is given in terms of inverse limits of discrete approximations. In [10], the authors construct a category that classifies compacta by their shape and finite topological spaces by their weak homotopy type.

The organization of this paper is as follows. In Section 2, we introduce the reformulation of the intrinsic description of shape theory [41] that was given in [3], we also recall the method to reconstruct algebraic invariants given in [9] and basic results in the theory of finite spaces. In Section 3, we describe our combinatorial approach of shape theory, define the notion of core for inverse sequences and obtain some shape invariants. In Section 4, we show the main result: the shape category of compact metric spaces is isomorphic to a category whose morphisms are described in terms of finite topological spaces (or finite partially ordered sets).

2 Preliminaries

Given a compact metric space (X, d), where d denotes the metric, we consider the so-called hyperspace of X, $2^X = \{C \subseteq X | C \text{ is non-empty and closed}\}$. Let $B(U) = \{C \in 2^X | C \subset U\}$ for every open set $U \subseteq X$. Then the family $B = \{B(U) | U \subseteq X \text{ is open}\}$ is a base for the upper semifinite topology on 2^X. Moreover, X can be embedded in 2^X because X is a T_1-space. It suffices to consider $\varphi : X \to 2^X$ defined by $\varphi(x) = \{x\}$. The image of X under φ is called *canonical copy* of X in 2^X. For a complete exposition on this topic, see [36]. Define $U_\epsilon = \{C \in 2^X | \text{diam}(C) < \epsilon\}$, where $\text{diam}(C)$ denotes the diameter of C and ϵ is a positive real value. We recall some properties of hyperspaces with the upper semifinite topology studied in [3] and [33].

Proposition 2.1. *Let (X,d) be a compact metric space. Then the family $\mathcal{U} = \{U_\epsilon\}_{\epsilon>0}$ is a base of open neighborhoods of the embedding of X in 2^X.*

Lemma 2.2. *Let X and Y be compact metric spaces and let $h : X \to 2^Y$ be a continuous map. Then the map $h^* : 2^X \to 2^Y$ defined by $h^*(C) = \bigcup_{c\in C} h(c)$ is well-defined and continuous.*

The following results use the construction U_ϵ described above for two compact metric spaces X and Y. We will denote them by $U_\epsilon(X)$ and $U_\epsilon(Y)$, respectively. Note that in the following result, we use the same notation as in [3, Theorem 4], although there is an abuse of formal language because we are denoting the canonical copy of X in 2^X and X as the same.

Theorem 2.3. *Let X and Y be compact metric spaces. If $H : X\times[0,1] \to 2^Y$ and $h : 2^X \to 2^Y$ are continuous maps such that $H(x,0) = h_{|X}(\{x\})$, then there exists a map $\overline{H} : 2^X \times [0,1] \to 2^Y$ satisfying the following properties:*

1. *$\overline{H}(C,0) = h(C)$ for all $C \in 2^X$.*
2. *$\overline{H}_{|X\times[0,1]} = H$.*
3. *$\overline{H}$ is continuous.*
4. *If $H(x,t) \in U_\epsilon(Y)$ for all $(x,t) \in X\times[0,1]$, then there exists $\gamma > 0$ such that $\overline{H}(U_\gamma(X)\times[0,1]) \subset U_\epsilon(Y)$.*

Now, we recall the description of shape theory provided in [3], using the results obtained in [41].

Definition 2.4. *Given two compact metric spaces X and Y, a sequence of continuous functions $f = \{f_k : X \to 2^Y\}_{k\in\mathbb{N}}$ is said to be an approximative map from X to Y if for every neighborhood U of the canonical copy Y in 2^Y there exists $k_0 \in \mathbb{N}$ such that f_k is homotopic to f_{k+1} in U for all $k \geq k_0$.*

Definition 2.5. *Given two approximative maps f and g from X to Y, f is homotopic to g if for each open neighborhood U of the canonical copy Y in 2^Y there exists n_0 such that f_n is homotopic to g_n in U for every $n \geq n_0$.*

Theorem 2.6. *The set of all homotopy classes of approximative maps from X to Y is in bijective correspondence with the set of shape morphisms from X to Y.*

We adapt the notion of composition of multivalued maps in [41] to approximative maps. Let $[\overline{f}] : X \to Y$ and $[\overline{g}] : Y \to Z$ be two classes of approximative maps, $f = \{f_k : X \to 2^Y\}_{k\in\mathbb{N}} \in [\overline{f}]$, $g = \{g_k : Y \to 2^Z\}_{k\in\mathbb{N}} \in [\overline{g}]$. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive real values such that g_n is homotopic to g_{n_0} in $U_{\epsilon_{n_0}}(Z)$ for every $n \geq n_0$ and let $\{\nu_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive real values such that $\operatorname{diam}(g_n(K)) < \epsilon_n$ whenever K is a set in Y with $\operatorname{diam}(K) < \nu_n$. Now consider a sequence of indices $k_1 < k_2 < ... < k_n < ...$ such that f_k is homotopic to f_{k_n} in $U_{\nu_n}(Y)$ for every $k \geq k_n$. We set $[\overline{g}] \circ [\overline{f}] = [\{g_n \circ f_{k_n} : X \to 2^Z\}_{n\in\mathbb{N}}]$. From [41, Proof of Theorem 3] and [3, Proposition 18] it may be concluded that the composition of classes of approximative maps is a class of approximative maps and it does not depend on the representatives of the homotopy classes $[\overline{f}]$ and $[\overline{g}]$.

We obtain a category whose objects are compact metric spaces and whose morphisms are the homotopy classes of approximative maps. The identity morphism is given by the homotopy class of $\{i_n : X \to 2^X\}_{n\in\mathbb{N}}$, where $i_n(x) = \{x\}$ for every $n \in \mathbb{N}$ and $x \in X$. Let HN denote this category.

Theorem 2.7. *The category HN is isomorphic to the shape category of compact metric spaces.*

Remark 2.8. It can be said that this approach substitutes the Hilbert cube for hyperspaces in the original approach of K. Borsuk [7].

We now recall basic concepts and set notation of the theory of finite topological spaces. We denote the category of finite partially ordered sets (or posets for short) by $Poset$. From now on, we assume that every finite topological space is a T_0-space without explicit mention, and we will refer to them as finite spaces for short. Let X be a finite space and $x \in X$. We denote by U_x the intersection of every open set containing x, which is again open. Similarly, F_x denotes the intersection of all closed sets containing x, which is a closed set. We say that $x \leq y$ if and only if $U_x \subseteq U_y$. With this relation we have that X is a finite poset. Suppose $(X, \leq)$ is a finite poset. Then the family of lower sets of $\leq$ forms a basis for a T_0 topology on X. A lower set S of X is a set satisfying that if $x \leq y \in S$, then $x \in S$. See [29] or [4] for a complete introduction to the theory of finite spaces. In what follows, we treat finite posets and finite spaces as the same object because indeed the category $Poset$ and the category of finite spaces are isomorphic (this can be deduced from [1]). If $x, y \in X$ and $x < y$, then we say that y covers x and x is covered by y. Note that a map $f : X \to Y$ between finite spaces is continuous if and only if it is order-preserving.

The Hasse diagram of a finite poset X is a directed graph: the vertices are the points of X and there is an edge (x, y) if and only if $x < y$ and there is no $z \in X$ satisfying $x < z < y$. We omit the orientation of the edges and assume an upward orientation in subsequent graphs.

An important concept in the theory of finite spaces is the notion of weak homotopy equivalence. We recall it here. A map $f : X \to Y$ is a weak homotopy equivalence if the induced maps $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ on the homotopy groups are isomorphisms for all $x \in X$ and all positive integer number i, where the map f_* induces a bijection in dimension 0. Notice that every weak homotopy equivalence induces isomorphisms on singular homology groups by a well-known theorem of J.H.C. Whitehead.

There is a functor $\mathcal{K} : Poset \to SimpComplex$, where $SimpComplex$ denotes the category of simplicial complexes whose morphisms are simplicial maps. As usual, $|K|$ stands for the geometric realization of a simplicial complex K. For a finite poset X, $\mathcal{K}(X)$ denotes the order complex of X. There is also a functor $\mathcal{X} : SimpComplex \to Poset$. For a simplicial complex L, $\mathcal{X}(L)$ denotes the face poset of L. The finite barycentric subdivision of a finite poset X is defined by $\mathcal{X}(\mathcal{K}(X))$. The following two theorems are from [30].

Theorem 2.9. *For each finite space X there exists a weak homotopy equivalence $f_X : |\mathcal{K}(X)| \to X$. For each finite simplicial complex K there exists a weak homotopy equivalence $f_K : |K| \to \mathcal{X}(K)$.*

Theorem 2.10. *Let $g : X \to Y$ be a continuous function between finite spaces and let $f_X : |\mathcal{K}(X)| \to X$ and $f_Y : |\mathcal{K}(Y)| \to Y$ denote the weak homotopy equivalences of Theorem 2.9. Then $g \circ f_X = f_Y \circ |\mathcal{K}(f)|$.*

We recall briefly the method obtained in [9] to reconstruct algebraic invariants of X. We say that a finite set $A \subset X$ is an ϵ-approximation of X if for every $x \in X$ there exists $a \in A$ such that $d(x, a) < \epsilon$, where ϵ is a positive real value. Given an ϵ-approximation A of X, we set $\mathcal{U}_\epsilon(A) = \{C \subset A | \text{diam}(C) < \epsilon\}$. Since A is a finite set, we obtain that $\mathcal{U}_\epsilon(A)$ is a finite poset with the following relation: $C \leq D$ if and only if $C \subseteq D$. Thus, $\mathcal{U}_\epsilon(A)$ is a finite space.

Let (X, d) be a compact metric space and let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive real values satisfying that $\epsilon_{n+1} < \frac{\epsilon_n}{2}$. For every $n \in \mathbb{N}$, consider an ϵ_n-approximation A_n of X. The map $q_{n,n+1} : \mathcal{U}_{4\epsilon_{n+1}}(A_{n+1}) \to \mathcal{U}_{4\epsilon_n}(A_n)$ defined by $q_{n,n+1}(C) = \bigcup_{c\in C} \mathcal{B}(c, \epsilon_n) \cap A_n$ is continuous (see [9]), where $\mathcal{B}(x, \epsilon)$ denotes the open ball of radius ϵ and center x. We say that the inverse sequence $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is a **finite approximation** of X. Let H_* denote the homological functor, where we consider the singular homology with integer coefficients.

Proposition 2.11. *([9, Proposition 3.2]) Given a compact metric space (X, d) and a finite approximation $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ of it. The inverse limit of $(H_l((\mathcal{U}_{4\epsilon_n}(A_n)), H_l(q_{n,n+1}))$ is isomorphic to the l-dimensional Čech homology group of X.*

Note that if X is a CW-complex, then the singular homology groups of X coincide with the Čech homology groups of X. As for prerequisites, the reader is expected to be familiar with the notion of pro-category. Particularly, we will use the categories pro-$HTop$ and pro-Top, where Top is the topological category and $HTop$ is the homotopy category of topological spaces. For more details about this topic, inverse systems and inverse sequences we refer the reader to [28].

Remark 2.12. There is also a similar construction $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ that uses other bonding maps . Let (X, d) be a compact metric space and let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive real values such that A_n is an ϵ_n-approximation of X, $\gamma_n = \sup\{d(x, A_n)|x \in X\}$ and $\epsilon_{n+1} < \frac{\epsilon_n - \gamma_n}{2}$. We define $p_{n,n+1} : \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1}) \to \mathcal{U}_{2\epsilon_n}(A_n)$ by

$$p_{n,n+1}(C) = \bigcup_{c\in C} \{a \in A_n | d(a, c) = d(A_n, c)\}. \quad (1)$$

We obtain that $\mathcal{U}_{2\epsilon_n}(A_n)$ is a finite poset with the subset relation and $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ is an inverse sequence. This inverse sequence is the so-called *Main Construction* introduced in [2] and is isomorphic to every finite approximation $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ of X in pro-$HTop$ (see[9, Section 5]).

3 Combinatorial description of shape theory and shape invariants

The idea of the combinatorial description of the shape theory is to use finite approximations of compact metric spaces to define morphisms between them. Given a compact metric space (X, d), we construct an inverse sequence of finite spaces from finite samples of X, i.e., the finite approximation of X introduced in Section 2. This construction is not unique since it depends on the finite samples of X and the values of ϵ_n. However, in [9, Section 5], it is shown that given two finite approximations of X, they are isomorphic in pro-$HTop$. Hence, we can say that a finite approximation $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ of X is unique in a suitable category and we denote it by $T(X)$.

We define the category $\mathbb{E}$ as follows. The objects of $\mathbb{E}$ are compact metric spaces. Given two compact metric spaces X and Y, set $\mathbb{E}(X,Y) = \{(f_n, f) : T(X) \to T(Y)|\ (f_n, f)$ is a morphism in pro-$HTop$ where $T(X) = (\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is a finite approximation of X and $T(Y) = (\mathcal{U}_{4\tau_n}(B_n), r_{n,n+1})$ is a finite approximation of $Y\}$. Let us recall that a morphism $(f_n, f) : T(X) \to T(Y)$ in pro-$HTop$ is an equivalence class of morphisms of systems. With a certain abuse of notation, we denote by (f_n, f) one of the representatives of this class. This representative consists of a function $f : \mathbb{N} \to \mathbb{N}$ and of morphisms $f_n : \mathcal{U}_{4\epsilon_{f(n)}}(A_{f(n)}) \to \mathcal{U}_{4\tau_n}(B_n)$ in $HTop$, one for each $n \in \mathbb{N}$, such that whenever $n \leq n'$, then there exists $m \in \mathbb{N}$, $m \geq f(n), f(n')$, for which $f_n \circ q_{f(n),m} = p_{n,n'} \circ f_{n'} \circ q_{f(n'),m}$.

It is easy to check that $\mathbb{E}$ is a category. This is because we are considering a subclass of objects (we identify X with $T(X)$) from the category pro-$HTop$ restricted to finite spaces, where the morphisms are the same as in the pro-$HTop$ category.

Let us illustrate in Figure 1 the situation described with the category $\mathbb{E}$. Let X and Y be two compact metric spaces. The space X is essentially a circle, and Y is a segment. For each space, we consider a finite approximation, $T(X) = (X_n, q_{n,n+1})$ and $T(Y) = (Y_n, r_{n,n+1})$. We have depicted the Hasse diagrams of the terms of the inverse sequence $T(X)$ and X (above) and $T(Y)$ and Y (below) . We also have represented a morphism $(f_n, f) : T(X) \to T(Y)$.

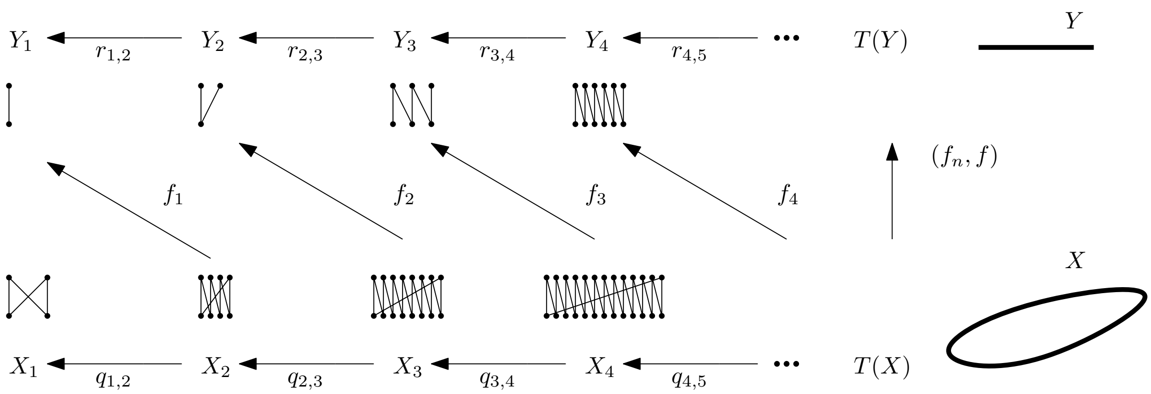


Figure 1: Schematic description of the combinatorial approach to shape theory.

We now state the main result and develop some notions and shape invariants.

Theorem 3.1. *The category $\mathbb{E}$ is isomorphic to the shape category of compact metric spaces.*

For finite spaces there exists the notion of core. We recall it for completeness.

Definition 3.2. *Let X be a finite space and $x \in X$. It is said that x is a down (up) beat point if $U_x \setminus \{x\}$ ($F_x \setminus \{x\}$) has a maximum (minimum).*

Proposition 3.3. *If X is a finite space and $x \in X$ is a beat point, then $X \setminus \{x\}$ is a strong deformation retract of X.*

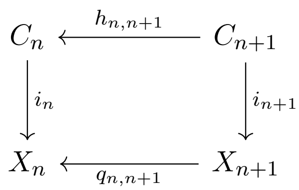
A finite space is a **minimal finite space** if it does not have beat points. The **core** of a finite space X is the resulting space after removing one by one beat points until there are no more. It is worth mentioning that the core of a finite space is unique up to homeomorphism, and that two finite spaces are homotopy equivalent if and only if their cores are homeomorphic. We generalize this notion to inverse sequences of finite spaces.

Definition 3.4. *Let $(X_n, t_{n,n+1})$ be an inverse sequence of finite spaces. Suppose C_n is the core of X_n and $r_n : X_n \to C_n$ is a retraction satisfying that $r_n \circ i_n = \mathrm{id}_{C_n}$ and $i_n \circ r_n$ is homotopic to id_{X_n}, where $i_n : C_n \to X_n$ denotes the inclusion. We say that the core of $(X_n, t_{n,n+1})$ is $(C_n, r_n \circ t_{n,n+1} \circ i_{n+1})$ and write $C(X_n, t_{n,n+1}) = (C_n, r_n \circ t_{n,n+1} \circ i_{n+1})$.*

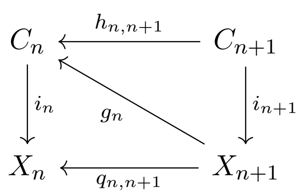
By Proposition 3.3, we deduce that every finite space and its core are isomorphic in $HTop$. We prove the analogue result for inverse sequences of finite spaces and their cores.

Theorem 3.5. *Let $(X_n, q_{n,n+1})$ be an inverse sequence of finite spaces. Then $(X_n, q_{n,n+1})$ is isomorphic to $C(X_n, q_{n,n+1})$ in pro-$HTop$.*

Proof. We fix notation, $C(X_n, q_{n,n+1}) = (C_n, h_{n,n+1})$ where $h_{n,n+1} = r_n \circ q_{n,n+1} \circ i_{n+1}$. There is a natural morphism (i_n, i) in pro-$HTop$ between $(C_n, h_{n,n+1})$ and $(X_n, q_{n,n+1})$ induced by the inclusions, that is, $i : \mathbb{N} \to \mathbb{N}$ is the identity map and $i_n : C_n \to X_n$ is the inclusion. It is trivial to check that (i_n, i) is a well-defined morphism since the following diagram is commutative up to homotopy.



We now construct a sequence $\{g_n : X_{n+1} \to C_n\}_{n\in\mathbb{N}}$ of continuous maps making the following diagram commutative up to homotopy.



For every $n \in \mathbb{N}$, define $g_n : X_{n+1} \to C_n$ by $g_n = r_n \circ q_{n,n+1}$. By the construction, we have $g_n \circ i_{n+1} = r_n \circ q_{n,n+1} \circ i_{n+1}$ and $h_{n,n+1} = r_n \circ q_{n,n+1} \circ i_{n+1}$, which yields the commutativity of the first triangle. We also have that $i_n \circ g_n = i_n \circ r_n \circ q_{n,n+1}$. Therefore, $i_n \circ g_n$ is homotopic to $q_{n,n+1}$ and we obtain the commutativity up to homotopy of the second triangle. By Morita's lemma (see [34] or [28, Chapter 2, Theorem 5]), we deduce the desired result. □

Remark 3.6. Let $(X_n, q_{n,n+1})$ be an inverse sequence of finite spaces. Suppose L_n is a strong deformation retract of X_n for every $n \in \mathbb{N}$. Following the same arguments used before, we can obtain an inverse sequence where the terms are given by L_n. Repeating the proof of Theorem 3.5, it can be deduced that this new inverse sequence is isomorphic to $(X_n, q_{n,n+1})$ in pro-$HTop$.

Example 3.7. We consider $X^1 = \{A, B\}$ with $A < B$. Let X^{n+1} denote the n-th finite barycentric subdivision of X^1. The finite barycentric subdivision of X^1 can be seen as the poset given by the chains of X^1, where the partial order is defined by the subset relation. We have a natural inverse sequence $(X^n, h_{n,n+1})$, where $h_{n,n+1} : X^{n+1} \to X^n$ is given by

$h(x_1 < ... < x_m) = x_m$ (see [4] or [29] for more details). It is easily seen that the core of X^n is a space with one point for every $n \in \mathbb{N}$. This implies that $(X^n, h_{n,n+1})$ is isomorphic to $(*, \mathrm{id})$. In Figure 2, we have the Hasse diagrams of the terms of both inverse sequences, as well as a grayscale representation of the inclusion morphism $(i_n, i) : C(X^n, h_{n,n+1}) \to (X^n, h_{n,n+1})$.

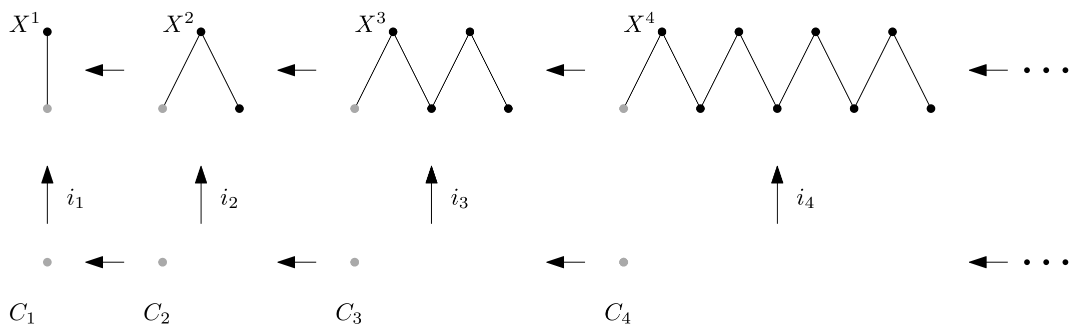


Figure 2: Schematic illustration of the inverse sequence $(X^n, h_{n,n+1})$ and its core, along with the inclusion morphism (i_n, i).

The core of an inverse sequence may be used to show that two compact metric spaces have the same shape, we give an example of this and also illustrate the way we can use different constructions of finite spaces (see Remark 2.12).

Example 3.8. We consider the computational model of the topologist's sine curve S, that is,

$$S = a_\infty \cup (\bigcup_{n\geq 1} \bar{b}_n) \cup (\bigcup_{n\geq 1} \underline{b}_n) \cup (\bigcup_{n\geq 0} a_n).$$

where

$$\begin{aligned}
\bar{b}_n &= (\frac{1}{2^{2n-1}}, \frac{1}{2}) - (\frac{1}{2^{2n-2}}, \frac{1}{2}) \quad n \geq 1, \\
\underline{b}_n &= (\frac{1}{2^{2n}}, 0) - (\frac{1}{2^{2n-1}}, 0) \quad n \geq 1, \\
a_n &= (\frac{1}{2^n}, \frac{1}{2}) - (\frac{1}{2^n}, 0) \quad n \geq 0, \\
a_\infty &= (0, \frac{1}{2}) - (0, 0),
\end{aligned}$$

$n \in \mathbb{N}$ and $(a, b) - (c, d)$ denotes the segment joining the point (a, b) with (c, d) (see Figure 3 for the topologist's sine curve with the segments introduced earlier). The metric of S is the one inherited as a subspace of $\mathbb{R}^2$. We obtain the *Main Construction* for S and study at the same time its core. It is important to note that in the subsequent steps, we will omit the computation of the sequence γ_n described in Remark 2.12. However, the sequence ϵ_n that we will obtain is within the hypothesis of the *Main Construction*. We prefer to omit this part for the sake of the exposition and to avoid extra computations.

Step 1. The diameter of S is $\frac{\sqrt{5}}{2}$, so we can consider $\epsilon_1 = \sqrt{5}$, $A_1 = \{(0, \frac{1}{4})\}$ and $\mathcal{U}_{2\epsilon_1}(A_1) = A_1$. The core of $\mathcal{U}_{2\epsilon_1}(A_1)$ is also A_1. Note that there is a certain abuse of the language to simplify the exposition when $\mathcal{U}_{2\epsilon_1}(A_1)$ and A_1 are considered.

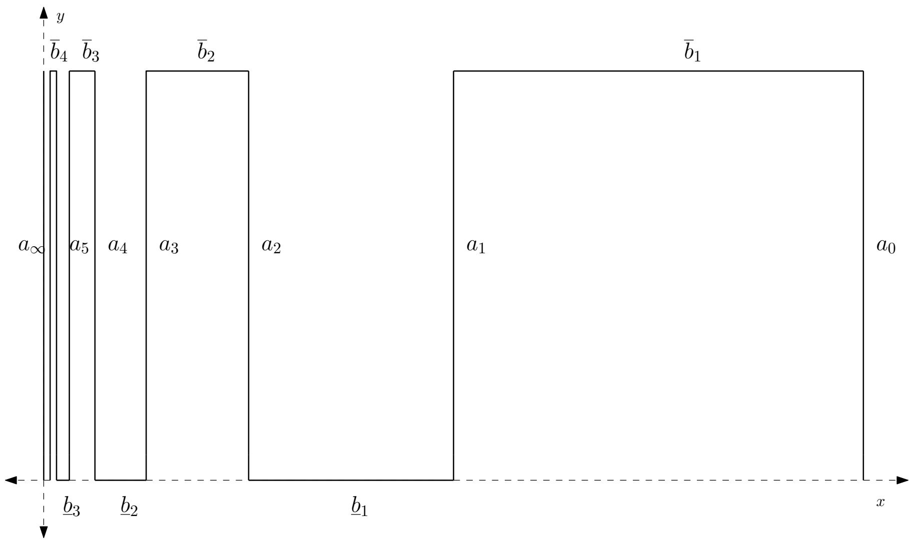


Figure 3: The computational topologist's sine, where the axis of $\mathbb{R}^2$ are with dashed lines.

Step 2. We consider $\epsilon_2 = \frac{\sqrt{2}}{2^3} < \frac{\epsilon_1}{2}$, the grid $G_2 = \{(\frac{l}{2^{3-1}}, \frac{k}{2^{3-1}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$ and the intersection of G_2 with S. There are two points in a_3 that are at distance ϵ_2 to $G_2 \cap S$, which are $b_1 = (\frac{1}{2^3}, \frac{1}{2^3})$ and $b_2 = (\frac{1}{2^3}, \frac{1}{2^3} + \frac{1}{2^2})$. If we add $(0, \frac{1}{2^3})$ and $(0, \frac{1}{2^3} + \frac{1}{2^2})$ to $G_2 \cap S$, then we get an ϵ_2-approximation

$$A_2 = G_2 \cap S \cup \{(0, \frac{1}{2^3}), (0, \frac{1}{2^3} + \frac{1}{2^2})\}.$$

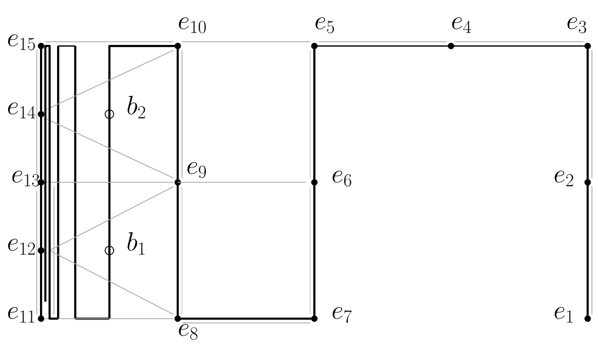


Figure 4: The ϵ_2-approximation A_2 of S.

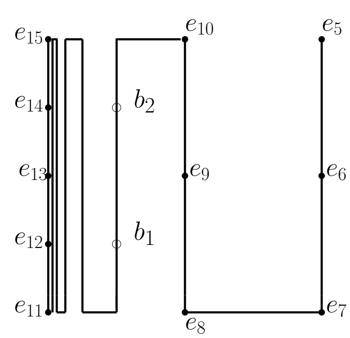


Figure 5: The set of points A'_2 in S.

We write $B_2 = \{x \in A_2 | x \in \bar{b}_1 \setminus \{a_1\} \text{ or } x \in a_0\}$, that is, the points of A_2 that lie to the right of a_1. It is easy to observe that $\mathcal{U}_{2\epsilon_2}(A'_2)$ is a strong deformation retract of $\mathcal{U}_{2\epsilon_2}(A_2)$, where $A'_2 = A_2 \setminus \{B_2\}$. The last assertion is an immediate consequence of the construction we made of A_2 and the value that we have chosen for ϵ_2. Suppose $C \in \mathcal{U}_{2\epsilon_2}(A_2)$ contains points of the approximation that lie in B_2, which means that C is of the form $C = \{e_k, e_{k+1}\}$ or $C = \{e_k\}$. Then $\{e_1\}$ is an up beat point because $F_{\{e_1\}} \setminus \{e_1\} = \{e_1, e_2\}$, so we can remove it without changing the homotopy type of $\mathcal{U}_{2\epsilon_2}(A_2)$. Now, $\{e_1, e_2\}$ is a down beat point since

$U_{\{e_1,e_2\}} \setminus \{e_1, e_2\} = \{e_2\}$. Therefore, we can remove it. We can proceed recursively until the point $\{e_5\}$. This point satisfies that $d(e_5, e_6), d(e_5, e_{10}) = \frac{1}{4} < 2\epsilon_2 = \frac{\sqrt{2}}{2^2}$, so $\{e_5, e_6\}$, $\{e_5, e_{10}\} \in \mathcal{U}_{2\epsilon_2}(A_2)$, which implies that $\{e_5\}$ is not an up beat point. On the other hand, $\{e_5, e_6\}$ and $\{e_5, e_{10}\}$ are clearly not down beat points. A similar argument can be made with the rest of the points in A_2 that lie in a_1. In addition, the map $p_{1,2}$ trivially sends every $C \in \mathcal{U}_{2\epsilon_2}(A'_2)$ to A_1 (see (1) in Remark 2.12). In Figure 4, we present A_2 and arcs in gray with less thickness than the topologist's sine curve to represent points of $\mathcal{U}_{2\epsilon_2}(A_2)$ that have cardinality equal to 2. In Figure 5, we present A'_2.

Step 3. We consider $\epsilon_3 = \frac{\sqrt{2}}{2^6} < \frac{\epsilon_2}{2}$, the grid $G_3 = \{(\frac{l}{2^{6-1}}, \frac{k}{2^{6-1}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$ and the intersection of G_3 with S. There are 16 points that are at distance ϵ_3 to A_3, these points lie in a_6. Specifically,

$$\{(\frac{1}{2^6}, \frac{2k+1}{2^6})|k = 0, 1, 2..., 15\}.$$

We add points of a_∞ to get an ϵ_3-approximation, i.e.,

$$A_2 = (G_3 \cap S) \cup \{(0, \frac{2k+1}{2^6})|k = 0, 1, 2..., 15\}.$$

We consider $B_3 = \{x \in A_2 | x \in a_l$ with $l = 0, 1, 2, 3$ or $x \in \overline{b}_i$ with $i = 1, 2$ or $x \in \underline{b}_1$ or $x \in \underline{b}_2 \setminus \{a_4\}$ $\}$, i.e., the points of A_3 that lie to the right of a_4. We enumerate from right to left the points of A_3, see Figure 6. We have that $\{e_1\}$ is only covered by $\{e_1, e_2\}$, so we can remove it. Now, $\{e_1, e_2\}$ only covers $\{e_2\}$, so it is a down beat point and we can remove it. If we continue in this fashion, we get that $\mathcal{U}_{2\epsilon_3}(A'_3)$ is a strong deformation retract of $\mathcal{U}_{2\epsilon_3}(A_3)$ where $A'_3 = A_3 \setminus \{B_3\}$. Suppose x is a point of A_3 that lie in a_4. Then there exist points in a_5 and a_4 that are at distance less than $2\epsilon_3$ to x. Notice that every $C \in \mathcal{U}_{2\epsilon_3}(A'_3)$ contains points lying in a_4 or a_5. In Figure 7, we have depicted A'_3.

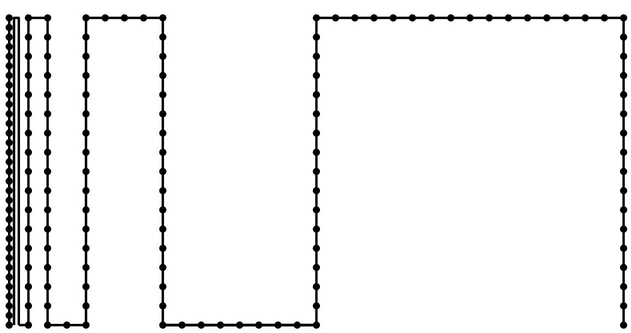


Figure 6: The ϵ_3-approximation A_3 of S.

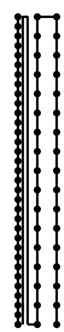


Figure 7: The set of points A'_3 in S.

Furthermore, the image of the map $p_{2,3} : \mathcal{U}_{2\epsilon_3}(A'_3) \to \mathcal{U}_{2\epsilon_2}(A_2)$ is $\mathcal{U}_{2\epsilon_2}(\overline{A}_2) \subset \mathcal{U}_{2\epsilon_2}(A'_2)$, where $\overline{A}_2 = \{x \in A_2 | x \in a_\infty\}$ (see (1) in Remark 2.12). On the other hand, after some routine computations, we may conclude that $\mathcal{U}_{2\epsilon_2}(\overline{A}_2)$ is contractible to $\{(0, \frac{1}{4})\}$.

Step n. We consider $\epsilon_n = \frac{\sqrt{2}}{2^{3n-3}}$, the grid $G_n = \{(\frac{l}{2^{3n-4}}, \frac{k}{2^{3n-4}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$ and the intersection of G_n with S. There are 2^{3n-5} points that lie in a_{3n-3} such that the distance to $G_n \cap S$ is exactly ϵ_n. If we add the following points to $G_n \cap S$, then we get an ϵ_n-approximation

$$A_n = (G_n \cap S) \cup \{(0, \frac{2k+1}{2^{3n-3}})|k = 0, 1..., 2^{3n-4} - 1\}.$$

It is simple to show that $\mathcal{U}_{2\epsilon_n}(A_n)$ is homotopy equivalent to $\mathcal{U}_{2\epsilon_n}(A'_n)$, where $A'_n = A_n \setminus \{B_n\}$ and B_n consists of points in A_n that lie to the right of a_{3n-5}.

In addition, $p_{n-1,n}$ sends $\mathcal{U}_{2\epsilon_n}(A'_n)$ to $\mathcal{U}_{2\epsilon_{n-1}}(\overline{A}_{n-1}) \subset \mathcal{U}_{2\epsilon_{n-1}}(A_{n-1})$, where $\overline{A}_{n-1} = \{x \in A_{n-1} | x \in a_\infty\}$. It is routine to check that $\mathcal{U}_{2\epsilon_{n-1}}(\overline{A}_{n-1})$ is contractible to $\{(0, \frac{1}{4})\}$.

Shape of the topologist's sine curve. By Remark 2.12, $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ is isomorphic to every finite approximation of S. Additionally, $(\mathcal{U}_{2\epsilon_n}(\overline{A}_n), p_{n,n+1_{|\mathcal{U}_{2\epsilon_n}(\overline{A}_n)}})$ is isomorphic to every finite approximation of $[0, \frac{1}{2}]$ in pro-$HTop$. Thus, S and $[0, \frac{1}{2}]$ are isomorphic in the shape category by Theorem 3.1.

We have obtained a more suitable description of shape theory. This description is based on finite spaces derived from finite samples of compact metric spaces. Due to the relationship between finite spaces and computational aspects (as pointed out in the introduction), this can be seen as a starting point towards a computational shape theory for future works. From an algebraic point of view, this result is not surprising (see Proposition 2.11). Nevertheless, every finite connected space has trivial shape (see [10, Proposition 2.6]). Given a compact metric space X and a finite approximation $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ of it, we can apply other functors. For instance, the functor $\mathcal{K}$ considered in Section 2. In that manner, more connections with shape theory can be found (see the notion of $HPol$-expansion in [28]).

Proposition 3.9. *Let X be a compact metric space and let $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ be a finite approximation of X. Then $(\mathcal{K}(\mathcal{U}_{4\epsilon_n}(A_n)), \mathcal{K}(q_{n,n+1}))$ is a $HPol$-expansion of X.*

This result is an immediate consequence of [33, Theorem 12] and [9, Section 5]. It also proves that we may obtain some shape invariants applying algebraic functors (see [28, Chapter II]). Particularly, we obtain homology pro-groups, Čech homology, Čech cohomology and shape groups.

By combining these techniques we may deduce whether two spaces are shape equivalent. For instance, let us consider the Cantor set C and the topological subspace of $\mathbb{R}^2$ given by the union along one edge of two squares, that is, Example 3.4 and Example 3.3, respectively, in [9]. After applying homological functors to the finite approximations of these spaces it may be observed that they are not shape equivalent (just looking at Table 1 and Table 2 in [9]).

To conclude this section, we define the notion of *height* for compact metric spaces. The height of a finite space X, denoted by $ht(X)$, is one less than the maximum number of elements in a chain of X. The dimension of a finite simplicial complex L, denoted by $\dim(L)$, is the maximum of dimension of the simplices of L. It is clear that $\dim(\mathcal{K}(X)) = ht(X)$.

Definition 3.10. *Let X be a compact metric space and let $U = (\mathcal{U}_{\epsilon_n}(A_n), q_{n,n+1})$ be a finite approximation of X. The height $ht(U)$ of U is less or equal to some natural number m if $ht(\mathcal{U}_{\epsilon_n}(A_n)) \leq m$ for every $n \in \mathbb{N}$. We say that the height of X, denoted by $ht(X)$, is less or equal to m provided there exists an inverse sequence of finite spaces $V = (V_n, t_{n,n+1})$ isomorphic in pro-$HTop$ to a finite approximation $(\mathcal{U}_{\epsilon_n}(A_n), q_{n,n+1})$ of X satisfying $ht(V) \leq m$. We write $ht(X) = n$ provided n is the least m for which $ht(X) \leq m$.*

Notice that if two compact metric spaces X and Y are isomorphic in $\mathbb{E}$, then $ht(X) = ht(Y)$. The shape dimension of a compact metric space X, denoted by $\mathrm{sd}(X)$, is defined similarly using the dimension of simplicial complexes and $HPol$-expansions, see [28, Chapter II, 1]. The following result is an immediate consequence of the definitions.

Proposition 3.11. *Let X be a compact metric space. Then* $\mathrm{sd}(X) \leq ht(X)$.

Example 3.12. Let us consider the topologist's sine curve S and the inverse sequences of finite posets considered in Example 3.8. Then $ht(S) = 0$, which coincides with the shape dimension of S.

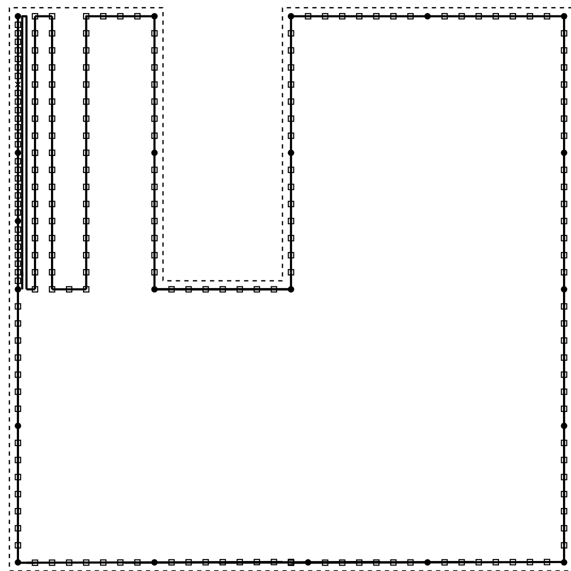


Figure 8: The ϵ_2-approximation D_2 represented with disks and the ϵ_3-approximation D_3 represented with both disks and squares.

Example 3.13. Let us consider the computational model of the Warsaw circle W, that is, consider the computational model of the topologist's sine curve S from Example 3.8 and define W as

$$W = S \cup ((0,0) - (0, -\frac{1}{2})) \cup ((0, -\frac{1}{2}) - (1, -\frac{1}{2})) \cup ((1, -\frac{1}{2}) - (1, 0)).$$

We can leverage the finite approximation obtained in Example 3.8 since $S \subset W$. Hence, we will consider points from $W \setminus \{S\}$ to obtain a finite approximation of W. Consider

$$D_n = A_n \cup (G_n \cap W \setminus \{S\}),$$

where $G_n = \{(\frac{l}{2^{3n-4}}, \frac{k}{2^{3n-4}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$. By the construction, it is trivial that D_n is an ϵ_n-approximation of W, where $\epsilon_n = \frac{\sqrt{2}}{2^{3n-3}}$. In Figure 8, we have represented the points of D_2 with disks and the points of D_3 with both disks and squares, along with the computational model of the Warsaw circle. Consequently, $U = (\mathcal{U}_{2\epsilon_n}(D_n), p_{n,n+1})$ is a finite approximation of W, and $ht(U) \leq 2$. On the other hand, by the construction (see (1) from Remark 2.12), one can deduce that the core of $p_{n,n+1}(\mathcal{U}_{2\epsilon_n}(D_n))$ has height 1 and it is not contractible. Thus, this space is weak homotopy equivalent to a wedge of circles. Let us analyze the case $p_{2,3}(\mathcal{U}_{2\epsilon_3}(D_3))$ to illustrate the previous comment. We enumerate the points of $D_2 \setminus \{A_2\}$ as $d_1, ..., d_7$ following the clockwise orientation. By the arguments given in Example 3.8,

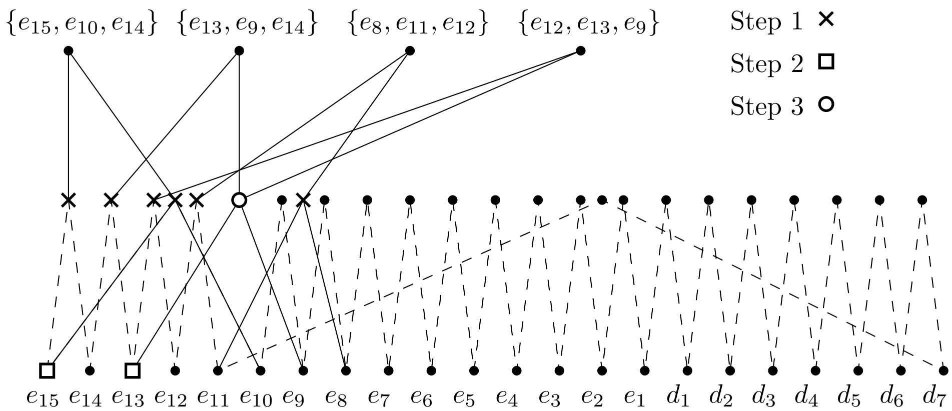


Figure 9: Hasse diagram of $p_{2,3}(\mathcal{U}_{2\epsilon_3}(D_3))$ showing the points removed step by step until de core is reached.

$p_{2,3}(\mathcal{U}_{2\epsilon_3}(D_3 \cap (a_3 \setminus \{(\frac{1}{8}, \frac{1}{2})\})))$ is a finite model of the circle (see the dashed line in Figure 8 and the dashed part of the Hasse diagram in Figure 9):

$$e_{15} < \{e_{15}, e_{14}\} > e_{14} < \{e_{14}, e_{13}\} > e_{13} < \cdots > e_{11} < \{e_{11}, d_7\} > d_7 < \{d_7, d_6\} > d_6 < \cdots$$
$$< \{d_2, d_1\} > d_1 < \{d_1, e_1\} > e_1 < \{e_1, e_2\} > e_2 < \cdots < \{e_9, e_{10}\} > e_{10} < \{e_{10}, e_{15}\} > e_{15}.$$

Note the singletons are denoted without brackets for simplicity. We only need to study $p_{2,3}(D_3 \cap a_3)$, and clearly $p_{2,3}(D_3 \cap a_3)) = \{\{e_8, e_{11}\}, \{e_8, e_{11}, e_{12}\}, \{e_{12}\}, \{e_{12}, e_{13}, e_9\}, \{e_{13}, e_9\}, \{e_{13}, e_9, e_{14}\}, \{e_{14}\}, \{e_{15}, e_{10}, e_{14}\}, \{e_{15}, e_{10}\}\}$. Thus, $p_{2,3}(\mathcal{U}_{2\epsilon_3}(D_3))$ is the poset represented by the Hasse diagram of Figure 9, where we have obtained its core step by step and represented the finite model of the previously mentioned circle in dashed lines. It is clear that $(p_{n,n+1}(\mathcal{U}_{2\epsilon_n}(D_n)), p_{n,n+1})$ is isomorphic to U in pro-$HTop$, and its core has height 1, so $ht(W) \leq 1$. On the other hand, $ht(W)$ cannot be zero; otherwise, we would get a contradiction. This would imply the existence of an inverse sequence whose terms are finite spaces with height zero (particularly, these are antichains) isomorphic to an inverse sequence whose terms are all weak homotopy equivalent to a wedge of circles. From this, we deduce that S and W do not have the same shape because $ht(S) = 0$ and $ht(W) = 1$.

Shape of the Warsaw circle. Observe that for every $n > 1$, $\mathcal{U}_{2\epsilon_n}(D_n)$ contains a finite model of the circle, denoted by K, which is maximal in terms of cardinality (see the dashed lines in Figure 8). Consider $m_n = |K|$. Let S^1 the unit circle in the complex plane with the geodesic distance. Set $E_n = \{e^{\frac{2\pi k i}{m_n}} | k = 0, ..., m_n - 1\}$. It is clear that E_n is a δ_n-approximation of S^1, where $\delta_n = \frac{2\pi}{m_n}$. Using the same map defined in Remark 2.12, we obtain a finite approximation of S^1. As it was done before, we have performed the Main Construction omitting the sequence γ_n from Remark 2.12 for simplicity. With a certain abuse of notation $p_{n,n+1}$ denotes the bonding maps of the finite approximations of S^1 and W. Moreover, $\mathcal{U}_{2\delta_n}(E_n)$ is clearly homeomorphic to K. We will identify both spaces without explicit mention and by abuse of notation we denote K by $\mathcal{U}_{2\delta_n}(E_n)$. Then, there is a natural morphism in pro-$HTop$ $(i_n, i) : (\mathcal{U}_{2\delta_n}(E_n), p_{n,n+1}) \to (\mathcal{U}_{2\epsilon_n}(D_n), p_{n,n+1})$, where $i : \mathbb{N} \to \mathbb{N}$ is the identity map and $i_n : \mathcal{U}_{2\delta_n}(E_n) \to \mathcal{U}_{2\epsilon_n}(D_n)$ is the inclusion map. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{U}_{2\delta_n}(E_n) & \xleftarrow{p_{n,n+1}} & \mathcal{U}_{2\delta_{n+1}}(E_{n+1}) \\ \Big\downarrow{\scriptstyle i_n} & & \Big\downarrow{\scriptstyle i_{n+1}} \\ \mathcal{U}_{2\epsilon_n}(D_n) & \xleftarrow[p_{n,n+1}]{} & \mathcal{U}_{2\epsilon_{n+1}}(D_{n+1}) \end{array}$$

Define $g_n : \mathcal{U}_{2\epsilon_{n+1}}(D_{n+1}) \to \mathcal{U}_{2\delta_n}(E_n)$ by $g_n(x) = p_{n,n+1}(x)$ if $x \in D_{n+1} \setminus \{a_{3n-3} \setminus \{(\frac{1}{2^{3n-3}}, \frac{1}{2})\}\}$ and $g_n(x) = \{y \in D_n \setminus \{a_{3n-4}\} | d(y,x) = d(y, D_n \setminus \{a_{3n-4}\})\}$ if $x \in a_{3n-3} \setminus \{(\frac{1}{2^{3n-3}}, \frac{1}{2})\}$, and for every $C \in \mathcal{U}_{2\epsilon_{n+1}}(D_{n+1})$, $g_n(C) = \bigcup_{x \in C} g_n(x)$. The map g_n is clearly well defined and continuous. Note that g_n coincides with $p_{n,n+1}$ on K. It is evident that $p_{n,n+1} = g_n \circ i_{n+1}$ and $i_n \circ g_n$ is homotopic to $p_{n,n+1}$ because $i_n(g_n(C)) \subseteq p_{n,n+1}(C)$ for every $C \in \mathcal{U}_{2\epsilon_{n+1}}(D_{n+1})$. By Morita's lemma ([28, Chapter 2, Theorem 5]), we deduce that $(\mathcal{U}_{2\delta_n}(E_n), p_{n,n+1})$ is isomorphic to $(\mathcal{U}_{2\epsilon_n}(D_n), p_{n,n+1})$ in pro-$HTop$, which gives that S^1 and W have the same shape, as expected.

4 Proof of Theorem 3.1

Given two compact metric spaces X and Y, we choose finite approximations for them, $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$ and $T(Y) = (\mathcal{U}_{4\epsilon_n}(B_n), q_{n,n+1})$. Note that, for the sake of exposition and to simplify notation, we denote the bonding maps for both finite approximations $T(X)$ and $T(Y)$ by $q_{n,n+1}$. This involves a certain abuse of notation, as the bonding maps for each finite approximation are not necessarily the same. We prove that the set of morphisms in pro-$HTop$ between their finite approximations is in bijective correspondence with the set of shape morphisms between X and Y. To this end, we first show some technical results.

Lemma 4.1. *Let (X,d) be a compact metric spaces and let A be an ϵ-approximation of X. Then, the map $p : X \to \mathcal{U}_{4\epsilon}(A)$ defined by $p(x) = \{a \in A | d(x,a) = d(x,A)\}$ is well-defined and continuous.*

Proof. The proof follows easily from [33, Lemma 2]. □

Lemma 4.2. *Let X be a topological space and Y be a compact metric space. Suppose A is a finite subset of Y. If $f, g : X \to U_\epsilon(Y)$ ($f, g : X \to \mathcal{U}_\epsilon(A)$) are continuous maps where ϵ is a positive real value and $f \cup g : X \to U_\epsilon(Y)$ ($f \cup g : X \to \mathcal{U}_\epsilon(A)$) defined by $(f \cup g)(x) = f(x) \cup g(x)$ is well-defined, then $f \cup g$ is continuous and homotopic to f and g.*

Proof. For simplicity, we denote the map $f \cup g$ by h. We prove the continuity of h. Let $x \in X$. If U is an open set containing $h(x)$, then it also contains $f(x)$ and $g(x)$. From the continuity of f and g, it follows that there exist open sets V_f and V_g containing x such that $f(V_f) \subseteq U$ and $g(V_g) \subseteq U$. From this, we deduce the continuity of h.

We show that h is homotopic to f. Consider $H : X \times [0,1] \to Z$ defined by $H(x,t) = f(x)$ if $t \in [0,1)$ and $H(x,1) = h(x)$, where Z is $U_\epsilon(Y)$ or $\mathcal{U}_\epsilon(A)$. It suffices to check the continuity of H at points of the form $(x,1)$ where $x \in X$. Since h is continuous, for every open set U containing $h(x)$ there exists an open set V containing x such that $f(V) \cup g(V) \subseteq U$. Particularly, $f(V) \subseteq U$. Hence, $V \times [0,1]$ is an open set of $X \times [0,1]$ containing $(x,1)$ and satisfying that $H(V \times [0,1]) \subseteq U$, which gives the desired result. The proof to show that g is homotopic to h is the same. □

In the following proposition, we present a constructive method to obtain a morphism in pro-$HTop$ induced by the homotopy class of an approximative map.

Proposition 4.3. *Let (X, d) and (Y, l) be compact metric spaces. If $[\overline{f}] : X \to Y$ is the homotopy class of an approximative map, then there exists a natural morphism $T([\overline{f}]) : T(X) \to T(Y)$ in pro-$HTop$.*

Proof. Set $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$ and $T(Y) = (\mathcal{U}_{4\epsilon_n}(B_n), q_{n,n+1})$. Firstly, we prove that for every $n \in \mathbb{N}$, the map $r_n : U_{2\epsilon_n}(Y) \to \mathcal{U}_{4\epsilon_n}(B_n)$ defined by

$$r_n(C) = \bigcup_{x \in C} \{b \in B_n | l(x, b) = l(x, B_n)\}$$

is well-defined and continuous. If $x, y \in r_n(C)$ for some $C \in U_{2\epsilon_n}(Y)$, then there exist $c_x, c_y \in C$ such that $x \in r_n(c_x)$ and $y \in r_n(c_y)$. We have $l(x, c_x), l(y, c_y) < \epsilon_n$ and $l(c_x, c_y) < 2\epsilon_n$. Hence, we obtain

$$l(x, y) < l(x, c_x) + l(c_x, c_y) + l(c_y, y) < \epsilon_n + 2\epsilon_n + \epsilon_n,$$

which implies that r_n is well-defined. We deduce that r_n is continuous because $r_n = p^*_{n|U_{2\epsilon_n}(Y)}$, where p_n is the continuous map considered in Lemma 4.1 and p^*_n denotes the extension of p_n to the hyperspace of Y defined in Lemma 2.2.

We now prove the commutativity up to homotopy of the following diagram, where i denotes the inclusion map.

$$\begin{array}{ccc} U_{2\epsilon_n}(Y) & \xrightarrow{\quad r_n \quad} & \mathcal{U}_{4\epsilon_n}(B_n) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle q_{n,n+1}} \\ U_{2\epsilon_{n+1}}(Y) & \xrightarrow{\quad r_{n+1} \quad} & \mathcal{U}_{4\epsilon_{n+1}}(B_{n+1}) \end{array}$$

Consider $C \in U_{2\epsilon_{n+1}}(Y)$. If $x \in q_{n,n+1}(r_{n+1}(C))$, then there exist $a_x \in B_{n+1}$ and $b_x \in C$ such that $x \in q_{n,n+1}(a_x)$ and $a_x \in r_{n+1}(b_x)$. We obtain

$$l(x, b_x) < l(x, a_x) + l(a_x, b_x) < \epsilon_n + \frac{\epsilon_n}{2}.$$

If $y \in r_n(i(C))$, then there exists $b_y \in C$ such that $y \in r_n(b_y)$. Since $l(b_x, b_y) < 2\epsilon_{n+1} < \epsilon_n$, we have

$$l(x, y) < l(x, b_x) + l(b_x, b_y) + l(b_y, y) < \epsilon_n + \frac{\epsilon_n}{2} + \epsilon_n + \epsilon_n.$$

Thus, the map $h = r_n \circ (i \cup q_{n,n+1}) \circ r_{n+1} : U_{2\epsilon_{n+1}}(Y) \to \mathcal{U}_{4\epsilon_n(B_n)}$ is well-defined. Consequently, the continuity of h and the commutativity up to homotopy of the previous diagram follow from Lemma 4.2.

Let us consider $f = \{f_k : X \to 2^Y\}_{k \in \mathbb{N}} \in [\overline{f}]$. By Proposition 2.1, $U_{2\epsilon_n}(Y)$ is an open neighborhood of Y in 2^Y for every $n \in \mathbb{N}$. By Definition 2.4, there exists a natural number, that we denote by s_n, such that f_m is homotopic to f_{m+1} in $U_{2\epsilon_n}(Y)$ for every $m \geq s_n$. Let H denote the homotopy between f_{s_n} and f_{s_n+1}. By Theorem 2.3, there exists $\gamma_n > 0$ such that $\overline{H}(U_{\gamma_n}(X) \times I) \subseteq U_{2\epsilon_n}(Y)$. This gives the key to be able to define a morphism $(f_n, f) : T(X) \to T(Y)$, where there is an abuse of notation: f_k is used in the definition of approximative map, while f_n will be used in the construction of a candidate to be a morphism in pro-$HTop$. With a certain abuse of notation, we also denote by f the map $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \min\{l \in \mathbb{N} | 4\delta_l < \gamma_n\}$, it is clear that f is well-defined and satisfies that $f(n) \leq f(m)$ for every $n \leq m$ in $\mathbb{N}$. For every natural number n, consider $f_n : \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \to \mathcal{U}_{4\epsilon_n}(B_n)$ given by $f_n = r_n \circ f^*_{s_n} \circ i$, where $f^*_{s_n}$ denotes the extension of f_{s_n} to the hyperspace of X given in Lemma 2.2 and $i : \mathcal{U}_{4\delta_{f(n)}}(B_n) \to U_{\gamma_n}(X)$ denotes the inclusion map. By the construction, it is immediate to conclude that f_n is well-defined and continuous for every $n \in \mathbb{N}$. Notice that the importance of the number γ_n can be seen in the construction of f and the importance of the homotopy $\overline{H}$ will be seen when we check that $(f_n, f) : T(X) \to T(Y)$ is a morphism in pro-$HTop$. To check that (f_n, f) is a morphism in pro-$HTop$ we need to verify the commutativity up to homotopy of the following diagram.

$$\begin{array}{ccccccc}
\mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) & \xrightarrow{\ i\ } & U_{\gamma_n}(X) & \xrightarrow{\ f^*_{s_n}\ } & U_{2\epsilon_n}(Y) & \xrightarrow{\ r_n\ } & \mathcal{U}_{4\epsilon_n}(B_n) \\
\uparrow{\scriptstyle q} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle q} \\
\mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)}) & \xrightarrow{\ i\ } & U_{\gamma_{n+1}}(X) & \xrightarrow{\ f^*_{s_{n+1}}\ } & U_{2\epsilon_{n+1}}(Y) & \xrightarrow{\ r_{n+1}\ } & \mathcal{U}_{4\epsilon_{n+1}}(B_{n+1})
\end{array}$$

We check the commutativity up to homotopy of the first square. Notice that we will omit the subscripts of the bonding maps $q_{n,n+1}$ for simplicity. Consider $C \in \mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)})$. If $x \in i(q(C)) = q(C)$, then there exists $a_x \in C$ such that $x \in q(a_x)$, which implies $d(x, a_x) < 2\delta_{f(n)} < \frac{\gamma_n}{2}$. If $y \in i(i(C)) = C$, then $d(y, a_x) < 4\delta_{f(n+1)} < \frac{\gamma_n}{2}$. We obtain $d(x, y) < \gamma_n$. Consequently, $q(C) \subseteq q(C) \cup C$ for every $C \in \mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)})$, where $q \cup i : \mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)}) \to U_{\gamma_n}(X)$ is well-defined and continuous. By Lemma 4.2, the commutativity up to homotopy of the first square can be deduced. The second square is commutative up to homotopy by the construction. The commutativity up to homotopy of the third square was proved at the beginning.

If $g \in [\overline{f}]$, then we can repeat the same construction to obtain $(g_n, g) : T(X) \to T(Y)$. The counterparts of γ_n and s_n in the construction of (f_n, f) here are τ_n and h_n, i.e., in the construction of (g_n, g) we obtain that τ_n and h_n play the same role as γ_n and s_n in the construction of (f_n, f). We prove that (f_n, f) is equivalent to (g_n, g) as morphisms in pro-$HTop$. To do this, given a natural number n, we need to verify the commutativity up to homotopy of the following diagram for some $m \geq f(n), g(n)$.

$$\begin{array}{ccccccccc}
 & & \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) & \xrightarrow{\ i\ } & U_{\gamma_n}(X) & \xrightarrow{\ f^*_{s_n}\ } & U_{2\epsilon_n}(Y) & & \\
 & \nearrow^{q} & & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} & \searrow^{r_n} & \\
\mathcal{U}_{4\delta_m}(A_m) & & & & & & & & \mathcal{U}_{4\epsilon_n}(B_n) \\
 & \searrow_{q} & & & & & & \nearrow_{r_n} & \\
 & & \mathcal{U}_{4\delta_{g(n)}}(A_{g(n)}) & \xrightarrow{\ i\ } & U_{\tau_n}(X) & \xrightarrow{\ g^*_{h_n}\ } & U_{2\epsilon_n}(Y) & &
\end{array}$$

We define $m = \max\{f(n), g(n)\}$. Without loss of generality we can assume that $\gamma_n \leq \tau_n$ and $m = f(n)$. We study the commutativity up to homotopy of the first square. Suppose $f(n) \neq g(n)$ because the other case follows easily. Consider $C \in \mathcal{U}_{4\delta_m}(A_m)$. If $y \in i(q(C)) =$

$q(C)$, then there exists $a_y \in C$ such that $y \in q(a_y)$. We have $d(a_y, y) < 2\delta_{g(n)} < \frac{\tau_n}{2}$ and $d(x, a_y) < 4\delta_{f(n)} < 2\delta_{g(n)} < \frac{\tau_n}{2}$ for every $x \in C$. Thus,

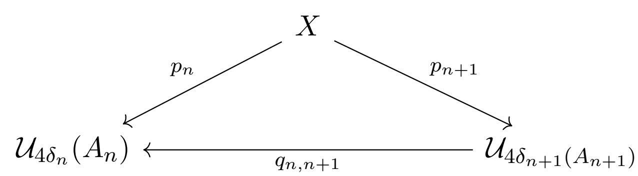
$$d(x, y) < d(a_x, a_y) + d(a_y, y) < \tau_n,$$

and consequently $i \circ (q \cup i) \circ i : \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \to U_{\tau_n}(X)$ is well-defined. By Lemma 4.2, we deduce the desired result. The commutativity up to homotopy of the second square follows from the fact that f and g are homotopic approximative maps (see Theorem 2.3) and the choices of s_n and h_n. For every $m \geq s_n, h_n$ we have that f_{s_n} is homotopic to f_m in $U_{2\epsilon_n}(Y)$ and g_{h_n} is homotopic to g_m in $U_{2\epsilon_n}(Y)$. The third square commutes trivially. □

Given a morphism in pro-$HTop$, we construct a homotopy class of an approximative map in the following result.

Proposition 4.4. *Let (X, d) and (Y, l) be compact metric spaces. If $(f_n, f) : T(X) \to T(Y)$ is a morphism in pro-$HTop$, then there exists a natural homotopy class of an approximative map $E(f_n) : X \to Y$ induced by (f_n, f).*

Proof. Set $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$ and $T(Y) = (\mathcal{U}_{4\epsilon_n}(B_n), q_{n,n+1})$. For the sake of the exposition, we use the same notation for the bonding maps of $T(X)$ and $T(Y)$. Firstly, we consider $p_n : X \to \mathcal{U}_{4\delta_n}(A_n)$ defined by $p_n(x) = \{a \in A_n | d(x, a) = d(x, A_n)\}$ for every $n \in \mathbb{N}$, that is, the continuous map considered in Lemma 4.1. For simplicity, we omit some of the subscripts of the maps. We prove that the following diagram commutes up to homotopy.

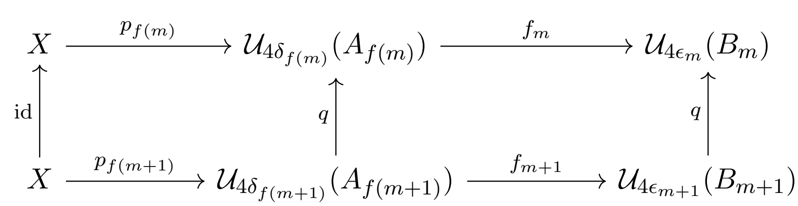


If $a_x \in q(p(x))$, then there exists $b_x \in A_{n+1}$ with $a_x \in q(b_x)$ and $b_x \in p(x)$. Hence, $d(x, b_x) < \delta_{n+1} < \frac{\delta_n}{2}$ and $d(a_x, b_x) < 2\delta_n$. If $c_x \in p(x)$, then we get $d(x, c_x) < \delta_n$. Thus,

$$d(c_x, a_x) < d(c_x, x) + d(x, b_x) + d(b_x, a_x) < 3\delta_n,$$

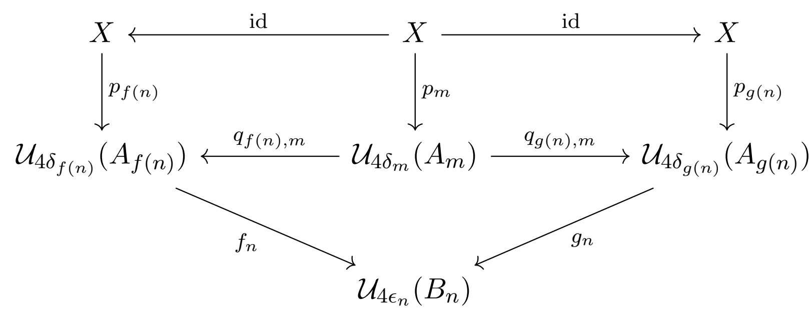
which implies that $q_{n,n+1} \circ (p_{n+1} \cup p_n) : X \to \mathcal{U}_{4\delta_n}(A_n)$ is well-defined. Applying Lemma 4.2 we get that the previous diagram commutes up to homotopy.

We construct a candidate to be an approximative map. Consider $F = \{F_k : X \to 2^Y\}_{k \in \mathbb{N}}$ given by $F_k = f_k \circ p_{f(k)} : X \to \mathcal{U}_{4\epsilon_k}(B_k)$. For every open neighborhood U of the canonical copy of Y in 2^Y, there exists $4\epsilon_m$ such that $\mathcal{U}_{4\epsilon_m}(B_m) \subset U_{4\epsilon_m}(Y) \subseteq U$ by Proposition 2.1. To prove that F is an approximative map we check that F_m is homotopic to F_{m+1} in U, which is equivalent to show the commutativity up to homotopy of the following diagram.



The commutativity up to homotopy of the first square was proved at the beginning. The second square commutes up to homotopy since (f_n, f) is a morphism in pro-$HTop$. Thus F is an approximative map. We denote by $E(f_n)$ the homotopy class generated by the approximative map F. We verify that E is well-defined, that is, if (g_n, g) is equivalent to (f_n, f) as morphisms in pro-$HTop$, then the induced approximative map $G = \{G_k = g_k \circ p_{g(k)} : X \to 2^Y\}_{k\in\mathbb{N}}$ is homotopic to $F = \{F_k = f_k \circ p_{f(k)} : X \to 2^Y\}_{k\in\mathbb{N}}$.

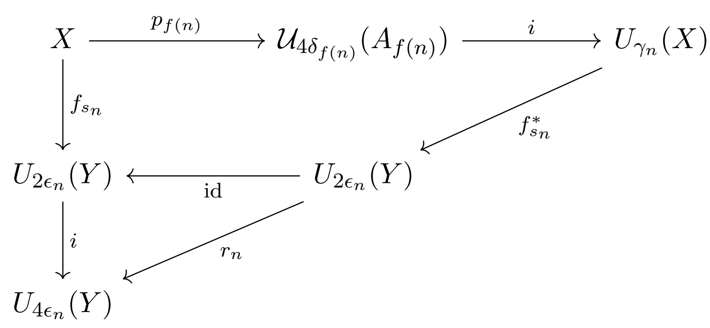
For every open neighborhood U of Y in 2^Y there exists $4\epsilon_n$ such that $U_{4\epsilon_n}(Y) \subseteq U$ by Proposition 2.1. By hypothesis, for every n there exists $m \geq f(n), g(n)$ such that $f_n \circ q_{f(n),m}$ is homotopic to $g_n \circ q_{g(n),m}$. We have the following diagram.



It is obvious that every square commutes up to homotopy. From here, we obtain that F and G are homotopic. □

Lemma 4.5. *Let (X, d) and (Y, l) be compact metric spaces. If $[\overline{f}] : X \to Y$ is the homotopy class of an approximative map, then $E(T([\overline{f}])) = [\overline{f}]$.*

Proof. Let $f = \{f_k : X \to 2^Y\}_{k\in\mathbb{N}}$ denote the approximative map that generates $[\overline{f}]$. We consider a representative (f_n, f) of $T([\overline{f}])$ induced by f and a representative $f' = \{f'_k : X \to 2^Y\}_{k\in\mathbb{N}}$ of $E(T(f_n))$ induced by (f_n, f). Note that, as before, there will be a certain abuse of notation, but the meaning of the maps can be deduced from the context. We will prove that f' and f are homotopic approximative maps. By the proof of Proposition 4.3, $f'_k = f_k \circ p_{f(k)}$, where $f_k = r_k \circ f^*_{s_k} \circ i$. By Proposition 2.1, for every open neigbhorhood U of Y in 2^Y there exists $U_{4\epsilon_n}(Y) \subset U$. We check that the following diagram is commutative up to homotopy.



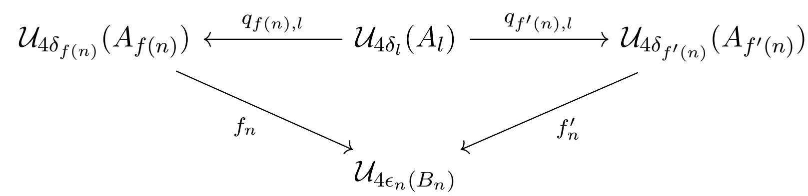
Let $x \in X$. Then we know that $\text{diam}(f_{s_n}(x)) < 2\epsilon_n$. We have $d(x, a) < \delta_{f(n)} < \gamma_n$ for every $a \in p(x)$. Thus, $2\epsilon_n > \text{diam}(f^*_{s_n}(i(p(x)) \cup \{x\})) = \text{diam}(f^*_{s_n}(i(p(x))) \cup f_{s_n}(x))$ so $h = f_{s_n} \cup f^*_{s_n} \circ i \circ p_{f(n)} : X \to U_{\epsilon_n}(Y)$ is continuous and well-defined. By Lemma 4.2, we get the commutativity up to homotopy of the first square. It is clear that $\text{diam}(i(C) \cup r_n(C)) < 4\epsilon_n$

for every $C \in U_{2\epsilon_n}(Y)$. Therefore, we can repeat the previous argument to show that r_n is homotopic to i.

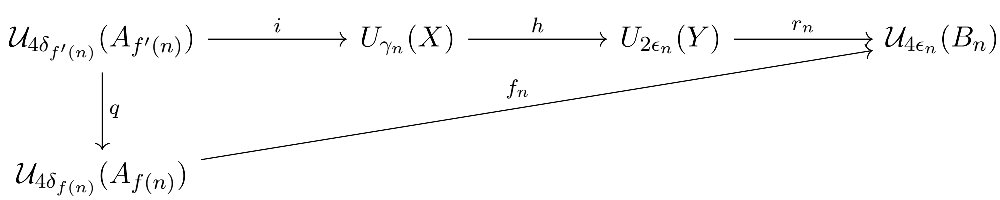
By the construction and hypothesis, for every $m \geq s_n, n$ we have that f_m is homotopic to f_{s_n} in U and f'_n is homotopic to f'_m in U, which gives the desired result. □

Lemma 4.6. *Let (X, d) and (Y, l) be compact metric spaces. If $(f_n, f) : T(X) \to T(Y)$ is a morphism in pro-HTop, then $T(E(f_n, f)) = (f_n, f)$.*

Proof. We consider a representative (f'_n, f') of $E(T(f_n))$, where (f'_n, f') is induced by an approximative map F induced by (f_n, f). Given $n \in \mathbb{N}$, we need to verify that for every $l \geq f'(n), f(n)$ the following diagram commutes up to homotopy.



Without loss of generality, we can assume that $f'(n) > f(n)$, $s_n \geq n$ and $f'(n) \geq f(s_n)$. Note that this is due to the construction provided in the proof of Proposition 4.3: s_n can be considered as large as we want (this guarantees $s_n > n$), and γ_n can be considered as small as needed (this guarantees that $f'(n)$ is as large as desired). Then, we need to check the commutativity up to homotopy of the following diagram, where $h = (f_{s_n} \circ p_{f(s_n)})^*$.



By the construction, $h(C) = \bigcup_{c \in C} f_{s_n}(p_{f(s_n)}(c))$ for every $C \in \mathcal{U}_{4\delta_{f'(n)}}(A_{f'(n)})$. By the assumptions made on s_n and γ_n earlier (that give $s_n \geq n$ and $f'(n) \geq f(s_n)$), it is easy to deduce that $q_{f(s_n),f'(n)}$ is homotopic to $p^*_{f(s_n)} \circ i$ because $p_{f(s_n)}(i(c)) \subseteq q_{f(s_n),f'(n)}(c)$ for every $c \in C$. In addition, r_n restricted to the image of $h \circ i$ is homotopic to q_{n,s_n} because $r_n(a) \subseteq q_{n,s_n}(a)$ for every $a \in C$ where $C \in \mathcal{U}_{4\epsilon_{s_n}}(B_{s_n})$. Since (f_n, f) is a morphism in pro-$HTop$, it follows the commutativity up to homotopy of the diagram. □

Theorem 4.7. *Let (X, d) and (Y, l) be compact metric spaces. The set of shape morphisms between X and Y is in bijective correspondence with the set of morphisms in pro-HTop between $T(X)$ and $T(Y)$.*

Proof. We consider the constructions made in Proposition 4.3 and Proposition 4.4. Thus, the result is an immediate consequence of Lemma 4.5 and Lemma 4.6. □

The task is now to prove that T is indeed a functor.

Lemma 4.8. *Let $[\overline{f}] : X \to Y$ and $[\overline{g}] : Y \to Z$ be two approximative maps. Then $T([\overline{g}] \circ [\overline{f}]) = T([\overline{g}]) \circ T([\overline{f}])$.*

Proof. We consider $\{f_n : X \to 2^Y\}_{n\in\mathbb{N}} \in [\overline{f}]$, $\{g_n : Y \to 2^Z\}_{n\in\mathbb{N}} \in [\overline{g}]$ and $\{g_n \circ f_{k_n} : X \to 2^Z\}_{n\in\mathbb{N}} \in [\overline{g}] \circ [\overline{f}]$. As done previously, we will omit some subscripts. With a certain abuse of notation, we will use the same letter to denote different maps. However, the meaning of the notation can be understood from the context. From the proof of Theorem 4.3, we obtain that $(r_n \circ f^*_{s_n} \circ i, f)$ is a morphism in pro-$HTop$ induced by $\{f_n : X \to 2^Y\}_{n\in\mathbb{N}}$, $(r_n \circ g^*_{h_n} \circ i, g)$ is a morphism in pro-$HTop$ induced by $\{g_n : Y \to 2^Z\}_{n\in\mathbb{N}}$ and $(r_n \circ h^*_{l_n} \circ i, h)$ is a morphism in pro-$HTop$ induced by $\{g_n \circ f_{k_n} : X \to 2^Z\}$, where $h_n = g^*_n \circ f^*_{k_n}$, $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$, $T(Y) = (\mathcal{U}_{4\epsilon_n}(B_n), q_{n,n+1})$ and $T(Z) = (\mathcal{U}_{4\psi_n}(C_n), q_{n,n+1})$. It suffices to show that $(r_n \circ g^*_{h_n} \circ i, g) \circ (r_n \circ f^*_{s_n} \circ i, f)$ is homotopic to $(r_n \circ h^*_{l_n} \circ i, h)$, that is, the following diagram commutes up to homotopy for some $m \geq h(n), g(f(n))$.

$$
\begin{array}{ccccccccc}
U_{\rho_n}(X) & \xleftarrow{\ i\ } & \mathcal{U}_{4\delta_{l(n)}}(A_{l(n)}) & \xleftarrow{\ q\ } & \mathcal{U}_{4\delta_m}(A_m) & \xrightarrow{\ q\ } & \mathcal{U}_{4\delta_{f(h(n))}}(A_{f(h(n))}) & \xrightarrow{\ i\ } & U_{\gamma_n}(X) \\
\downarrow{\scriptstyle h^*_{l_n}} & & & & & & & & \downarrow{\scriptstyle f^*_{s_{h_n}}} \\
U_{2\psi_n}(Z) & & & & & & & & U_{2\epsilon_{h_n}}(Y) \\
\downarrow{\scriptstyle r_n} & & & & & & & & \downarrow{\scriptstyle r_{h_n}} \\
\mathcal{U}_{4\psi_n}(C_n) & & \xleftarrow{\ r_n\ } & & U_{2\psi_n}(Z) & \xleftarrow{\ g^*_{h_n}\ } & U_{\tau_n}(Y) & \xleftarrow{\ i\ } & \mathcal{U}_{4\epsilon_{h_n}}(B_{h_n})
\end{array}
$$

Without loss of generality we can assume that $m = max\{l(n), f(g(n))\} = l(n)$ and $\gamma_n \geq \rho_n$. Hence, the commutativity up to homotopy of the following diagram follows trivially.

$$
\begin{array}{ccc}
\mathcal{U}_{4\delta_{f(h(n))}}(A_{f(h(n))}) & \xrightarrow{\ i\ } & U_{\gamma_n}(X) \\
{\scriptstyle q}\uparrow & & {\scriptstyle i}\uparrow \\
\mathcal{U}_{4\delta_{l(n)}}(A_{l(n)}) & \xrightarrow{\ i\ } & U_{\rho_n}(X)
\end{array}
$$

We verify the commutative up to homotopy of the following diagram.

$$
\begin{array}{ccc}
U_{2\epsilon_{h_n}}(Y) & \xrightarrow{\ r_{h_n}\ } & \mathcal{U}_{4\epsilon_{h_n}}(B_{h_n}) \\
 & \searrow^{g^*_{h_n}} & \downarrow{\scriptstyle g^*_{h_n}} \\
 & & U_{2\psi_n}(Z)
\end{array}
$$

For every $C \in U_{2\epsilon_{h_n}}(Y)$, we obtain that $\mathrm{diam}(C \cup r_{h_n}(C)) < 4\epsilon_{h_n}$. Hence, $g^*_{h_n} \cup g^*_{h_n} \circ r_{h_n} : U_{2\epsilon_{h_n}}(Y) \to U_{2\psi_n}(Z)$ given by $g^*_{h_n} \cup g^*_{h_n} \circ r_{h_n}(C) = g^*_{h_n}(C) \cup g^*_{h_n}(r_{h_n}(C))$ is well-defined. By Lemma 4.2, we deduce that the above diagram is commutative up to homotopy.

From the commutativity of the previous diagrams and the properties of approximative maps, it follows that the first diagram commutes up to homotopy. □

Lemma 4.9. *Let* [id] *be the class of the identity morphism* $\{\mathrm{id}_n : X \to 2^X\}$. *Then* $T([\mathrm{id}]) : T(X) \to T(X)$ *is homotopic to the identity morphism in pro-HTop.*

Proof. Set $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$. Consider $(I_n, I) \in T([\mathrm{id}])$ given by Proposition 4.3, that is, $I_n = r_n$ and $I : \mathbb{N} \to \mathbb{N}$ is the identity map. By the definition of r_n it is easily seen that I_n is the identity map. Therefore, it can be deduced that $T([\mathrm{id}])$ is the identity morphism in pro-$HTop$. □

By combining previous results the proof of Theorem 3.1 is straightforward.

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