VARIATIONAL PRINCIPLE FOR MEAN DIMENSION WITH POTENTIAL OF $\mathbb{R}^d$-ACTIONS: I

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Abstract. We develop a variational principle for mean dimension with potential of $\mathbb{R}^d$-actions. We prove that mean dimension with potential is bounded from above by the supremum of the sum of rate distortion dimension and a potential term. A basic strategy of the proof is the same as the case of $\mathbb{Z}$-actions. However measure theoretic details are more involved because $\mathbb{R}^d$ is a continuous group. We also establish several basic properties of metric mean dimension with potential and mean Hausdorff dimension with potential for $\mathbb{R}^d$-actions.

1. Introduction

1.1. **Background: the case of $\mathbb{Z}$-actions.** The purpose of this paper is to develop a theory of variational principle for mean dimension with potential of $\mathbb{R}^d$-actions. First we review the theory already established in the case of $\mathbb{Z}$-actions.

Mean dimension is a topological invariant of dynamical systems introduced by Gromov [Gro99] at the end of the last century. It is the number of parameters *per unit time* for describing given dynamical systems. Mean dimension has several applications to topological dynamics, most notably in the embedding problem of dynamical systems [LW00, Lin99, GT20, GQT19].

Lindenstrauss and the author [LT18, LT19, Tsu20] began to develop the variational principle in mean dimension theory. Let $\mathcal{X}$ be a compact metrizable space, and let $T\colon \mathcal{X} \to \mathcal{X}$ be a homeomorphism of $\mathcal{X}$. The classical variational principle [Goodw69, Din70, Goodm71] states that the topological entropy $h_{\mathrm{top}}(T)$ is equal to the supremum of the Kolmogorov–Sinai entropy $h_\mu(T)$ over all invariant probability measures μ:

$$h_{\mathrm{top}}(T) = \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} h_\mu(T), \tag{1·1}$$

where $\mathscr{M}^T(\mathcal{X})$ denotes the set of all T-invariant Borel probability measures on $\mathcal{X}$. Ruelle [Rue73] and then Walters [Wal75] generalized (1·1) to pressure: Let $\varphi\colon \mathcal{X} \to \mathbb{R}$ be a

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continuous function, and we denote by $P_T(\varphi)$ the topological pressure of (X, T, φ). Then

$$P_T(\varphi) = \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(h_\mu(T) + \int_{\mathcal{X}} \varphi \, d\mu \right). \tag{1·2}$$

In the classical variational principles (1·1) and (1·2), the quantities $h_{\mathrm{top}}(T)$ and $P_T(\varphi)$ in the left-hand sides are topological invariants of dynamical systems. The Kolomogorov–Sinai entropy in the right-hand side is an information theoretic quantity. Therefore (1·1) and (1·2) connect topological dynamics to information theory. Lindenstrauss and the author tried to find an analogous structure in mean dimension theory. (See also the paper of Gutman–Śpiewak [GŚ20] for a connection between mean dimension and information theory.) In the papers [LT18, LT19] they found that *rate distortion theory* provides a fruitful framework for the problem. This is a branch of information theory studying *lossy* data compression method under a distortion constraint.

Let $T : \mathcal{X} \to \mathcal{X}$ be a homeomorphism on a compact metrizable space $\mathcal{X}$ as in the above. We denote the mean dimension of $(\mathcal{X}, T)$ by $\mathrm{mdim}(\mathcal{X}, T)$. We would like to connect it to some information theoretic quantity. We define $\mathscr{D}(\mathcal{X})$ as the set of all metrics (distance functions) on $\mathcal{X}$ compatible with the given topology. Let $\mathbf{d} \in \mathscr{D}(\mathcal{X})$ and $\mu \in \mathcal{M}^T(\mathcal{X})$. We randomly choose a point $x \in \mathcal{X}$ according to the distribution μ and consider the orbit $\{T^n x\}_{n\in\mathbb{Z}}$. For $\varepsilon > 0$, we define the *rate distortion function* $R(\mathbf{d}, \mu, \varepsilon)$ as the minimum number of bits per unit time for describing $\{T^n x\}_{n\in\mathbb{Z}}$ with average distortion bounded by ε with respect to $\mathbf{d}$. See §2.3 for the precise definition of $R(\mathbf{d}, \mu, \varepsilon)$ in the case of $\mathbb{R}^d$-actions.

We define the **upper and lower rate distortion dimensions** by

$$\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) = \limsup_{\varepsilon \to 0} \frac{R(\mathbf{d}, \mu, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) = \liminf_{\varepsilon \to 0} \frac{R(\mathbf{d}, \mu, \varepsilon)}{\log(1/\varepsilon)}.$$

Rate distortion dimension was first introduced by Kawabata–Dembo [KD94].

Lindenstrauss and the author [LT19, Corollary 3.13] proved that

$$\mathrm{mdim}(\mathcal{X}, T) \le \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) \tag{1·3}$$

for any metric $\mathbf{d}$ on $\mathcal{X}$ compatible with the given topology. Moreover they proved that if $(\mathcal{X}, T)$ is a free minimal dynamical system then [LT19, Theorem 1.1]

$$\begin{aligned} \mathrm{mdim}(\mathcal{X}, T) &= \min_{\mathbf{d} \in \mathscr{D}(\mathcal{X})} \left(\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) \right) \\ &= \min_{\mathbf{d} \in \mathscr{D}(\mathcal{X})} \left(\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) \right). \end{aligned} \tag{1·4}$$

They called this "double variational principle" because it involves a minimax problem with respect to the *two* variables $\mathbf{d}$ and μ. We conjecture that (1·4) holds for all dynamical systems without any additional assumption.

The author [Tsu20] generalized (1·3) and (1·4) to *mean dimension with potential*, which is a mean dimension analogue of topological pressure. Let $\varphi\colon \mathcal{X} \to \mathbb{R}$ be a continuous function. The paper [Tsu20] introduced mean dimension with potential (denoted by $\mathrm{mdim}(\mathcal{X}, T, \varphi)$) and proved that [Tsu20, Corollary 1.7]

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) \leq \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi\, d\mu \right). \tag{1·5}$$

Moreover, if $(\mathcal{X}, T)$ is a free minimal dynamical system then [Tsu20, Theorem 1.1]

$$\begin{aligned} \mathrm{mdim}(\mathcal{X}, T, \varphi) &= \min_{\mathbf{d} \in \mathscr{D}(\mathcal{X})} \left\{ \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi\, d\mu \right) \right\} \\ &= \min_{\mathbf{d} \in \mathscr{D}(\mathcal{X})} \left\{ \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi\, d\mu \right) \right\}. \end{aligned} \tag{1·6}$$

We also conjecture that this holds for all dynamical systems.

The main purpose of this paper is to generalize the above (1·5) to $\mathbb{R}^d$-actions. We think that we can also generalize the *double variational principle* (1·6) to free minimal $\mathbb{R}^d$-actions. However it requires a technically heavy work. We postpone it to Part II of this series of papers. In this paper we concentrate on the inequality (1·5).

The motivation to generalize (1·5) and (1·6) to $\mathbb{R}^d$-actions comes from the fact that many natural examples of mean dimension theory are rooted in geometric analysis [Gro99, MT15, Tsu18]. In geometric analysis we usually consider actions of groups more complicated than $\mathbb{Z}$. Maybe $\mathbb{R}^d$-actions are the most basic case. We plan to apply the results of this paper to geometric examples of [Gro99, MT15, Tsu18] in a future paper.

Since $\mathbb{R}^d$ is a continuous group, several new technical difficulties appear. Especially measure theoretic details are more complicated in the case of $\mathbb{R}^d$-actions than in the case of $\mathbb{Z}$-actions. A main task of this paper is to establish such details.

We would like to mention the paper of Huo–Yuan [HY]. They develop the variational principle for mean dimension of $\mathbb{Z}^d$-actions. In §4 and §5 we also touch the case of $\mathbb{Z}^d$-actions. Some results in these sections were already studied in [HY].

1.2. **Mean dimension with potential of $\mathbb{R}^d$-actions.** In this subsection we introduce mean dimension with potential for $\mathbb{R}^d$-actions. Let P be a finite simplicial complex. (Here "finite" means that the number of faces is finite. In this paper we do not consider infinite simplicial complexes. Simplicial complexes are always finite.) For a point $a \in P$ we define $\dim_a P$ as the maximum of $\dim \Delta$ where Δ runs over all simplices of P containing a. We call $\dim_a P$ the **local dimension** of P at a. See Figure 1. (This is the same as [Tsu20, Fig. 1].)

Let $(\mathcal{X}, \mathbf{d})$ be a compact metric space. Let $\mathcal{Y}$ be a topological space and $f\colon \mathcal{X} \to \mathcal{Y}$ a continuous map. For a positive number ε we call f an **ε-embedding** if we have

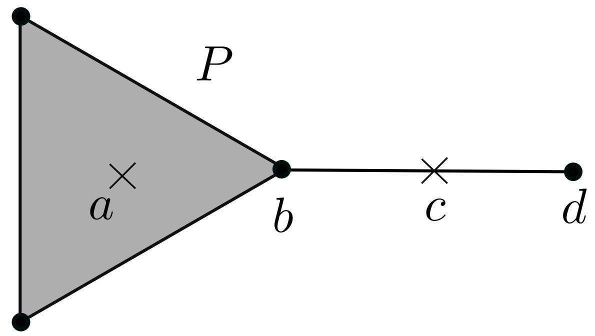

FIGURE 1. Here P has four vertexes (denoted by dots), four 1-dimensional simplexes and one 2-dimensional simplex. The points b and d are vertexes of P whereas a and c are not. We have $\dim_a P = \dim_b P = 2$ and $\dim_c P = \dim_d P = 1$.

$\mathrm{Diam} f^{-1}(y) < \varepsilon$ for all $y \in \mathcal{Y}$. Let $\varphi : \mathcal{X} \to \mathbb{R}$ be a continuous function. We define the **ε-width dimension with potential** by

$$\begin{aligned} &\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) \\ &= \inf \left\{ \max_{x\in\mathcal{X}} \left(\dim_{f(x)} P + \varphi(x)\right) \middle| \begin{array}{c} P \text{ is a finite simplicial complex and} \\ f : \mathcal{X} \to P \text{ is an } \varepsilon\text{-embedding} \end{array} \right\}. \end{aligned} \tag{1·7}$$

Let d be a natural number. We consider that $\mathbb{R}^d$ is equipped with the Euclidean topology and standard additive group structure. We denote the standard Lebesgue measure on $\mathbb{R}^d$ by $\mathbf{m}$. Let $T: \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ compatible with the topology, and let $\varphi: \mathcal{X} \to \mathbb{R}$ be a continuous function. For a bounded Borel subset $A \subset \mathbb{R}^d$ we define a new metric $\mathbf{d}_A$ and a new function $\varphi_A: \mathcal{X} \to \mathbb{R}$ by

$$\mathbf{d}_A(x,y) = \sup_{u\in A} \mathbf{d}(T^u x, T^u y), \quad \varphi_A(x) = \int_A \varphi(T^u x)\, d\mathbf{m}(u).$$

If $\varphi(x) \geq 0$ for all $x \in \mathcal{X}$ then we have:

(1) **Subadditivity:** For bounded Borel subsets $A, B \subset \mathbb{R}^d$

$$\mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_{A\cup B}, \varphi_{A\cup B}\right) \leq \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_A, \varphi_A\right) + \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_B, \varphi_B\right).$$

(2) **Monotonicity:** If $A \subset B$ then

$$0 \leq \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_A, \varphi_A\right) \leq \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_B, \varphi_B\right).$$

(3) **Invariance:** For $a \in \mathbb{R}^d$ and a bounded Borel subset $A \subset \mathbb{R}^d$

$$\mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_{a+A}, \varphi_{a+A}\right) = \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_A, \varphi_A\right),$$

where $a + A = \{a + u \mid u \in A\}$.

Notice that we need to assume the nonnegativity of φ for the properties (1) and (2).

For a positive number L we denote $\mathbf{d}_{[0,L)^d}$ and $\varphi_{[0,L)^d}$ by $\mathbf{d}_L$ and φ_L respectively for simplicity. We define the **mean dimension with potential** of $(\mathcal{X}, T, \varphi)$ by

$$\mathrm{mdim}\,(\mathcal{X}, T, \varphi) = \lim_{\varepsilon\to 0}\left(\lim_{L\to\infty}\frac{\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_L, \varphi_L)}{L^d}\right). \tag{1·8}$$

This is a topological invariant, namely its value is independent of the choice of the metric $\mathbf{d}$. Notice that we do not assume the nonnegativity of φ in the definition (1·8).

We need to check that the limits in the definition (1·8) exist. The limit with respect to ε exists because $\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_L, \varphi_L)$ is monotone in ε. We prove the existence of the limit with respect to L in the next lemma.

Lemma 1.1. *The limit* $\lim_{L\to\infty}\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_L, \varphi_L)/L^d$ *exists in the definition (1·8).*

Proof. Let c be the minimum of $\varphi(x)$ over $x \in \mathcal{X}$ and set $\psi(x) = \varphi(x) - c$. Then ψ is a nonnegative function with

$$\mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_A, \psi_A\right) = \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_A, \varphi_A\right) - c\,\mathbf{m}(A).$$

Set $h(A) = \mathrm{Widim}_\varepsilon\left(\mathcal{X}, \mathbf{d}_A, \psi_A\right)$. It is enough to prove that the limit $\lim_{L\to\infty} h\left([0,L)^d\right)/L^d$ exists. For $0 < L < R$, let $n = \lfloor R/L \rfloor$ be the integer part of R/L. We have

$$[0,R)^d \subset \bigcup_{u\in\mathbb{Z}^d\cap[0,n]^d}\left(Lu + [0,L)^d\right).$$

Since ψ is nonnegative, $h(A)$ satisfies the subadditivity, monotonicity and invariance. Hence

$$h\left([0,R)^d\right) \le (n+1)^d \cdot h\left([0,L)^d\right).$$

Dividing this by R^d and letting $R \to \infty$, we get

$$\limsup_{R\to\infty}\frac{h\left([0,R)^d\right)}{R^d} \le \frac{h\left([0,L)^d\right)}{L^d}.$$

Then letting $L \to \infty$ we get

$$\limsup_{R\to\infty}\frac{h\left([0,R)^d\right)}{R^d} \le \liminf_{L\to\infty}\frac{h\left([0,L)^d\right)}{L^d}.$$

Therefore the limit $\lim_{L\to\infty} h\left([0,L)^d\right)/L^d$ exists. □

Remark 1.2. By the Ornstein–Weiss quasi-tiling argument ([OS87], [Gro99, §1.3.1]) we can also prove that for any Følner sequence $A_1, A_2, A_3, \dots$ of $\mathbb{R}^d$ the limit

$$\lim_{n\to\infty}\frac{\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_{A_n}, \varphi_{A_n})}{\mathbf{m}(A_n)}$$

exists and that its value is independent of the choice of a Følner sequence. In particular, we can define the mean dimension with potential by

$$\mathrm{mdim}\,(\mathcal{X}, T, \varphi) = \lim_{\varepsilon\to 0}\left(\lim_{R\to\infty}\frac{\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_{B_R}, \varphi_{B_R})}{\mathbf{m}(B_R)}\right),$$

where $B_R = \{u \in \mathbb{R}^d \mid |u| \le R\}$.

1.3. **Main result.** Let $\mathcal{X}$ be a compact metrizable space. Recall that we have denoted by $\mathscr{D}(\mathcal{X})$ the set of metrics $\mathbf{d}$ on $\mathcal{X}$ compatible with the given topology. Let $T\colon \mathbb{R}^d\times\mathcal{X}\to\mathcal{X}$ be a continuous action. A Borel probability measure μ on $\mathcal{X}$ is said to be T**-invariant** if $\mu(T^{-u}A) = \mu(A)$ for all $u\in\mathbb{R}^d$ and all Borel subsets $A\subset\mathcal{X}$. We define $\mathscr{M}^T(\mathcal{X})$ as the set of all T-invariant Borel probability measures μ on $\mathcal{X}$.

Take a metric $\mathbf{d}\in\mathscr{D}(\mathcal{X})$ and a measure $\mu\in\mathscr{M}^T(\mathcal{X})$. We randomly choose a point $x\in\mathcal{X}$ according to the distribution μ and consider the orbit $\{T^u x\}_{u\in\mathbb{R}^d}$. For a positive number ε we define the rate distortion function $R(\mathbf{d},\mu,\varepsilon)$ as the minimum bits per unit volume for describing $\{T^u x\}_{u\in\mathbb{R}^d}$ with average distortion bounded by ε with respect to $\mathbf{d}$. The precise definition of $R(\mathbf{d},\mu,\varepsilon)$ is given in §2.3.

We define the **upper and lower rate distortion dimensions** by

$$\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) = \limsup_{\varepsilon\to 0}\frac{R(\mathbf{d},\mu,\varepsilon)}{\log(1/\varepsilon)},\quad \underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) = \liminf_{\varepsilon\to 0}\frac{R(\mathbf{d},\mu,\varepsilon)}{\log(1/\varepsilon)}.$$

The following is the main result of this paper.

Theorem 1.3 (Main theorem)**.** *Let $T\colon\mathbb{R}^d\times\mathcal{X}\to\mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\varphi\colon\mathcal{X}\to\mathbb{R}$ be a continuous function. Then for any metric $\mathbf{d}\in\mathscr{D}(\mathcal{X})$*

$$\mathrm{mdim}\,(\mathcal{X}, T, \varphi) \le \sup_{\mu\in\mathscr{M}^T(\mathcal{X})}\left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}}\varphi\, d\mu\right).$$

We propose a conjecture:

Conjecture 1.4. *In the setting of Theorem 1.3 we have*

$$\begin{aligned}\mathrm{mdim}(\mathcal{X}, T, \varphi) &= \min_{\mathbf{d}\in\mathscr{D}(\mathcal{X})}\left\{\sup_{\mu\in\mathscr{M}^T(\mathcal{X})}\left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}}\varphi\, d\mu\right)\right\}\\ &= \min_{\mathbf{d}\in\mathscr{D}(\mathcal{X})}\left\{\sup_{\mu\in\mathscr{M}^T(\mathcal{X})}\left(\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}}\varphi\, d\mu\right)\right\}.\end{aligned}$$

We think that probably we can prove this conjecture if $T\colon\mathbb{R}^d\times\mathcal{X}\to\mathcal{X}$ is a free minimal action. The proof will be rather lengthy and technically heavy. We postpone it to Part II of this series of papers.

Along the way to prove Theorem 1.3, we will introduce *mean Hausdorff dimension with potential* and *metric mean dimension with potential* for $\mathbb{R}^d$-actions and establish their basic properties. In particular we prove:

(1) Mean Hausdorff dimension with potential bounds $\mathrm{mdim}\,(\mathcal{X}, T, \varphi)$ from above (Theorem 3.4).
(2) We can construct invariant probability measures which capture the complexity of dynamics expressed by mean Hausdorff dimension with potential (*Dynamical Frostman's lemma*; Theorem 3.7).
(3) Metric mean dimension with potential bounds $\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \displaystyle\int_{\mathcal{X}} \varphi\, d\mu$ from above (Proposition 3.5).
(4) Metric mean dimension with potential can be calculated by using only "local" information (Theorem 7.1).

The results (1) and (2) will be used in the proof of Theorem 1.3. The results (3) and (4) are not used in the proof of Theorem 1.3. We plan to use (3) in Part II of this series of papers. The result (4) may be useful when we study geometric examples of [Gro99, MT15, Tsu18] in a future.

1.4. **Organization of the paper.** In §2 we prepare basic definitions and results on mutual information and rate distortion theory. In §3 we introduce mean Hausdorff dimension with potential and metric mean dimension with potential for $\mathbb{R}^d$-actions. We also state their fundamental properties in §3. The proofs will be given in §5 and §6. Theorem 1.3 (Main Theorem) follows from the properties of mean Hausdorff dimension with potential stated in §3. In §4 we prepare some basic results on mean dimension theory of $\mathbb{Z}^d$-actions. They will be used in §5. In §5 we prove that $\mathrm{mdim}(\mathcal{X}, T, \varphi)$ is bounded from above by mean Hausdorff dimension with potential. In §6 we prove dynamical Frostman's lemma. In §7 we prove that metric mean dimension with potential can be calculated by using certain local information. §7 is independent of the proof of Theorem 1.3.

2. Mutual information and rate distortion theory

We prepare basics of rate distortion theory in this section. Throughout this paper $\log x$ denotes the logarithm of base two. The natural logarithm is denoted by $\ln x$:

$$\log x = \log_2 x, \quad \ln x = \log_e x.$$

This section is rather long. This is partly because we have to be careful of measure theoretic details. Hopefully this section will become a useful reference in a future study of mean dimension of $\mathbb{R}^d$-actions. At the first reading, readers may skip the whole of Subsection 2.1 and most of Subsection 2.2. The crucial parts of this section are only the definition of mutual information in §2.2 and the definition of rate distortion function in §2.3. All the rest of this section are technical details.

2.1. **Measure theoretic preparations.** We need to prepare some basic results on measure theory. A **measurable space** is a pair $(\mathcal{X}, \mathcal{A})$ of a set $\mathcal{X}$ and its σ-algebra $\mathcal{A}$. Two measurable spaces $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ are said to be **isomorphic** if there exists a bijection $f\colon \mathcal{X} \to \mathcal{Y}$ such that both f and f^{-1} are measurable (i.e. $f(\mathcal{A}) = \mathcal{B}$).

For a topological space $\mathcal{X}$, its **Borel σ-algebra** $\mathcal{B}_\mathcal{X}$ is the minimum σ-algebra containing all open subsets of $\mathcal{X}$. A **Polish space** is a topological space $\mathcal{X}$ admitting a metric $\mathbf{d}$ for which $(\mathcal{X}, \mathbf{d})$ is a complete separable metric space.

A measurable space $(\mathcal{X}, \mathcal{A})$ is said to be a **standard Borel space** if there exists a Polish space $\mathcal{Y}$ for which $(\mathcal{X}, \mathcal{A})$ is isomorphic to $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ as measurable spaces. It is known that any two uncountable standard Borel spaces are isomorphic to each other (the Borel isomorphism theorem [Sri98, Theorem 3.3.13]). Therefore every standard Borel space is isomorphic to one of the following measurable spaces:

- A finite set A with its discrete σ-algebra $2^A := \{\text{subset of } A\}$.
- The set of natural numbers $\mathbb{N}$ with its discrete σ-algebra $2^\mathbb{N} := \{\text{subset of } \mathbb{N}\}$.
- The Cantor set $\mathcal{C} = \{0,1\}^\mathbb{N}$ with its Borel σ-algebra $\mathcal{B}_\mathcal{C}$. (Here $\{0,1\}$ is endowed with the discrete topology and the topology of $\mathcal{C}$ is the product topology.)

An importance of standard Borel spaces is that we can prove the existence of *regular conditional distribution* under the assumption of "standard Borel". Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be measurable spaces. A **transition probability** on $\mathcal{X} \times \mathcal{Y}$ is a map $\nu : \mathcal{X} \times \mathcal{B} \to [0,1]$ such that

- for every $x \in \mathcal{X}$, the map $\mathcal{B} \ni B \mapsto \nu(x, B) \in [0,1]$ is a probability measure on $(\mathcal{Y}, \mathcal{B})$,
- for every $B \in \mathcal{B}$, the map $\mathcal{X} \ni x \mapsto \nu(x, B) \in [0,1]$ is measurable.

We often denote $\nu(x, B)$ by $\nu(B|x)$.

For two measurable spaces $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ we denote their product by $(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B})$ where $\mathcal{A} \otimes \mathcal{B}$ is the minimum σ-algebra containing all the rectangles $A \times B$ ($A \in \mathcal{A}, B \in \mathcal{B}$). For any $E \in \mathcal{A} \otimes \mathcal{B}$, it is known that the **section** $E_x := \{y \in \mathcal{Y} \mid (x, y) \in E\}$ belongs to $\mathcal{B}$ for every $x \in \mathcal{X}$. (This fact is a part of the Fubini theorem. It can be easily proved by using Dynkin's π-λ theorem [Dur10, p.402 Theorem A.1.4].) Moreover, if $(\mathcal{Y}, \mathcal{B})$ is a standard Borel space, then for any transition probability ν on $\mathcal{X} \times \mathcal{Y}$ and any $E \in \mathcal{A} \otimes \mathcal{B}$ the map $\mathcal{X} \ni x \mapsto \nu(E_x|x) \in [0,1]$ is measurable [Sri98, Proposition 3.4.24].

A **probability space** is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where $(\Omega, \mathcal{F})$ is a measurable space and $\mathbb{P}$ is a probability measure defined on it. Let $X\colon \Omega \to \mathcal{X}$ be a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a Borel space $(\mathcal{X}, \mathcal{A})$. We denote the push-forward measure $X_*\mathbb{P}$ by $\mathrm{Law}X$ and call it the **law of** X or the **distribution of** X. (Here $X_*\mathbb{P}(A) = \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A))$ for $A \in \mathcal{A}$.)

The next theorem is a fundamental result. It guarantees the existence of regular conditional probability. For the proof, see [IW81, p.15 Theorem 3.3 and its Corollary] or Gray [Gra09, p. 182 Corollary 6.2].

Theorem 2.1 (Existence of regular conditional distribution)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ standard Borel spaces. Let $X : \Omega \to \mathcal{X}$ and $Y : \Omega \to \mathcal{Y}$ be measurable maps, and set $\mu := \mathrm{Law} X$. Then there exists a transition probability ν on $\mathcal{X} \times \mathcal{Y}$ such that for any $E \in \mathcal{A} \otimes \mathcal{B}$ we have*

$$\mathbb{P}\left((X, Y) \in E\right) = \int_{\mathcal{X}} \nu(E_x|x)\, d\mu(x).$$

If a transition probability ν' on $\mathcal{X} \times \mathcal{Y}$ satisfies the same property then there exists a μ-null set $N \in \mathcal{A}$ such that $\nu(B|x) = \nu'(B|x)$ for all $x \in \mathcal{X} \setminus N$ and $B \in \mathcal{B}$.

The transition probability $\nu(\cdot|x)$ in this theorem is called the **regular conditional distribution of Y given $X = x$**. We sometimes denote $\nu(B|x)$ by $\mathbb{P}(Y \in B|X = x)$ for $x \in \mathcal{X}$ and $B \in \mathcal{B}$. If $\mathcal{X}$ and $\mathcal{Y}$ are finite sets, then this coincides with the elementary notion of conditional probability:

$$\mathbb{P}(Y \in B|X = x) = \frac{\mathbb{P}(X = x, Y \in B)}{\mathbb{P}(X = x)} \quad (\text{if } \mathbb{P}(X = x) \neq 0).$$

In this case we usually denote $\nu\left(\{y\}|x\right)$ by $\nu(y|x)$ $(x \in \mathcal{X}, y \in \mathcal{Y})$ and call it a **conditional probability mass function**[1].

By using the notion of regular conditional distribution, we can introduce the definition of *conditional independence* of random variables. Let $(\Omega, \mathbb{P})$ be a probability space and $(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B}), (\mathcal{Z}, \mathcal{C})$ standard Borel spaces. Let $X \colon \Omega \to \mathcal{X}$, $Y \colon \Omega \to \mathcal{Y}$ and $Z \colon \Omega \to \mathcal{Z}$ be measurable maps. We say that X and Y are **conditionally independent given Z** if we have

$$\mathbb{P}\left((X, Y) \in A \times B|Z = z\right) = \mathbb{P}\left(X \in A|Z = z\right) \cdot \mathbb{P}\left(Y \in B|Z = z\right) \tag{2·1}$$

for $Z_*\mathbb{P}$-a.e. $z \in \mathcal{Z}$ and all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Here $Z_*\mathbb{P}$ is the push-forward measure of $\mathbb{P}$ by Z. The left-hand side of (2·1) is the conditional regular distribution of $(X, Y) \colon \Omega \to \mathcal{X} \times \mathcal{Y}$ given $Z = z$. The right-hand side is the multiple of the conditional distribution of X given $Z = z$ and the conditional distribution of Y given $Z = z$.

At the end of this subsection we explain the log-sum inequality. This will be used in the next subsection.

Lemma 2.2 (Log-sum inequality)**.** *Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. Let μ be a measure on it with $0 < \mu(\mathcal{X}) < \infty$. Let f and g be nonnegative measurable functions defined on $\mathcal{X}$. Suppose that g is μ-integrable and $g(x) > 0$ for μ-a.e. $x \in \mathcal{X}$. Then*

$$\left(\int_{\mathcal{X}} f(x)\, d\mu(x)\right) \log \frac{\int_{\mathcal{X}} f(x)\, d\mu(x)}{\int_{\mathcal{X}} g(x)\, d\mu(x)} \leq \int_{\mathcal{X}} f(x) \log \frac{f(x)}{g(x)}\, d\mu(x).$$

In particular, if the left-hand side is infinite then the right-hand side is also infinite.

[1] For convenience in the sequel, we define this notion more precisely. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets. A map $\mathcal{X} \times \mathcal{Y} \ni (x, y) \mapsto \nu(y|x) \in [0, 1]$ is called a conditional probability mass function if $\sum_{y \in \mathcal{Y}} \nu(y|x) = 1$ for every $x \in \mathcal{X}$.

Here we assume $0 \log \frac{0}{a} = 0$ for all $a > 0$.

Proof. Set $\phi(t) = t \log t$ for $t \geq 0$. Since $\phi''(t) = \log e/t > 0$ for $t > 0$, this is a convex function. We define a probability measure w on $\mathcal{X}$ by

$$w(A) = \frac{\int_A g\, d\mu}{\int_{\mathcal{X}} g\, d\mu} \quad (A \subset \mathcal{X}).$$

The Radon–Nikodim derivative of w by μ is given by

$$\frac{dw}{d\mu} = \frac{g}{\int_{\mathcal{X}} g\, d\mu}.$$

By Jensen's inequality

$$\phi\left(\int_{\mathcal{X}} \frac{f}{g}\, dw\right) \leq \int_{\mathcal{X}} \phi\left(\frac{f}{g}\right) dw. \tag{2·2}$$

Here, if the left-hand side is infinite, then the right-hand side is also infinite. We have

$$\phi\left(\int_{\mathcal{X}} \frac{f}{g}\, dw\right) = \phi\left(\frac{\int_{\mathcal{X}} f\, d\mu}{\int_{\mathcal{X}} g\, d\mu}\right) = \frac{\int_{\mathcal{X}} f\, d\mu}{\int_{\mathcal{X}} g\, d\mu} \log \frac{\int_{\mathcal{X}} f\, d\mu}{\int_{\mathcal{X}} g\, d\mu}.$$

The right-hand side of (2·2) is

$$\int_{\mathcal{X}} \phi\left(\frac{f}{g}\right) dw = \frac{1}{\int_{\mathcal{X}} g\, d\mu} \int_{\mathcal{X}} g\phi\left(\frac{f}{g}\right) d\mu = \frac{1}{\int_{\mathcal{X}} g\, d\mu} \int_{\mathcal{X}} f \log \frac{f}{g}\, d\mu.$$

Therefore (2·2) provides

$$\left(\int_{\mathcal{X}} f\, d\mu\right) \log \frac{\int_{\mathcal{X}} f\, d\mu}{\int_{\mathcal{X}} g\, d\mu} \leq \int_{\mathcal{X}} f \log \frac{f}{g}\, d\mu.$$

□

The following is the finitary version of the log-sum inequality:

Corollary 2.3. *Let $a_1, \dots, a_n$ be nonnegative numbers and $b_1, \dots, b_n$ positive numbers. Then*

$$\left(\sum_{i=1}^n a_i\right) \log \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \leq \sum_{i=1}^n a_i \log \frac{a_i}{b_i}.$$

Proof. Apply Lemma 2.2 to the finite set $\mathcal{X} = \{1, 2, \dots, n\}$ with the discrete σ-algebra and the counting measure. □

2.2. **Mutual information.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We assume that all random variables in this subsection are defined on $(\Omega, \mathcal{F}, \mathbb{P})$. In this paper a finite set is always assumed to be endowed with the discrete topology and the discrete σ-algebra (i.e. the set of all subsets). The purpose of this subsection is to define and study mutual information. A basic reference of mutual information is the book of Cover–Thomas [CT06]. A mathematically sophisticated presentation is given in the book of Gray [Gra11].

First we define the Shannon entropy. Let $(\mathcal{X}, \mathcal{A})$ be a finite set with the discrete σ-algebra, and let $X\colon \Omega \to \mathcal{X}$ be a measurable map. We define the **Shannon entropy of** X by

$$H(X) = -\sum_{x \in \mathcal{X}} \mathbb{P}(X = x) \log \mathbb{P}(X = x).$$

Here we assume $0 \log 0 = 0$ as usual.

Next we define the mutual information. Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be two measurable spaces and let $X\colon \Omega \to \mathcal{X}$ and $Y\colon \Omega \to \mathcal{Y}$ be measurable maps. We want to define the mutual information $I(X;Y)$. Intuitively $I(X;Y)$ measure the amount of information shared by the random variables X and Y.

- **Case I:** Suppose $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ are finite sets with the discrete σ-algebras. Then we define

$$I(X;Y) = H(X) + H(Y) - H(X,Y), \tag{2·3}$$

  where $H(X,Y)$ is the Shannon entropy of the measurable map $(X,Y) : \Omega \to \mathcal{X} \times \mathcal{Y}$. Since $H(X,Y) \le H(X) + H(Y)$, the mutual information $I(X;Y)$ is always nonnegative. The explicit formula is given by

$$I(X;Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \mathbb{P}(X = x, Y = y) \log \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)\mathbb{P}(Y = y)}.$$

  Here we assume $0 \log \frac{0}{a} = 0$ for any $a \ge 0$. The mutual information $I(X;Y)$ satisfies the following natural monotonicity[2]: Let $\mathcal{X}'$ and $\mathcal{Y}'$ be finite sets (endowed with the discrete σ-algebras), and let $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$ be any maps. Then it follows from the log-sum inequality (Corollary 2.3) that

$$I\left(f(X); g(Y)\right) \le I(X;Y). \tag{2·4}$$

- **Case II:** Here we define $I(X;Y)$ for general random variables X and Y. (Namely $\mathcal{X}$ and $\mathcal{Y}$ may be infinite sets.) Let $\mathcal{X}'$ and $\mathcal{Y}'$ be arbitrary finite sets, and let $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$ be any measurable maps. Then we can consider $I\left(f(X); g(Y)\right)$ by Case I. We define $I(X;Y)$ as the supremum of $I\left(f(X); g(Y)\right)$ over all finite sets $\mathcal{X}', \mathcal{Y}'$ and measurable maps $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$. The mutual information $I(X,Y)$ is always nonnegative and symmetric: $I(X;Y) = I(Y;X) \ge 0$.

  If $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$ are measurable maps to some other measurable spaces $(\mathcal{X}', \mathcal{A}')$ and $(\mathcal{Y}', \mathcal{B}')$ (not necessarily finite sets) then we have $I\left(f(X); g(Y)\right) \le I\left(X; Y\right)$.

  If $\mathcal{X}$ and $\mathcal{Y}$ are finite sets, then the definition of Case II is compatible with Case I by the monotonicity (2·4).

[2]This is a special case of the data-processing inequality [CT06, Theorem 2.8.1].

If $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ are standard Borel spaces, then we can consider the regular conditional distribution of Y given $X = x$. We denote

$$\nu(B|x) = \mathbb{P}(Y \in B|X = x) \quad (x \in \mathcal{X}, B \in \mathcal{B}).$$

Let $\mu = X_*\mathbb{P}$ be the push-forward measure of $\mathbb{P}$ by X. The distribution of (X, Y) is determined by μ and ν. Hence the mutual information $I(X;Y)$ is also determined by μ and ν. Therefore we sometimes denote $I(X;Y)$ by $I(\mu, \nu)$. An importance of this description comes from the fact that $I(\mu, \nu)$ is a concave function in μ and a convex function in ν (Proposition 2.10 below).

In the rest of this subsection we prepare several basic properties of mutual information. They are rather heavy. Readers may skip to the next subsection at the first reading.

If $(\mathcal{X}, \mathcal{A})$ is a standard Borel space and *if $\mathcal{Y}$ is a finite set*, then we can express $I(X;Y)$ in another convenient way. For $x \in \mathcal{X}$ we set

$$H(Y|X = x) = -\sum_{y \in \mathcal{Y}} \mathbb{P}(Y = y|X = x) \log \mathbb{P}(Y = y|X = x).$$

We define the **conditional entropy of Y given X** by

$$H(Y|X) = \int_{\mathcal{X}} H(Y|X = x)\, d\mu(x) \quad (\mu := X_*\mathbb{P}).$$

The next theorem is given in the book of Gray [Gra11, p. 213, Lemma 7.20].

Theorem 2.4. *Let X and Y be random variables taking values in a standard Borel space $(\mathcal{X}, \mathcal{A})$ and a finite set $\mathcal{Y}$ respectively. Then we have*

$$I(X;Y) = H(Y) - H(Y|X).$$

When both $\mathcal{X}$ and $\mathcal{Y}$ are finite sets, this theorem is a very well-known result. A point of the theorem is that we do not need to assume that $\mathcal{X}$ is a finite set.

The following is also a basic result. This is given in [Gra11, p. 211, Lemma 7.18].

Theorem 2.5. *Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be standard Borel spaces. Then there exist sequences of measurable maps $f_n \colon \mathcal{X} \to \mathcal{X}_n$ and $g_n \colon \mathcal{Y} \to \mathcal{Y}_n$ to some finite sets $\mathcal{X}_n$ and $\mathcal{Y}_n$ $(n \geq 1)$ for which the following statement holds: If X and Y are random variables taking values in $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ respectively, then*

$$I(X;Y) = \lim_{n \to \infty} I(f_n(X); g_n(Y)).$$

Sketch of the proof. Standard Borel spaces are isomorphic to either countable sets or the Cantor sets. The case of countable sets is easier. So we assume that both $\mathcal{X}$ and $\mathcal{Y}$ are the Cantor set $\{0,1\}^{\mathbb{N}}$. Let $f_n \colon \mathcal{X} \to \{0,1\}^n$ and $g_n \colon \mathcal{Y} \to \{0,1\}^n$ be the natural projections to the first n coordinates. Then we can check that f_n and g_n satisfy the statement. $\square$

Lemma 2.6. *Let X_n and Y_n $(n \geq 1)$ be sequences of random variables taking values in finite sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Suppose (X_n, Y_n) converges to (X, Y) in law. (Namely $\mathbb{P}(X_n = x, Y_n = y) \to \mathbb{P}(X = x, Y = y)$ as $n \to \infty$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.) Then $I(X_n; Y_n)$ converges to $I(X; Y)$ as $n \to \infty$.*

Proof. This immediately follows from the definition (2·3) in Case I above. $\square$

Lemma 2.7 (Subadditivity of mutual information)**.** *Let X, Y, Z be random variables taking values in standard Borel spaces $(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B}), (\mathcal{Z}, \mathcal{C})$ respectively. Suppose that X and Y are conditionally independent given Z. Then*

$$I(X, Y; Z) \leq I(X; Z) + I(Y; Z),$$

where $I(X, Y; Z) = I((X, Y); Z)$ is the mutual information between the random variables (X, Y) and Z.

Proof. Let $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$ be measurable maps to some finite sets $\mathcal{X}'$ and $\mathcal{Y}'$. Then by Theorem 2.4

$$I(f(X), g(Y); Z) = H(f(X), g(Y)) - H(f(X), g(Y)|Z).$$

We have [CT06, Theorem 2.6.6]

$$H(f(X), g(Y)) \leq H(f(X)) + H(g(Y)).$$

The random variables $f(X)$ and $g(Y)$ are conditionally independent given Z. Hence

$$H(f(X), g(Y)|Z) = H(f(X)|Z) + H(g(Y)|Z).$$

Therefore

$$\begin{aligned} I(f(X), g(Y); Z) &\leq \{H(f(X)) - H(f(X)|Z)\} + \{H(g(Y)) - H(g(Y)|Z)\} \\ &= I(f(X); Z) + I(g(Y); Z). \end{aligned}$$

We have $I(X, Y; Z) = \sup_{f,g} I(f(X), g(Y); Z)$ where $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$ run over all measurable maps to some finite sets. This follows from the fact that $\mathcal{A} \otimes \mathcal{B}$ is generated by rectangles $A \times B$ $(A \in \mathcal{A}, B \in \mathcal{B})$ [Gra11, p.175 Lemma 7.3]. Therefore we get

$$I(X, Y; Z) \leq I(X; Z) + I(Y; Z).$$

$\square$

As we briefly mentioned above, the mutual information $I(\mu, \nu)$ is a concave function in a probability measure μ and a convex function in a transition probability ν. Next we are going to establish this fact. We need some preparations.

For a finite set $\mathcal{Y}$, a **probability mass function** p on $\mathcal{Y}$ is a nonnegative function on $\mathcal{Y}$ satisfying $\sum_{y \in \mathcal{Y}} p(y) = 1$. For a probability mass function p on $\mathcal{Y}$ we define

$$H(p) = -\sum_{y \in \mathcal{Y}} p(y) \log p(y).$$

Lemma 2.8 (Concavity of the Shannon entropy)**.** *Let $\mathcal{Y}$ be a finite set and let $(\mathcal{Z}, \mathcal{C}, m)$ be a probability space. Suppose that we are given a probability mass function p_z on $\mathcal{Y}$ for each $z \in \mathcal{Z}$ and that the map $\mathcal{Z} \ni z \mapsto p_z(y) \in [0,1]$ is measurable for each $y \in \mathcal{Y}$. We define a probability mass function p on $\mathcal{Y}$ by*

$$p(y) = \int_{\mathcal{Z}} p_z(y)\, dm(z).$$

Then

$$H(p) \geq \int_{\mathcal{Z}} H(p_z)\, dm(z).$$

Proof. From the log-sum inequality (Lemma 2.2),

$$-p(y) \log p(y) \geq -\int_{\mathcal{Z}} p_z(y) \log p_z(y)\, dm(z).$$

Summing this over $y \in \mathcal{Y}$, we get the statement. □

Lemma 2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets and $(\mathcal{Z}, \mathcal{C}, m)$ a probability space. Let μ be a probability mass function on $\mathcal{X}$. Suppose that, for each $z \in \mathcal{Z}$, we are given a conditional probability mass function $\nu_z(y|x)$ in $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that the map $\mathcal{Z} \ni z \mapsto \nu_z(y|x) \in [0,1]$ is measurable for each $(x,y) \in \mathcal{X} \times \mathcal{Y}$. We define*

$$\nu(y|x) = \int_{\mathcal{Z}} \nu_z(y|x)\, dm(z) \quad (x \in \mathcal{X}, y \in \mathcal{Y}).$$

Then

$$I(\mu, \nu) \leq \int_{\mathcal{Z}} I(\mu, \nu_z)\, dm(z).$$

Proof. For $y \in \mathcal{Y}$ we set

$$p_z(y) = \sum_{x \in \mathcal{X}} \mu(x) \nu_z(y|x), \quad p(y) = \sum_{x \in \mathcal{X}} \mu(x) \nu(y|x).$$

We have

$$I(\mu, \nu) = \sum_{x,y} \mu(x) \nu(y|x) \log \frac{\mu(x)\nu(y|x)}{\mu(x)p(y)}, \quad I(\mu, \nu_z) = \sum_{x,y} \mu(x) \nu_z(y|x) \log \frac{\mu(x)\nu_z(y|x)}{\mu(x)p_z(y)}.$$

Here we assume $0 \log \frac{a}{0} = 0$ for all $a \geq 0$.

We estimate each summand of $I(\mu, \nu)$ and $I(\mu, \nu_z)$. We fix $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with $\mu(x)p(y) > 0$. We define a subset $\mathcal{Z}' \subset \mathcal{Z}$ by

$$\mathcal{Z}' = \{z \mid p_z(y) > 0\} \supset \{z \mid \nu_z(y|x) > 0\}.$$

Since $\mu(x)p(y) > 0$, we have $m(\mathcal{Z}') > 0$. We have

$$\mu(x)\nu(y|x) = \int_{\mathcal{Z}'} \mu(x) \nu_z(y|x)\, dm(z), \quad \mu(x)p(y) = \int_{\mathcal{Z}'} \mu(x) p_z(y)\, dm(z).$$

By the log-sum inequality (Lemma 2.2)

$$\mu(x)\nu(y|x)\log\frac{\mu(x)\nu(y|x)}{\mu(x)p(y)} \le \int_{\mathcal{Z}'} \mu(x)\nu_z(y|x)\log\frac{\mu(x)\nu_z(y|x)}{\mu(x)p_z(y)}dm(z)$$
$$= \int_{\mathcal{Z}} \mu(x)\nu_z(y|x)\log\frac{\mu(x)\nu_z(y|x)}{\mu(x)p_z(y)}dm(z).$$

Taking sums over $(x,y)\in\mathcal{X}\times\mathcal{Y}$, we get the statement. $\square$

Proposition 2.10 ($I(\mu,\nu)$ is concave in μ and convex in ν)**.** *Let $(\mathcal{X},\mathcal{A})$ and $(\mathcal{Y},\mathcal{B})$ be standard Borel spaces, and let $(\mathcal{Z},\mathcal{C},m)$ be a probability space.*

(1) *Let ν be a transition probability on $\mathcal{X}\times\mathcal{Y}$. Suppose that we are given a probability measure μ_z on $\mathcal{X}$ for each $z\in\mathcal{Z}$ such that the map $\mathcal{Z}\ni z\mapsto\mu_z(A)\in[0,1]$ is measurable for every $A\in\mathcal{A}$. We define a probability measure μ on $(\mathcal{X},\mathcal{A})$ by*

$$\mu(A) = \int_{\mathcal{Z}} \mu_z(A)\,dm(z) \quad (A\in\mathcal{A}).$$

Then we have

$$I(\mu,\nu) \ge \int_{\mathcal{Z}} I(\mu_z,\nu)\,dm(z). \tag{2·5}$$

(2) *Let μ be a probability measure on $\mathcal{X}$. Suppose that we are given a transition probability ν_z on $\mathcal{X}\times\mathcal{Y}$ for each $z\in\mathcal{Z}$ such that the map $\mathcal{X}\times\mathcal{Z}\ni(x,z)\mapsto\nu_z(B|x)\in[0,1]$ is measurable with respect to $\mathcal{A}\otimes\mathcal{C}$ for each $B\in\mathcal{B}$. We define a transition probability ν on $\mathcal{X}\times\mathcal{Y}$ by*

$$\nu(B|x) = \int_{\mathcal{Z}} \nu_z(B|x)dm(z) \quad (x\in\mathcal{X}, B\in\mathcal{B}).$$

Then we have

$$I(\mu,\nu) \le \int_{\mathcal{Z}} I(\mu,\nu_z)\,dm(z).$$

Proof. (1) By Theorem 2.5, there exists a sequence of measurable maps $g_n\colon\mathcal{Y}\to\mathcal{Y}_n$ to finite sets $\mathcal{Y}_n$ such that

$$I(\mu_z,\nu) = \lim_{n\to\infty} I\left(\mu_z,(g_n)_*\nu\right), \quad I(\mu,\nu) = \lim_{n\to\infty} I\left(\mu,(g_n)_*\nu\right).$$

Here $(g_n)_*\nu$ is a transition probability on $\mathcal{X}\times\mathcal{Y}_n$ defined by

$$(g_n)_*\nu(B|x) = \nu\left((g_n)^{-1}B|x\right) \quad (B\subset\mathcal{Y}_n).$$

It is enough to prove that for each n we have

$$I\left(\mu,(g_n)_*\nu\right) \ge \int_{\mathcal{Z}} I\left(\mu_z,(g_n)_*\nu\right)\,dm(z).$$

If this is proved then we get the above (2·5) by Fatou's lemma. Therefore we can assume that $\mathcal{Y}$ itself is a finite set from the beginning.

We define probability mass functions $p(y)$ and $p_z(y)$ $(z \in \mathcal{Z})$ on $\mathcal{Y}$ by

$$p(y) = \int_{\mathcal{X}} \nu(y|x)\, d\mu(x), \quad p_z(y) = \int_{\mathcal{X}} \nu(y|x)\, d\mu_z(x).$$

We have $p(y) = \int_{\mathcal{Z}} p_z(y)\, dm(z)$. Then by Theorem 2.4

$$I(\mu, \nu) = H(p) - \int_{\mathcal{X}} H\left(\nu(\cdot|x)\right)\, d\mu(x), \quad I\left(\mu_z, \nu\right) = H(p_z) - \int_{\mathcal{X}} H\left(\nu(\cdot|x)\right)\, d\mu_z(x).$$

Here $H\left(\nu(\cdot|x)\right) = -\sum_{y \in \mathcal{Y}} \nu(y|x) \log \nu(y|x)$. Notice that in particular this shows that $I\left(\mu_z, \nu\right)$ is a measurable function in the variable $z \in \mathcal{Z}$. By Lemma 2.8, we have $H(p) \geq \int_{\mathcal{Z}} H(p_z)\, dm(z)$. We also have

$$\int_{\mathcal{X}} H\left(\nu(\cdot|x)\right) d\mu(x) = \int_{\mathcal{Z}} \left(\int_{\mathcal{X}} H\left(\nu(\cdot|x)\right)\, d\mu_z(x)\right) dm(z).$$

Thus

$$I(\mu, \nu) \geq \int_{\mathcal{Z}} I\left(\mu_z, \nu\right) dm(z).$$

(2) Let $f\colon \mathcal{X} \to \mathcal{X}'$ and $g\colon \mathcal{Y} \to \mathcal{Y}'$ be measurable maps to finite sets $\mathcal{X}'$ and $\mathcal{Y}'$. We define a probability mass function μ' on $\mathcal{X}'$ by

$$\mu'(x') = \mu\left(f^{-1}(x')\right) \quad (x' \in \mathcal{X}').$$

We also define conditional probability mass functions ν' and ν'_z $(z \in \mathcal{Z})$ on $\mathcal{X}' \times \mathcal{Y}'$ by

$$\nu'(y'|x') = \frac{\int_{f^{-1}(x')} \nu\left(g^{-1}(y')|x\right) d\mu(x)}{\mu\left(f^{-1}(x')\right)}, \quad \nu'_z(y'|x') = \frac{\int_{f^{-1}(x')} \nu_z\left(g^{-1}(y')|x\right) d\mu(x)}{\mu\left(f^{-1}(x')\right)}$$

where $x' \in \mathcal{X}'$ and $y' \in \mathcal{Y}'$. We have

$$\nu'(y'|x') = \int_{\mathcal{Z}} \nu'_z(y'|x') dm(z).$$

Then by Lemma 2.9

$$I(\mu', \nu') \leq \int_{\mathcal{Z}} I(\mu', \nu'_z)\, dm(z).$$

It follows from the definition of mutual information that we have $I(\mu', \nu'_z) \leq I(\mu, \nu_z)$ for all $z \in \mathcal{Z}$. Hence

$$I(\mu', \nu') \leq \int_{\mathcal{Z}} I(\mu, \nu_z)\, dm(z).$$

(It follows from Theorem 2.5 that $I(\mu, \nu_z)$ is measurable in $z \in \mathcal{Z}$.) Taking the supremum over f and g, we get

$$I(\mu, \nu) \leq \int_{\mathcal{Z}} I(\mu, \nu_z)\, dm(z).$$

$\square$

Next we will establish a method to prove a lower bound on mutual information (Proposition 2.13 below). We need to use the following integral representation of $I(X; Y)$. This is given in [Gra11, p. 176 Lemma 7.4, p. 206 Equation (7.31)].

Theorem 2.11. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be measurable spaces. Let $X \colon \Omega \to \mathcal{X}$ and $Y \colon \Omega \to \mathcal{Y}$ be measurable maps with distributions $\mu = \mathrm{Law}(X) = X_*\mathbb{P}$ and $\nu = \mathrm{Law}(Y) = Y_*\mathbb{P}$ respectively. Let $p = \mathrm{Law}(X, Y) = (X, Y)_*\mathbb{P}$ be the distribution of $(X, Y) \colon \Omega \to \mathcal{X} \times \mathcal{Y}$. Suppose that the mutual information $I(X; Y)$ is finite. Then p is absolutely continuous with respect to the product measure $\mu \otimes \nu$. Moreover, letting $f = dp/d(\mu \otimes \nu)$ be the Radon–Nikodim derivative, we have*

$$I(X;Y) = \int_{\mathcal{X}\times\mathcal{Y}} \log f \, dp = \int_{\mathcal{X}\times\mathcal{Y}} f \log f \, d(\mu \otimes \nu).$$

We learnt the next result from [KD94, Lemma A.1]. This is a kind of duality of convex programming.

Proposition 2.12. *Let $\varepsilon > 0$ and $a \geq 0$ be real numbers. Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ be measurable spaces and $\rho \colon \mathcal{X} \times \mathcal{Y} \to [0, +\infty)$ a measurable map. Let μ be a probability measure on $\mathcal{X}$. Suppose a measurable map $\lambda \colon \mathcal{X} \to [0, +\infty)$ satisfies*

$$\forall y \in \mathcal{Y}: \quad \int_{\mathcal{X}} \lambda(x) 2^{-a\rho(x,y)} \, d\mu(x) \leq 1.$$

If X and Y are random variables taking values in $\mathcal{X}$ and $\mathcal{Y}$ respectively and satisfying $\mathrm{Law}(X) = \mu$ and $\mathbb{E}\rho(X, Y) < \varepsilon$ then we have

$$I(X;Y) \geq -a\varepsilon + \int_{\mathcal{X}} \log \lambda(x) \, d\mu(x). \tag{2·6}$$

Proof. Let $\nu = \mathrm{Law}(Y)$ and $p = \mathrm{Law}(X, Y)$ be the distributions of Y and (X, Y) respectively. If $I(X; Y)$ is infinite then the statement is trivial. So we assume $I(X; Y) < \infty$. Then by Theorem 2.11 the measure p is absolutely continuous with respect to $\mu \otimes \nu$. Let $f = dp/d(\mu \otimes \nu)$ be the Radon–Nikodim derivative. We have

$$I(X;Y) = \int_{\mathcal{X}\times\mathcal{Y}} \log f \, dp.$$

Set $g(x, y) = \lambda(x) 2^{-a\rho(x,y)}$. Since $-\varepsilon < -\mathbb{E}\rho(X, Y) = -\int_{\mathcal{X}\times\mathcal{Y}} \rho(x, y) dp(x, y)$, we have

$$\begin{aligned}\int_{\mathcal{X}\times\mathcal{Y}} \log g(x,y) \, dp(x,y) &\geq -a\varepsilon + \int_{\mathcal{X}\times\mathcal{Y}} \log \lambda(x) \, dp(x,y) \\ &= -a\varepsilon + \int_{\mathcal{X}} \log \lambda(x) \, d\mu(x).\end{aligned}$$

Therefore

$$I(X;Y) + a\varepsilon - \int_{\mathcal{X}} \log \lambda(x) \, d\mu(x) \geq \int_{\mathcal{X}\times\mathcal{Y}} (\log f - \log g) \, dp = \int_{\mathcal{X}\times\mathcal{Y}} f \log(f/g) \, d\mu(x) d\nu(y).$$

Since $\ln t \le t-1$, we have $\ln(1/t) \ge 1-t$ and hence $f\ln(f/g) \ge f-g$. Then

$$\begin{aligned}(\ln 2)\int_{\mathcal{X}\times\mathcal{Y}} f\log(f/g)\,d\mu(x)d\nu(y) &= \int_{\mathcal{X}\times\mathcal{Y}} f\ln(f/g)\,d\mu(x)d\nu(y)\\ &\ge \int_{\mathcal{X}\times\mathcal{Y}} (f(x,y)-g(x,y))\,d\mu(x)d\nu(y)\\ &= 1-\int_{\mathcal{Y}}\left(\int_{\mathcal{X}} g(x,y)\,d\mu(x)\right)d\nu(y) \ge 0.\end{aligned}$$

In the last inequality we have used the assumption $\int_{\mathcal{X}} g(x,y)\,d\mu(x) \le 1$. $\square$

The next proposition is a key result. We will use it for connecting geometric measure theory to rate distortion theory. This result is essentially due to Kawabata–Dembo [KD94, Proposition 3.2]. Recall that, for a metric space $(\mathcal{X}, \mathbf{d})$, we use the notation

$$\mathrm{Diam}E = \sup\{\mathbf{d}(x,y) \mid x, y \in E\} \quad (E \subset \mathcal{X}).$$

Proposition 2.13 (Kawabata–Dembo estimate)**.** *Let ε and δ be positive numbers with $2\varepsilon\log(1/\varepsilon) \le \delta$. Let s be a nonnegative real number. Let $(\mathcal{X}, \mathbf{d})$ be a separable metric space with a Borel probability measure μ satisfying*

$$\mu(E) \le (\mathrm{Diam}E)^s \quad \textit{for all Borel sets } E \subset \mathcal{X} \textit{ with } \mathrm{Diam}E < \delta. \tag{2·7}$$

Let X and Y be random variables taking values in $\mathcal{X}$ and satisfying $\mathrm{Law}X = \mu$ and $\mathbb{E}\mathbf{d}(X,Y) < \varepsilon$. Then

$$I(X;Y) \ge s\log(1/\varepsilon) - K(s+1).$$

Here K is a universal positive constant independent of the given data (i.e. $\varepsilon, \delta, s, (\mathcal{X}, \mathbf{d}), \mu$).

Proof. The proof is almost identical with [LT19, Lemma 2.10]. But we repeat it for completeness. If $s = 0$ then the statement is trivial. So we can assume $s > 0$. We use Proposition 2.12. Set $a = s/\varepsilon$ and we estimate $\int_{\mathcal{X}} 2^{-ad(x,y)}d\mu(x)$ for each $y \in \mathcal{X}$. By the Fubini theorem (see [Mat95, 1.15 Theorem])

$$\int_{\mathcal{X}} 2^{-ad(x,y)}d\mu(x) = \int_0^1 \mu\{x \mid 2^{-ad(x,y)} \ge u\}\,du.$$

Changing the variable $u = 2^{-av}$, we have $du = -a(\ln 2)2^{-av}dv$ and hence

$$\begin{aligned}\int_0^1 \mu\{x \mid 2^{-ad(x,y)} \ge u\}\,du &= \int_0^\infty \mu\{x \mid d(x,y) \le v\}a(\ln 2)2^{-av}\,dv\\ &= a\ln 2\left(\int_0^{\delta/2} + \int_{\delta/2}^\infty\right)\mu\{x \mid d(x,y) \le v\}2^{-av}\,dv.\end{aligned}$$

By using (2·7)

$$
\begin{aligned}
a\ln 2\int_0^{\delta/2}\mu\{x \mid d(x,y)\le v\}2^{-av}\,dv &\le a\ln 2\int_0^{\delta/2}(2v)^s 2^{-av}\,dv \\
&= \int_0^{\frac{a\delta\ln 2}{2}}\left(\frac{2t}{a\ln 2}\right)^s e^{-t}\,dt \quad (t = a(\ln 2)v) \\
&\le \left(\frac{2}{a\ln 2}\right)^s\int_0^\infty t^s e^{-t}\,dt \\
&= \left(\frac{2\varepsilon}{\ln 2}\right)^s s^{-s}\Gamma(s+1) \quad \left(a = \frac{s}{\varepsilon}\right).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
a\ln 2\int_{\delta/2}^{\infty}\mu\{x \mid d(x,y)\le v\}2^{-av}\,dv &\le a\ln 2\int_{\delta/2}^\infty 2^{-av}dv \\
&= 2^{-a\delta/2} \\
&= \left(2^{-\delta/(2\varepsilon)}\right)^s \quad \left(a = \frac{s}{\varepsilon}\right).
\end{aligned}
$$

Since $\delta \ge 2\varepsilon\log(1/\varepsilon)$, we have $-\frac{\delta}{2\varepsilon}\le \log\varepsilon$. Hence $2^{-\delta/(2\varepsilon)}\le\varepsilon$.

Summing the above estimates, we get

$$
\int_{\mathcal{X}} 2^{-ad(x,y)}d\mu(x) \le \varepsilon^s\left\{1 + \left(\frac{2}{\ln 2}\right)^s s^{-s}\Gamma(s+1)\right\}.
$$

Using the Stirling formula $s^{-s}\Gamma(s+1)\sim e^{-s}\sqrt{2\pi s}$, we can find a constant $c>1$ such that the term $\{\cdots\}$ is bounded by c^{s+1} from above and hence

$$
\int_{\mathcal{X}} 2^{-ad(x,y)}d\mu(x) \le c^{s+1}\varepsilon^s.
$$

We set $\lambda(x) = c^{-1-s}\varepsilon^{-s}$ for $x\in\mathcal{X}$. (This is a constant function.) Then for all $y\in\mathcal{X}$

$$
\int_{\mathcal{X}}\lambda(x)2^{-ad(x,y)}d\mu(x)\le 1.
$$

We apply Proposition 2.12 and get

$$
\begin{aligned}
I(X;Y) &\ge -a\varepsilon + \int_{\mathcal{X}}\log\lambda\,d\mu \\
&= -s + \log\lambda \quad \left(a = \frac{s}{\varepsilon}\right) \\
&= s\log(1/\varepsilon) - (1+\log c)s - \log c.
\end{aligned}
$$

Then the constant $K := 1 + \log c$ satisfies the statement. $\square$

2.3. **Rate distortion theory.** In this subsection we introduce a rate distortion function. A basic of rate distortion theory can be found in the book of Cover–Thomas [CT06, Chapter 10]. The rate distortion theory for continuous-time stochastic processes are presented in the paper of Pursley–Gray [PG77].

Recall that we have denoted the Lebesgue measure on $\mathbb{R}^d$ by $\mathbf{m}$. For a measurable function $f(u)$ on $\mathbb{R}^d$ we usually denote its integral with respect to $\mathbf{m}$ by

$$\int_{\mathbb{R}^d} f(u)du.$$

Let $(\mathcal{X}, \mathbf{d})$ be a compact metric space. Let A be a Borel subset of $\mathbb{R}^d$ of finite measure $\mathbf{m}(A) < \infty$. We define $L^1(A, \mathcal{X})$ as the space of all measurable maps $f\colon A \to \mathcal{X}$. We identify two maps if they coincide $\mathbf{m}$-almost everywhere. We define a metric on $L^1(A, \mathcal{X})$ by

$$D(f, g) = \int_A \mathbf{d}\left(f(u), g(u)\right) du \quad \left(f, g \in L^1(A, \mathcal{X})\right).$$

We need to check the following technical fact.

Lemma 2.14. *$(L^1(A, \mathcal{X}), D)$ is a complete separable metric space. Hence it is a standard Borel space with respect to the Borel σ-algebra.*

Proof. First we need to understand what happens if we change the metric $\mathbf{d}$ on $\mathcal{X}$. Let $\mathbf{d}'$ be another metric on $\mathcal{X}$ compatible with the given topology. We define a metric D' on $L^1(A, \mathcal{X})$ by

$$D'(f, g) = \int_A \mathbf{d}'\left(f(u), g(u)\right) du.$$

Let ε be a positive number. There exists $\delta > 0$ such that $\mathbf{d}(x, y) < \delta \Longrightarrow \mathbf{d}'(x, y) < \varepsilon$.

Suppose $f, g \in L^1(A, \mathcal{X})$ satisfy $D(f, g) < \varepsilon\delta$. Then

$$\mathbf{m}\{u \in A \mid \mathbf{d}\left(f(u), g(u)\right) \geq \delta\} \leq \frac{1}{\delta}\int_A \mathbf{d}\left(f(u), g(u)\right) du < \varepsilon.$$

We have $\mathbf{d}'\left(f(u), g(u)\right) < \varepsilon$ on $\{u \in A \mid \mathbf{d}\left(f(u), g(u)\right) < \delta\}$. Hence

$$D'(f, g) < \varepsilon\left(\mathrm{Diam}(\mathcal{X}, \mathbf{d}') + \mathbf{m}(A)\right).$$

So the identity map $\mathrm{id}\colon (L^1(A, \mathcal{X}), D) \to (L^1(A, \mathcal{X}), D')$ is uniformly continuous. The same is true if we exchange D and D'. Therefore if $(L^1(A, \mathcal{X}), D')$ is complete and separable then so is $(L^1(A, \mathcal{X}), D)$.

Every compact metric space topologically embeds into the Hilbert cube $[0, 1]^{\mathbb{N}}$. We define a metric $\mathbf{d}'$ on $[0, 1]^{\mathbb{N}}$ by

$$\mathbf{d}'(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|.$$

Let $L^1\left(A, [0, 1]^{\mathbb{N}}\right)$ be the space of measurable maps from A to $[0, 1]^{\mathbb{N}}$. We define a metric D' on $L^1\left(A, [0, 1]^{\mathbb{N}}\right)$ as above. The space $L^1(A, \mathcal{X})$ is identified with a closed subspace of $L^1\left(A, [0,1]^{\mathbb{N}}\right)$. So it is enough to show that $\left(L^1\left(A, [0,1]^{\mathbb{N}}\right), D'\right)$ is a complete separable metric space. This follows from the standard fact that $L^1(A, [0,1])$ is complete and separable with respect to the L^1-norm. $\square$

In the following we always assume that $L^1(A, \mathcal{X})$ is endowed with the Borel σ-algebra (and hence it is a standard Borel space).

Let $(\mathcal{X}, \mathbf{d})$ be a compact metric space, and let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$. Let μ be a T-invariant Borel probability measure on $\mathcal{X}$.

Let $\varepsilon > 0$ and let A be a bounded Borel subset of $\mathbb{R}^d$ with $\mathbf{m}(A) > 0$. We define $R(\varepsilon, A)$ as the infimum of the mutual information $I(X; Y)$ where X and Y are random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- X takes values in $\mathcal{X}$ and its distribution is given by μ,
- Y takes values in $L^1(A, \mathcal{X})$ and satisfies

$$\mathbb{E}\left(\frac{1}{\mathbf{m}(A)} \int_A \mathbf{d}(T^u X, Y_u)\, du\right) < \varepsilon.$$

Here $Y_u = Y_u(\omega)$ $(\omega \in \Omega)$ is the value of $Y(\omega) \in L^1(A, \mathcal{X})$ at $u \in A$. We set $R(\varepsilon, A) = 0$ if $\mathbf{m}(A) = 0$.

Remark 2.15. In the above definition of $R(\varepsilon, A)$, we can assume that Y takes only finitely many values. Indeed let X and Y be random variables satisfying the conditions in the definition of $R(\varepsilon, A)$. We take a positive number τ satisfying

$$\mathbb{E}\left(\frac{1}{\mathbf{m}(A)} \int_A \mathbf{d}(T^u X, Y_u)\, du\right) < \varepsilon - 2\tau.$$

Since $L^1(A, \mathcal{X})$ is separable, it contains a dense countable subsets $\{f_1, f_2, f_3, \dots\}$. We define a map $F\colon L^1(A, \mathcal{X}) \to \{f_1, f_2, f_3, \dots\}$ by $F(f) = f_n$ where n is the smallest natural number satisfying $D(f, f_n) < \tau \cdot \mathbf{m}(A)$. Set $Y' = F(Y)$. Then we have

$$\mathbb{E}\left(\frac{1}{\mathbf{m}(A)} \int_A \mathbf{d}(T^u X, Y'_u)\, du\right) < \varepsilon - \tau.$$

Define $p_n = \mathbb{P}(Y' = f_n)$. We choose n_0 such that

$$\sum_{n > n_0} p_n \mathrm{Diam}(\mathcal{X}, \mathbf{d}) < \tau.$$

We define $G\colon \{f_1, f_2, f_3, \dots\} \to \{f_1, f_2, \dots, f_{n_0}\}$ by

$$G(f) = \begin{cases} f & \text{if } f \in \{f_1, f_2, \dots, f_{n_0}\} \\ f_{n_0} & \text{otherwise} \end{cases}.$$

Set $Y'' = G(Y')$. Then Y'' takes only finitely many values (i.e. $f_1, \dots, f_{n_0}$) and we have

$$\mathbb{E}\left(\frac{1}{\mathbf{m}(A)} \int_A \mathbf{d}(T^u X, Y''_u)\, du\right) < \varepsilon,$$

$$I(X;Y'') \le I(X;Y') \le I(X;Y).$$

Therefore, when we consider the infimum in the definition of $R(\varepsilon, A)$, we only need to take into account such random variables Y''.

For a bounded Borel subset $A \subset \mathbb{R}^d$ and $r > 0$ we define $N_r(A)$ as the r-neighborhood of A with respect to the ℓ^∞-norm, i.e. $N_r(A) = \{u + v \mid u \in A, v \in (-r, r)^d\}$.

Lemma 2.16. *We have:*

(1) $R(\varepsilon, A) \le \log \#\left(\mathcal{X}, \mathbf{d}_A, \varepsilon\right) \le \mathbf{m}\left(N_{1/2}(A)\right) \log \#(\mathcal{X}, \mathbf{d}_{(-1,1)^d}, \varepsilon)$.
(2) $R(\varepsilon, a + A) = R(\varepsilon, A)$ *for any* $a \in \mathbb{R}^d$.
(3) *If* $A \cap B = \emptyset$ *then* $R(\varepsilon, A \cup B) \le R(\varepsilon, A) + R(\varepsilon, B)$.

Proof. (1) Let $\mathcal{X} = U_1 \cup U_2 \cup \dots \cup U_n$ be an open cover with $n = \#\left(\mathcal{X}, \mathbf{d}_A, \varepsilon\right)$ and $\mathrm{Diam}(U_k, \mathbf{d}_A) < \varepsilon$ for all $1 \le k \le n$. Take a point $x_k \in U_k$ for each k and define a map $f\colon \mathcal{X} \to \{x_1, \dots, x_n\}$ by $f(x) = \{x_k\}$ for $x \in U_k \setminus (U_1 \cup \dots \cup U_{k-1})$. Let X be a random variable taking values in $\mathcal{X}$ according to μ. We set $Y_u = T^u f(X)$ for $u \in A$. Then X and Y satisfy the conditions of the definition of $R(\varepsilon, A)$. We have

$$R(\varepsilon, A) \le I(X;Y) \le H(Y) \le \log n = \log \#\left(\mathcal{X}, \mathbf{d}_A, \varepsilon\right).$$

We estimate $\log \#\left(\mathcal{X}, \mathbf{d}_A, \varepsilon\right)$. Let $\{u_1, \dots, u_a\}$ be a maximal 1-separated subset of A where "1-separated" means $\|u_i - u_j\|_\infty \ge 1$ for $i \ne j$. Then $A \subset \bigcup_{i=1}^a \left(u_i + (-1,1)^d\right)$ and hence

$$\log \#\left(\mathcal{X}, \mathbf{d}_A, \varepsilon\right) \le a \log \#\left(\mathcal{X}, \mathbf{d}_{(-1,1)^d}, \varepsilon\right).$$

The sets $u_i + (-1/2, 1/2)^d$ $(1 \le i \le a)$ are mutually disjoint and contained in $N_{1/2}(A)$. Therefore $a \le \mathbf{m}\left(N_{1/2}(A)\right)$.

(2) Let X and Y be random variables satisfying the conditions of the definition of $R(\varepsilon, A)$ (i.e. X is distributed according to μ and the average distance between $\{T^u X\}_{u \in A}$ and Y is bounded by ε). We define new random variables X' and Y' by

$$X' = T^{-a} X, \quad Y'_v = Y_{v-a} \quad (v \in a + A).$$

Since μ is T-invariant, we have $\mathrm{Law}X' = \mathrm{Law}X = \mu$. The random variable Y' takes values in $L^1(a + A, \mathcal{X})$ and

$$\begin{aligned} \int_{a+A} \mathbf{d}(T^v X', Y'_v)\, dv &= \int_{a+A} \mathbf{d}(T^{v-a} X, Y_{v-a})\, dv \\ &= \int_A \mathbf{d}(T^u X, Y_u)\, du, \quad (u = v - a). \end{aligned}$$

We have $I(X';Y') = I(X;Y)$. Therefore $R(\varepsilon, a + A) = R(\varepsilon, A)$.

(3) Let X and Y be random variables satisfying the conditions of the definition of $R(\varepsilon, A)$ as above, and let X' and Y' be random variables satisfying the conditions of the definition of $R(\varepsilon, B)$. We denote by $\mathbb{P}(Y \in E \mid X = x)$ $(E \subset L^1(A, \mathcal{X}))$ the regular conditional distribution of Y given $X = x$. Similarly for $\mathbb{P}(Y' \in F \mid X' = x)$.

We naturally identify $L^1(A\cup B,\mathcal{X})$ with $L^1(A,\mathcal{X})\times L^1(B,\mathcal{X})$. We define a transition probability ν on $\mathcal{X}\times L^1(A\cup B,\mathcal{X})$ by

$$\nu(E\times F|x) = \mathbb{P}(Y\in E\mid X=x)\mathbb{P}(Y'\in F\mid X'=x),$$

for $E\times F\subset L^1(A,\mathcal{X})\times L^1(B,\mathcal{X}) = L^1(A\cup B,\mathcal{X})$ and $x\in\mathcal{X}$. We define a probability measure Q on $\mathcal{X}\times L^1(A\cup B,\mathcal{X})$ by

$$Q(G) = \int_{\mathcal{X}} \nu(G_x|x)\,d\mu(x), \quad (G\subset \mathcal{X}\times L^1(A\cup B,\mathcal{X})),$$

where $G_x = \{f\in L^1(A\cup B,\mathcal{X})\mid (x,f)\in G\}$. Let (X'',Y'') be the random variable taking values in $\mathcal{X}\times L^1(A\cup B,\mathcal{X})$ according to Q. Then $\mathrm{Law}X''=\mu$ and

$$\begin{aligned}\mathbb{E}\left(\int_{A\cup B}\mathbf{d}(T^uX'',Y''_u)\,du\right) &= \mathbb{E}\left(\int_{A}\mathbf{d}(T^uX,Y_u)\,du\right)+\mathbb{E}\left(\int_{B}\mathbf{d}(T^uX',Y'_u)\,du\right)\\ &< \varepsilon\,\mathbf{m}(A)+\varepsilon\,\mathbf{m}(B) = \varepsilon\,\mathbf{m}(A\cup B).\end{aligned}$$

The random variables $Y''|_A$ and $Y''|_B$ is conditionally independent given X''. Therefore by Lemma 2.7

$$I(X'';Y'') = I(X'';Y''|_A,Y''|_B)\le I(X'';Y''|_A)+I(X'';Y''|_B) = I(X;Y)+I(X',Y').$$

The statement (3) follows from this. $\square$

Lemma 2.17. *The limit of* $\dfrac{R\left(\varepsilon,[0,L)^d\right)}{L^d}$ *as* $L\to\infty$ *exists and is equal to the infimum of* $\dfrac{R\left(\varepsilon,[0,L)^d\right)}{L^d}$ *over* $L>0$.

Proof. Let $0<\ell<L$. We divide L by ℓ and let $L=q\ell+r$ where q is a natural number and $0\le r<\ell$. Set

$$\Gamma = \{(\ell n_1,\dots,\ell n_d)\mid n_i\in\mathbb{Z},\ 0\le n_i<q\ (1\le i\le d)\}.$$

The cubes $u+[0,\ell)^d$ $(u\in\Gamma)$ are disjoint and contained in $[0,L)^d$. Let A be the complement:

$$A = [0,L)^d\setminus\bigcup_{u\in\Gamma}\left(u+[0,\ell)^d\right).$$

The volume of the 1/2-neighborhood of A is $O(L^{d-1})$:

$$\mathbf{m}\left(N_{1/2}(A)\right)\le d(r+1)(L+1)^{d-1}.$$

By Lemma 2.16

$$\begin{aligned}R\left(\varepsilon,[0,L)^d\right) &\le \sum_{u\in\Gamma}R\left(\varepsilon,u+[0,\ell)^d\right)+R(\varepsilon,A)\\ &\le q^dR\left(\varepsilon,[0,\ell)^d\right)+C(L+1)^{d-1}.\end{aligned}$$

By dividing this by L^d and letting $L \to \infty$, we get

$$\limsup_{L\to\infty} \frac{R\left(\varepsilon, [0,L)^d\right)}{L^d} \le \frac{R\left(\varepsilon, [0,\ell)^d\right)}{\ell^d}.$$

Then

$$\limsup_{L\to\infty} \frac{R\left(\varepsilon, [0,L)^d\right)}{L^d} \le \inf_{\ell>0} \frac{R\left(\varepsilon, [0,\ell)^d\right)}{\ell^d} \le \liminf_{\ell\to\infty} \frac{R\left(\varepsilon, [0,\ell)^d\right)}{\ell^d}$$

□

Recall that $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ is a continuous action of $\mathbb{R}^d$ on a compact metric space $(\mathcal{X}, \mathbf{d})$ with an invariant probability measure μ. For $\varepsilon > 0$ we define the **rate distortion function** $R(\mathbf{d}, \mu, \varepsilon)$ $(\varepsilon > 0)$ by

$$R(\mathbf{d}, \mu, \varepsilon) = \lim_{L\to\infty} \frac{R\left(\varepsilon, [0,L)^d\right)}{L^d} = \inf_{L>0} \frac{R\left(\varepsilon, [0,L)^d\right)}{L^d}.$$

We define the **upper/lower rate distortion dimensions** of $(\mathcal{X}, T, \mathbf{d}, \mu)$ by

$$\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) = \limsup_{\varepsilon\to 0} \frac{R(\mathbf{d}, \mu, \varepsilon)}{\log(1/\varepsilon)}, \quad \underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) = \liminf_{\varepsilon\to 0} \frac{R(\mathbf{d}, \mu, \varepsilon)}{\log(1/\varepsilon)}.$$

Remark 2.18. A tiling argument similar to the proof of Lemma 2.17 shows that if $\Lambda_1, \Lambda_2, \Lambda_3, \dots$ are a sequence of rectangles of $\mathbb{R}^d$ such that the minimum side length of Λ_n diverges to infinity then we have

$$R(\mathbf{d}, \mu, \varepsilon) = \lim_{n\to\infty} \frac{R(\varepsilon, \Lambda_n)}{\mathbf{m}(\Lambda_n)}.$$

With a bit more effort we can also prove that

$$R(\mathbf{d}, \mu, \varepsilon) = \lim_{r\to\infty} \frac{R(\varepsilon, B_r)}{\mathbf{m}(B_r)}$$

where B_r is the Euclidean r-ball of $\mathbb{R}^d$ centered at the origin. But we are not sure whether, for any Følner sequence $A_1, A_2, A_3, \dots$ of $\mathbb{R}^d$, the limit of $\frac{R(\varepsilon, A_n)}{\mathbf{m}(A_n)}$ exists or not. (Maybe not.) Probably we need to modify the definition of rate distortion function when we study the rate distortion theory for actions of general amenable groups.

3. Metric mean dimension with potential and mean Hausdorff dimension with potential

The purpose of this section is to introduce *metric mean dimension with potential* and *mean Hausdorff dimension with potential* for $\mathbb{R}^d$-actions. These are dynamical versions of Minkowski dimension and Hausdorff dimension. Mean Hausdorff dimension with potential is a main ingredient of the proof of Theorem 1.3. Metric mean dimension with potential is not used in the proof of Theorem 1.3. But it is also an indispensable tool of mean dimension theory. Therefore we develop its basic theory. We plan to use it in Part II of this series of papers.

Let $(\mathcal{X}, \mathbf{d})$ be a compact metric space and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. Let ε be a positive number. We define **covering number with potential** by

$$\#\left(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon\right) = \inf\left\{ \sum_{i=1}^{n} (1/\varepsilon)^{\sup_{U_i}\varphi} \,\middle|\, \begin{array}{c} \mathcal{X} = U_1 \cup \cdots \cup U_n \text{ is an open cover with} \\ \operatorname{Diam} U_i < \varepsilon \text{ for all } 1 \le i \le n \end{array} \right\}.$$

Let s be a real number larger than the maximum value of φ. We define

$$\mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) = \inf\left\{ \sum_{n=1}^{\infty} (\mathrm{Diam} E_n)^{s-\sup_{E_n}\varphi} \,\middle|\, \mathcal{X} = E_1 \cup E_2 \cup E_3 \cup \dots \text{ with } \mathrm{Diam} E_n < \varepsilon \right\}.$$

Here we assume that the empty set has diameter zero. We define **Hausdorff dimension with potential at the scale** ε by

$$\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon) = \inf\{s \mid \mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) < 1,\ s > \max\varphi\}.$$

When the function φ is identically zero ($\varphi \equiv 0$), we denote $\#\left(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon\right)$, $\mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi)$ and $\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon)$ by $\#\left(\mathcal{X}, \mathbf{d}, \varepsilon\right)$, $\mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d})$ and $\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varepsilon)$ respectively:

$$\begin{aligned}
\#\left(\mathcal{X}, \mathbf{d}, \varepsilon\right) &= \min\left\{ n \,\middle|\, \begin{array}{c} \mathcal{X} = U_1 \cup \cdots \cup U_n \text{ is an open cover with} \\ \operatorname{Diam} U_i < \varepsilon \text{ for all } 1 \le i \le n \end{array} \right\}, \\
\mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d}) &= \inf\left\{ \sum_{n=1}^{\infty} (\mathrm{Diam} E_n)^{s} \,\middle|\, \mathcal{X} = E_1 \cup E_2 \cup E_3 \cup \dots \text{ with } \mathrm{Diam} E_n < \varepsilon \right\}, \\
\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varepsilon) &= \inf\{s \mid \mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d}) < 1,\ s > 0\}.
\end{aligned}$$

We assume that $\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varepsilon) = -\infty$ if $\mathcal{X}$ is the empty set.

Lemma 3.1. *For* $0 < \varepsilon < 1$*, we have* $\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon) \le \dfrac{\log \#(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon)}{\log(1/\varepsilon)}$.

Proof. Suppose $\dfrac{\log \#(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon)}{\log(1/\varepsilon)} < s$. We have $s > \max\varphi$. There is an open cover $\mathcal{X} = U_1 \cup \cdots \cup U_n$ with $\mathrm{Diam} U_i < \varepsilon$ and

$$\sum_{i=1}^{n} (1/\varepsilon)^{\sup_{U_i}\varphi} < (1/\varepsilon)^s.$$

Then

$$\sum_{i=1}^{n} (\mathrm{Diam} U_i)^{s-\sup_{U_i}\varphi} \le \sum_{i=1}^{n} \varepsilon^{s-\sup_{U_i}\varphi} < 1.$$

Hence $\mathcal{H}^s_\varepsilon(\mathcal{X}, \mathbf{d}, \varepsilon) < 1$ and we have $\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon) \le s$. □

Let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of the group $\mathbb{R}^d$. For a bounded Borel subset A of $\mathbb{R}^d$, as in §1.2, we define a metric $\mathbf{d}_A$ and a function φ_A on $\mathcal{X}$ by

$$\mathbf{d}_A(x, y) = \sup_{u \in A} \mathbf{d}(T^u x, T^u y), \quad \varphi_A(x) = \int_A \varphi(T^u x)\, du.$$

In particular, for a positive number L, we set $\mathbf{d}_L = \mathbf{d}_{[0,L)^d}$ and $\varphi_L = \varphi_{[0,L)^d}$. We define **upper/lower metric mean dimension with potential** by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \limsup_{\varepsilon \to 0} \left(\lim_{L \to \infty} \frac{\log \#\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right)}{L^d \log(1/\varepsilon)} \right),$$
$$(3\text{·}1) \qquad \underline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \liminf_{\varepsilon \to 0} \left(\lim_{L \to \infty} \frac{\log \#\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right)}{L^d \log(1/\varepsilon)} \right).$$

Here the limit with respect to L exists. The proof is similar to the proof of Lemma 1.1. (The quantity $\log \#\left(\mathcal{X}, \mathbf{d}_A, \varphi_A, \varepsilon\right)$ satisfies the natural subadditivity, monotonicity and invariance if $\varphi(x)$ is a nonnegative function.)

We define **upper/lower mean Hausdorff dimension with potential** by

$$\overline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\limsup_{L \to \infty} \frac{\dim_{\mathrm{H}}\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right)}{L^d} \right),$$
$$\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\liminf_{L \to \infty} \frac{\dim_{\mathrm{H}}\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right)}{L^d} \right).$$

Remark 3.2. We are not sure whether or not these definitions of the upper and lower mean Hausdorff dimensions with potential coincide with the following:

$$\lim_{\varepsilon \to 0} \left(\limsup_{r \to \infty} \frac{\dim_{\mathrm{H}}\left(\mathcal{X}, \mathbf{d}_{B_r}, \varphi_{B_r}, \varepsilon\right)}{\mathbf{m}(B_r)} \right), \quad \lim_{\varepsilon \to 0} \left(\liminf_{r \to \infty} \frac{\dim_{\mathrm{H}}\left(\mathcal{X}, \mathbf{d}_{B_r}, \varphi_{B_r}, \varepsilon\right)}{\mathbf{m}(B_r)} \right).$$

(Maybe not in general.) Here B_r is the Euclidean r-ball of $\mathbb{R}^d$ centered at the origin.

The next lemma is a dynamical version of the fact that Hausdorff dimension is smaller than or equal to Minkowski dimension.

Lemma 3.3. $\overline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \underline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi)$.

Proof. This follows from Lemma 3.1. □

For the proof of Theorem 1.3 we need another version of mean Hausdorff dimension. For a positive number L we define a metric $\overline{\mathbf{d}}_L$ on $\mathcal{X}$ by

$$\overline{\mathbf{d}}_L(x, y) = \frac{1}{L^d} \int_{[0,L)^d} d(T^u x, T^u y)\, du.$$

This is also compatible with the given topology of $\mathcal{X}$. We define **upper/lower L^1-mean Hausdorff dimension with potential** by

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\limsup_{L \to \infty} \frac{\dim_{\mathrm{H}}\left(\mathcal{X}, \overline{\mathbf{d}}_L, \varphi_L, \varepsilon\right)}{L^d} \right),$$
$$(3\text{·}2) \qquad \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\liminf_{L \to \infty} \frac{\dim_{\mathrm{H}}\left(\mathcal{X}, \overline{\mathbf{d}}_L, \varphi_L, \varepsilon\right)}{L^d} \right).$$

We have $\overline{\mathbf{d}}_L(x, y) \leq \mathbf{d}_L(x, y)$ and hence

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \overline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi), \quad \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi).$$

It is well-known that topological dimension is smaller than or equal to Hausdorff dimension. The next result is its dynamical version. The proof will be given in §5.

Theorem 3.4. $\mathrm{mdim}(\mathcal{X}, T, \varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi)$.

Notice that this also implies $\mathrm{mdim}(\mathcal{X}, T, \varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi)$.

Metric mean dimension with potential is related to rate distortion dimension by the following result.

Proposition 3.5. *For any T-invariant Borel probability measure μ on $\mathcal{X}$ we have*

$$\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi\, d\mu \le \overline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi),$$

$$\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi\, d\mu \le \underline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi).$$

Proof. We will use the following well-known inequality. For the proof see [Wal75, §9.3, Lemma 9.9].

Lemma 3.6. *Let $a_1, \dots, a_n$ be real numbers and $(p_1, \dots, p_n)$ a probability vector. For a positive number ε we have*

$$\sum_{i=1}^{n} \left(-p_i \log p_i + p_i a_i \log(1/\varepsilon)\right) \le \log \sum_{i=1}^{n} (1/\varepsilon)^{a_i}.$$

Let L and ε be positive numbers. Let $\mathcal{X} = U_1 \cup \dots \cup U_n$ be an open cover with $\mathrm{Diam}(U_i, \mathbf{d}_L) < \varepsilon$. Take a point $x_i \in U_i$ for each $1 \le i \le n$. Set $E_i = U_i \setminus (U_1 \cup \dots \cup U_{i-1})$ and $p_i = \mu(E_i)$. We define $f\colon \mathcal{X} \to \{x_1, \dots, x_n\}$ by $f(E_i) = \{x_i\}$. Let X be a random variable taking values in $\mathcal{X}$ according to μ. Set $Y_u = T^u f(X)$ for $u \in [0, L)^d$. Then almost surely we have $\mathbf{d}(T^u X, Y_u) < \varepsilon$ for all $u \in [0, L)^d$. Therefore

$$R\left(\varepsilon, [0, L)^d\right) \le H(Y) = H\left(f(X)\right) = -\sum_{i=1}^{n} p_i \log p_i.$$

On the other hand

$$\int_{\mathcal{X}} \varphi\, d\mu = \int_{\mathcal{X}} \frac{\varphi_L}{L^d}\, d\mu \le \frac{1}{L^d} \sum_{i=1}^{n} p_i \sup_{U_i} \varphi_L.$$

By Lemma 3.6

$$\begin{aligned} R\left(\varepsilon, [0, L)^d\right) + L^d \log(1/\varepsilon) \int_{\mathcal{X}} \varphi\, d\mu &\le \sum_{i=1}^{n} \left(-p_i \log p_i + p_i \sup_{U_i} \varphi_L \log(1/\varepsilon)\right) \\ &\le \log \sum_{i=1}^{n} (1/\varepsilon)^{\sup_{U_i} \varphi_L}. \end{aligned}$$

Therefore

$$\frac{R\left(\varepsilon, [0, L)^d\right)}{L^d \log(1/\varepsilon)} + \int_{\mathcal{X}} \varphi\, d\mu \le \frac{\log \#\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right)}{L^d \log(1/\varepsilon)}.$$

Letting $L \to \infty$

$$\frac{R(\mathbf{d}, \mu, \varepsilon)}{\log(1/\varepsilon)} + \int_{\mathcal{X}} \varphi \, d\mu \leq \lim_{L \to \infty} \frac{\log \# (\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon)}{L^d \log(1/\varepsilon)}.$$

Letting $\varepsilon \to 0$ we get the statement. □

L^1-mean Hausdorff dimension with potential is related to rate distortion dimension by the next theorem. We call this result "dynamical Frostman's lemma" because the classical Frostman's lemma [Mat95, Sections 8.14-8.17] plays an essential role in its proof. Recall that we have denoted by $\mathscr{M}^T(\mathcal{X})$ the set of T-invariant Borel probability measures on $\mathcal{X}$.

Theorem 3.7 (Dynamical Frostman's lemma)**.**

$$\overline{\mathrm{mdim}}_{\mathrm{H}, L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right).$$

The proof of this theorem is the most important step of the proof of Theorem 1.3. It will be given in §6.

By combining Theorems 3.4 and 3.7, we get

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) \leq \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right).$$

This is the statement of Theorem 1.3. Therefore the proof of Theorem 1.3 is reduced to the proofs of Theorems 3.4 and 3.7.

Conjecture 3.8. *Let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. For any continuous function $\varphi\colon \mathcal{X} \to \mathbb{R}$ there exists a metric $\mathbf{d}$ on $\mathcal{X}$ compatible with the given topology satisfying*

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) = \overline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi). \tag{3·3}$$

Suppose that this conjecture is true and let $\mathbf{d}$ be a metric satisfying (3·3). Then by Proposition 3.5

$$\begin{aligned} \mathrm{mdim}(\mathcal{X}, T, \varphi) &= \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right) \\ &= \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\overline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right). \end{aligned}$$

Namely Conjecture 3.8 implies Conjecture 1.4 in §1.3. Conjecture 3.8 is widely open in general. We plan to prove it for free minimal $\mathbb{R}^d$-actions in Part II of this series of papers.

At the end of this section we present a small technical result on L^1-mean Hausdorff dimension with potential. This will be used in §4.

Lemma 3.9. *In the definition (3·2) we can restrict the parameter L to natural numbers. Namely we have*

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi) = \lim_{\varepsilon\to 0}\left(\limsup_{\substack{N\in\mathbb{N}\\ N\to\infty}} \frac{\dim_{\mathrm{H}}\left(\mathcal{X},\overline{\mathrm{d}}_N,\varphi_N,\varepsilon\right)}{N^d}\right)$$

$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi) = \lim_{\varepsilon\to 0}\left(\liminf_{\substack{N\in\mathbb{N}\\ N\to\infty}} \frac{\dim_{\mathrm{H}}\left(\mathcal{X},\overline{\mathrm{d}}_N,\varphi_N,\varepsilon\right)}{N^d}\right).$$

Here the parameter N runs over natural numbers. A similar result also holds for upper and lower mean Hausdorff dimensions with potential.

Proof. We prove the lower case. The upper case is similar. By adding a positive constant to φ, we can assume that φ is a nonnegative function. Set

$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi)' = \lim_{\varepsilon\to 0}\left(\liminf_{\substack{N\in\mathbb{N}\\ N\to\infty}} \frac{\dim_{\mathrm{H}}\left(\mathcal{X},\overline{\mathrm{d}}_N,\varphi_N,\varepsilon\right)}{N^d}\right).$$

It is obvious that $\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi)'$. We prove the reverse inequality. Let L be a positive real number. We assume that it is sufficiently large so that

$$\left(\frac{L}{L-1}\right)^d < 2.$$

Let $N := \lfloor L \rfloor$ be the natural number not greater than L. Then for any $x, y \in \mathcal{X}$ we have $\overline{\mathbf{d}}_N(x,y) \le 2\,\overline{\mathbf{d}}_L(x,y)$. We also have $\varphi_N(x) \le \varphi_L(x)$.

Let $0 < \varepsilon < 1/2$. We prove that for any $s > \max \varphi_L$

$$\mathcal{H}_\varepsilon^{s-\frac{s}{\log\varepsilon}}\left(\mathcal{X},\overline{\mathbf{d}}_N,\varphi_N\right) \le \mathcal{H}^s_{\varepsilon/2}\left(\mathcal{X},\overline{\mathbf{d}}_L,\varphi_L\right). \tag{3·4}$$

Indeed let E be a subset of $\mathcal{X}$ with $\mathrm{Diam}(E,\overline{\mathbf{d}}_L) < \varepsilon/2$. Then

$$\mathrm{Diam}(E,\overline{\mathbf{d}}_N) \le 2\,\mathrm{Diam}(E,\overline{\mathbf{d}}_L) < \varepsilon.$$

Moreover

$$\begin{aligned}
\mathrm{Diam}(E,\overline{\mathbf{d}}_N)^{s-\frac{s}{\log\varepsilon}-\sup_E\varphi_N} &\le \mathrm{Diam}(E,\overline{\mathbf{d}}_N)^{s-\frac{s}{\log\varepsilon}-\sup_E\varphi_L} \quad (\text{by } \varphi_N \le \varphi_L)\\
&\le \left(2\,\mathrm{Diam}(E,\overline{\mathbf{d}}_L)\right)^{s-\frac{s}{\log\varepsilon}-\sup_E\varphi_L}\\
&\le \varepsilon^{-\frac{s}{\log\varepsilon}}\cdot\left(2\,\mathrm{Diam}(E,\overline{\mathbf{d}}_L)\right)^{s-\sup_E\varphi_L}\\
&= 2^{-s}\left(2\,\mathrm{Diam}(E,\overline{\mathbf{d}}_L)\right)^{s-\sup_E\varphi_L}\\
&\le \mathrm{Diam}(E,\overline{\mathbf{d}}_L)^{s-\sup_E\varphi_L} \quad (\text{by } \varphi_L \ge 0).
\end{aligned}$$

Therefore we have (3·4) and hence

$$\dim_{\mathrm{H}}(\mathcal{X},\overline{\mathbf{d}}_N,\varphi_N,\varepsilon) \le \left(1-\frac{1}{\log\varepsilon}\right)\dim_{\mathrm{H}}\left(\mathcal{X},\overline{\mathbf{d}}_L,\varphi_L,\frac{\varepsilon}{2}\right).$$

We divide this by L^d and let $L \to \infty$ and $\varepsilon \to 0$. Then we get $\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi)' \leq \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi)$. $\square$

Remark 3.10. In the definitions (1·8) and (3·1) of mean dimension with potential and (upper/lower) metric mean dimension with potential, the limits with respect to L exist. Therefore we can also restrict the parameter L to natural numbers when we take the limits.

4. Mean dimension of $\mathbb{Z}^d$-actions

In this section we prepare some basic results on mean dimension theory of $\mathbb{Z}^d$-actions. We need it in the proof of Theorem 3.4. This is a rather technical and indirect approach. It is desirable to find a more direct proof of Theorem 3.4. However we have not found it so far[3].

The paper of Huo–Yuan [HY] studies the variational principle for mean dimension of $\mathbb{Z}^d$-actions. Proposition 4.3 and Theorem 5.4 below were already mentioned in their paper [HY, Lemma 2.12 and Lemma 2.15] in the case that the potential function is zero.

4.1. Definitions of various mean dimensions for $\mathbb{Z}^d$-actions.

For a natural number N we set

$$[N]^d = \{0, 1, 2, \dots, N-1\}^d.$$

Let $T\colon \mathbb{Z}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of the group $\mathbb{Z}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. For a natural number N we define metrics $\mathbf{d}_N$ and $\overline{\mathbf{d}}_N$ and a function φ_N on $\mathcal{X}$ by

$$\mathbf{d}_N(x, y) = \max_{u \in [N]^d} \mathbf{d}(T^u x, T^u y), \quad \overline{\mathbf{d}}_N(x, y) = \frac{1}{N^d} \sum_{u \in [N]^d} \mathbf{d}(T^u x, T^u y),$$

$$\varphi_N(x) = \sum_{u \in [N]^d} \varphi(T^u x).$$

In the sequel we will sometimes consider $\mathbb{Z}^d$-actions and $\mathbb{R}^d$-actions simultaneously. In that case we use the notations $\mathbf{d}_N^{\mathbb{Z}}, \overline{\mathbf{d}}_N^{\mathbb{Z}}, \varphi_N^{\mathbb{Z}}$ for clarifying that these quantities are defined with respect to $\mathbb{Z}^d$-actions. (On the other hand, we will use the notations $\mathbf{d}_N^{\mathbb{R}}, \overline{\mathbf{d}}_N^{\mathbb{R}}, \varphi_N^{\mathbb{R}}$ when they are defined with respect to $\mathbb{R}^d$-actions.)

We define mean dimension with potential by

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) = \lim_{\varepsilon \to 0} \left(\lim_{N \to \infty} \frac{\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_N, \varphi_N)}{N^d} \right).$$

[3]The difficulty lies in Proposition 4.3 below. It is unclear for the author how to formulate and prove an analogous result for $\mathbb{R}^d$-actions.

This is a topological invariant (i.e. independent of the choice of $\mathbf{d}$). We define upper/lower mean Hausdorff dimension with potential by

$$\overline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\limsup_{N \to \infty} \frac{\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}_N, \varphi_N, \varepsilon)}{N^d} \right),$$

$$\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\liminf_{N \to \infty} \frac{\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}_N, \varphi_N, \varepsilon)}{N^d} \right).$$

We define upper/lower L^1-mean Hausdorff dimension with potential by

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\limsup_{N \to \infty} \frac{\dim_{\mathrm{H}}(\mathcal{X}, \overline{\mathbf{d}}_N, \varphi_N, \varepsilon)}{N^d} \right),$$

$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) = \lim_{\varepsilon \to 0} \left(\liminf_{N \to \infty} \frac{\dim_{\mathrm{H}}(\mathcal{X}, \overline{\mathbf{d}}_N, \varphi_N, \varepsilon)}{N^d} \right).$$

Since $\overline{\mathbf{d}}_N(x, y) \leq \mathbf{d}_N(x, y)$, we have

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \overline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi), \quad \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi).$$

We can also consider upper/lower metric mean dimension with potential for $\mathbb{Z}^d$-actions. But we do not need them in this paper.

4.2. **Tame growth of covering numbers.** The purpose of this subsection is to establish a convenient sufficient condition under which mean Hausdorff dimension with potential and L^1-mean Hausdorff dimension with potential coincide.

The following is a key definition [LT18, Condition 3].

Definition 4.1. A compact metric space $(\mathcal{X}, \mathbf{d})$ is said to have **tame growth of covering numbers** if for any positive number δ we have

$$\lim_{\varepsilon \to 0} \varepsilon^{\delta} \log \# (\mathcal{X}, \mathbf{d}, \varepsilon) = 0.$$

Recall that $\# (\mathcal{X}, \mathbf{d}, \varepsilon)$ is the minimum number n such that there is an open cover $\mathcal{X} = U_1 \cup U_2 \cup \cdots \cup U_n$ with $\mathrm{Diam} U_i < \varepsilon$ for all $1 \leq i \leq n$. Notice that this is purely a condition on metric geometry. It does not involve dynamics.

For example, every compact subset of the Euclidean space $\mathbb{R}^n$ has the tame growth of covering numbers with respect to the Euclidean metric. The Hilbert cube $[0, 1]^{\mathbb{N}}$ has the tame growth of covering numbers with respect to the metric

$$\mathbf{d}\left((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}\right) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$

The next lemma shows that every compact metrizable space admits a metric having the tame growth of covering numbers [LT19, Lemma 3.10].

Lemma 4.2. *For any compact metric space $(\mathcal{X}, \mathbf{d})$ there exists a metric $\mathbf{d}'$ on $\mathcal{X}$ compatible with the given topology satisfying the following two conditions.*

- *For all $x, y \in \mathcal{X}$ we have $\mathbf{d}'(x,y) \le \mathbf{d}(x,y)$.*
- *The space $(\mathcal{X}, \mathbf{d}')$ has the tame growth of covering numbers.*

Proof. Take a countable dense subset $\{x_1, x_2, x_3, \dots\}$ of $\mathcal{X}$. We define a metric $\mathbf{d}'$ by

$$\mathbf{d}'(x,y) = \sum_{n=1}^{\infty} 2^{-n} \left| d(x, x_n) - d(y, x_n) \right|.$$

It is easy to check that this satisfies the statement. $\square$

Proposition 4.3. *Let $(\mathcal{X}, \mathbf{d})$ be a compact metric space having the tame growth of covering numbers. Let $T\colon \mathbb{Z}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of the group $\mathbb{Z}^d$. For any continuous function $\varphi\colon \mathcal{X} \to \mathbb{R}$ we have*

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) = \overline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi),$$
$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) = \underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi).$$

Proof. The case of $d = 1$ was proved in [Tsu20, Lemma 4.3]. The following argument is its simple generalization. We prove the lower case. The upper case is similar.

It is obvious that $\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi)$. We prove the reverse inequality. By adding a positive constant to φ, we can assume that φ is a nonnegative function. For a finite subset A of $\mathbb{Z}^d$ we define a metric $\mathbf{d}_A$ on $\mathcal{X}$ by

$$\mathbf{d}_A(x,y) = \max_{u \in A} \mathbf{d}(T^u x, T^u y).$$

Let s be an arbitrary positive number with $\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) < s$. Let $0 < \delta < 1/2$ be arbitrary. For each positive number τ we take an open cover $\mathcal{X} = W_1^\tau \cup \dots \cup W_{M(\tau)}^\tau$ with $M(\tau) = \#(\mathcal{X}, \mathbf{d}, \tau)$ and $\mathrm{Diam}(W_i^\tau, \mathbf{d}) < \tau$ for all $1 \le i \le M(\tau)$. From the condition of tame growth of covering numbers, we can find $0 < \varepsilon_0 < 1$ satisfying

$$M(\tau)^{\tau^\delta} < 2 \quad \text{for all } 0 < \tau < \varepsilon_0, \tag{4·1}$$

$$2^{2+(1+2\delta)s}\, \varepsilon_0^{s\delta(1-2\delta)} < 1. \tag{4·2}$$

Let $0 < \varepsilon < \varepsilon_0$. We have $\dim_{\mathrm{H}}(\mathcal{X}, \overline{\mathbf{d}}_N, \varphi_N, \varepsilon) < sN^d$ for infinitely many N. Pick up such an N. Then there is a covering $\mathcal{X} = \bigcup_{n=1}^{\infty} E_n$ with $\tau_n := \mathrm{Diam}(E_n, \overline{\mathbf{d}}_N) < \varepsilon$ for all $n \ge 1$ and

$$\sum_{n=1}^{\infty} \tau_n^{sN^d - \sup_{E_n} \varphi_N} < 1. \tag{4·3}$$

Set $L_n = \tau_n^{-\delta}$. Pick $x_n \in E_n$ for each n. Every $x \in E_n$ satisfies $\overline{\mathbf{d}}_N(x, x_n) \le \tau_n$ and hence

$$\left| \{ u \in [N]^d \mid \mathbf{d}(T^u x, T^u x_n) \ge L_n \tau_n \} \right| \le \frac{N^d}{L_n}.$$

Namely there is $A \subset [N]^d$ (depending on x) such that $|A| \leq N^d/L_n$ and $\mathbf{d}_{[N]^d \setminus A}(x, x_n) < L_n \tau_n$. Therefore

$$E_n \subset \bigcup_{\substack{A \subset [N]^d \\ |A| \leq N^d/L_n}} B_{L_n \tau_n}(x_n, \mathbf{d}_{[N]^d \setminus A}).$$

Here $B_{L_n\tau_n}(x_n, \mathbf{d}_{[N]^d \setminus A})$ is the ball of radius $L_n \tau_n$ with respect to $\mathbf{d}_{[N]^d \setminus A}$ centered at x_n. For $A = \{a_1, \dots, a_r\} \subset [N]^d$ and $1 \leq i_1, \dots, i_r \leq M(\tau_n)$ we set

$$W(A, \tau_n, i_1, \dots, i_r) = T^{-a_1} W_{i_1}^{\tau_n} \cap \dots \cap T^{-a_r} W_{i_r}^{\tau_n}.$$

We have

$$\mathcal{X} = \bigcup_{1 \leq i_1, \dots, i_r \leq M(\tau_n)} W(A, \tau_n, i_1, \dots, i_r), \quad \text{(here A and τ_n are fixed)},$$

and hence

$$B_{L_n\tau_n}(x_n, \mathbf{d}_{[N]^d \setminus A}) = \bigcup_{1 \leq i_1, \dots, i_r \leq M(\tau_n)} B_{L_n\tau_n}(x_n, \mathbf{d}_{[N]^d \setminus A}) \cap W(A, \tau_n, i_1, \dots, i_r).$$

Then

$$\mathcal{X} = \bigcup_{n=1}^{\infty} \bigcup_{\substack{A \subset [N]^d \\ r := |A| \leq N^d/L_n}} \bigcup_{1 \leq i_1, \dots, i_r \leq M(\tau_n)} E_n \cap B_{L_n\tau_n}(x_n, \mathbf{d}_{[N]^d \setminus A}) \cap W(A, \tau_n, i_1, \dots, i_r).$$

The diameter of $E_n \cap B_{L_n\tau_n}(x_n, \mathbf{d}_{[N]^d \setminus A}) \cap W(A, \tau_n, i_1, \dots, i_r)$ with respect to $\mathbf{d}_N$ is less than or equal to $2L_n\tau_n = 2\tau_n^{1-\delta} < 2\varepsilon^{1-\delta}$. Hence

$$\mathcal{H}_{2\varepsilon^{1-\delta}}^{(1+2\delta)sN^d}(\mathcal{X}, \mathbf{d}_N, \varphi_N) \leq \sum_{n=1}^{\infty} 2^{N^d} M(\tau_n)^{N^d/L_n} \left(2\tau_n^{1-\delta}\right)^{(1+2\delta)sN^d - \sup_{E_n} \varphi_N}.$$

Here the factor 2^{N^d} comes from the choice of $A \subset [N]^d$. By $L_n = \tau_n^{-\delta}$ and (4·1)

$$M(\tau_n)^{N^d/L_n} = \left(M(\tau_n)^{\tau_n^\delta}\right)^{N^d} < 2^{N^d}.$$

Since φ is a nonnegative function,

$$2^{(1+2\delta)sN^d - \sup_{E_n} \varphi_N} \leq 2^{(1+2\delta)sN^d}.$$

Hence

$$\mathcal{H}_{2\varepsilon^{1-\delta}}^{(1+2\delta)sN^d}(\mathcal{X}, \mathbf{d}_N, \varphi_N) \leq \sum_{n=1}^{\infty} \left(2^{2+(1+2\delta)s}\right)^{N^d} \left(\tau_n^{1-\delta}\right)^{(1+2\delta)sN^d - \sup_{E_n} \varphi_N}.$$

We have

$$\begin{aligned}
\left(\tau_n^{1-\delta}\right)^{(1+2\delta)sN^d - \sup_{E_n} \varphi_N} &= \tau_n^{-\delta\left\{(1+2\delta)sN^d - \sup_{E_n} \varphi_N\right\}} \cdot \tau_n^{(1+2\delta)sN^d - \sup_{E_n} \varphi_N} \\
&= \tau_n^{\delta\left\{(1-2\delta)sN^d + \sup_{E_n} \varphi_n\right\}} \cdot \tau_n^{sN^d - \sup_{E_n} \varphi_N}.
\end{aligned}$$

Since φ is nonnegative and $\tau_n < \varepsilon < \varepsilon_0 < 1$

$$\tau_n^{\delta\left\{(1-2\delta)sN^d+\sup_{E_n}\varphi_n\right\}} \leq \tau_n^{\delta(1-2\delta)sN^d} < \varepsilon_0^{\delta(1-2\delta)sN^d}.$$

Therefore

$$\begin{aligned}\mathcal{H}^{(1+2\delta)sN^d}_{2\varepsilon^{1-\delta}}(\mathcal{X},\mathbf{d}_N,\varphi_N) &\leq \sum_{n=1}^{\infty} \underbrace{\left(2^{2+(1+2\delta)s}\cdot\varepsilon_0^{\delta(1-2\delta)s}\right)^{N^d}}_{<1 \text{ by } (4\cdot 2)} \tau_n^{sN^d-\sup_{E_n}\varphi_N} \\ &\leq \sum_{n=1}^{\infty}\tau_n^{sN^d-\sup_{E_n}\varphi_N} < 1 \quad \text{by } (4\cdot 3).\end{aligned}$$

Thus

$$\dim_{\mathrm{H}}(\mathcal{X},\mathbf{d}_N,\varphi_N,2\varepsilon^{1-\delta}) \leq (1+2\delta)sN^d.$$

This holds for infinitely many N. Hence

$$\liminf_{N\to\infty}\frac{\dim_{\mathrm{H}}(\mathcal{X},\mathbf{d}_N,\varphi_N,2\varepsilon^{1-\delta})}{N^d} \leq (1+2\delta)s.$$

Letting $\varepsilon\to 0$

$$\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X},T,\mathbf{d},\varphi) \leq (1+2\delta)s.$$

Letting $\delta\to 0$ and $s\to\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi)$, we get

$$\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X},T,\mathbf{d},\varphi) \leq \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi).$$

□

4.3. **$\mathbb{R}^d$-actions and $\mathbb{Z}^d$-actions.** We naturally consider $\mathbb{Z}^d$ as a subgroup of $\mathbb{R}^d$. Let $T\colon \mathbb{R}^d\times\mathcal{X}\to\mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. We denote by $T|_{\mathbb{Z}^d}\colon\mathbb{Z}^d\times\mathcal{X}\to\mathcal{X}$ the restriction of T to the subgroup $\mathbb{Z}^d$. In this subsection we study relations between various mean dimensions of T and $T|_{\mathbb{Z}^d}$.

Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon\mathcal{X}\to\mathbb{R}$ a continuous function.

Lemma 4.4. *We have:*

$$\begin{aligned}&\mathrm{mdim}(\mathcal{X},T,\varphi) = \mathrm{mdim}(\mathcal{X},T|_{\mathbb{Z}^d},\varphi_1^{\mathbb{R}}),\\ &\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi) = \overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T|_{\mathbb{Z}^d},\overline{\mathbf{d}}_1^{\mathbb{R}},\varphi_1^{\mathbb{R}}),\\ &\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi) = \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T|_{\mathbb{Z}^d},\overline{\mathbf{d}}_1^{\mathbb{R}},\varphi_1^{\mathbb{R}}).\end{aligned}$$

Here $\overline{\mathbf{d}}_1^{\mathbb{R}}$ and $\varphi_1^{\mathbb{R}}$ are a metric and a function on $\mathcal{X}$ defined by

$$\overline{\mathbf{d}}_1^{\mathbb{R}}(x,y) = \int_{[0,1)^d}\mathbf{d}(T^ux,T^uy)\,du,\quad \varphi_1^{\mathbb{R}}(x) = \int_{[0,1)^d}\varphi(T^ux)\,du.$$

We also have a similar result for mean Hausdorff dimension with potential by replacing $\overline{\mathbf{d}}_1^{\mathbb{R}}(x,y)$ by $\mathbf{d}_1^{\mathbb{R}}(x,y) = \sup_{u\in[0,1)^d}\mathbf{d}(T^ux,T^uy)$.

Proof. Set $\rho = \overline{\mathbf{d}}_1^{\mathbb{R}}$ and $\psi = \varphi_1^{\mathbb{R}}$. For a natural number N we have

$$\overline{\rho}_N^{\mathbb{Z}}(x,y) = \frac{1}{N^d}\sum_{u\in[N]^d}\rho(T^u x, T^u y) = \frac{1}{N^d}\sum_{u\in[N]^d}\int_{v\in[0,1)^d}\mathbf{d}(T^{u+v}x, T^{u+v}y)\,dv$$
$$= \frac{1}{N^d}\int_{[0,N)^d}\mathbf{d}(T^v x, T^v y)\,dv = \overline{\mathbf{d}}_N^{\mathbb{R}}(x,y).$$

Similarly

$$\psi_N^{\mathbb{Z}}(x) = \sum_{u\in[N]^d}\psi(T^u x) = \int_{[0,N)^d}\varphi(T^v x)\,dv = \varphi_N^{\mathbb{R}}.$$

By using Lemma 3.9

$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T,\mathbf{d},\varphi) = \lim_{\varepsilon\to 0}\left(\liminf_{\substack{N\in\mathbb{N}\\ N\to\infty}}\frac{\dim_{\mathrm{H}}\left(\mathcal{X},\overline{\mathbf{d}}_N^{\mathbb{R}},\varphi_N^{\mathbb{R}},\varepsilon\right)}{N^d}\right)$$
$$= \lim_{\varepsilon\to 0}\left(\liminf_{\substack{N\in\mathbb{N}\\ N\to\infty}}\frac{\dim_{\mathrm{H}}\left(\mathcal{X},\overline{\rho}_N^{\mathbb{Z}},\psi_N^{\mathbb{Z}},\varepsilon\right)}{N^d}\right)$$
$$= \underline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X},T|_{\mathbb{Z}^d},\rho,\psi).$$

We can prove the case of upper L^1-mean Hausdorff dimension with potential in the same way. The case of (topological) mean dimension with potential can be also proved similarly by using $\left(\mathbf{d}_1^{\mathbb{R}}\right)_N^{\mathbb{Z}} = \mathbf{d}_N^{\mathbb{R}}$. □

5. Mean dimension is bounded by mean Hausdorff dimension: proof of Theorem 3.4

In this section we prove Theorem 3.4.

5.1. A variation of the definition of mean dimension with potential. This subsection is a simple generalization of [Tsu20, §3.2]. Here we introduce a variation of the definition of mean dimension with potential. Let P be a finite simplicial complex and $a \in P$. We define **small local dimension** $\dim'_a P$ as the minimum of $\dim\Delta$ where Δ is a simplex of P containing a. See Figure 2. (This is the same as [Tsu20, Figure 2].)

Let $(\mathcal{X},\mathbf{d})$ be a compact metric space and $\varphi\colon \mathcal{X}\to\mathbb{R}$ a continuous function. For $\varepsilon > 0$ we set

$$\mathrm{Widim}'_\varepsilon(\mathcal{X},\mathbf{d},\varphi) = \inf\left\{\sup_{x\in\mathcal{X}}\left(\dim'_{f(x)}P + \varphi(x)\right)\,\middle|\, \begin{array}{c}P \text{ is a finite simplicial complex and}\\ f:\mathcal{X}\to P \text{ is an } \varepsilon\text{-embedding}\end{array}\right\}.$$

We also set

$$\mathrm{var}_\varepsilon(\varphi,\mathbf{d}) = \sup\{|\varphi(x)-\varphi(y)|\,|\,\mathbf{d}(x,y)<\varepsilon\}.$$

The following lemma is given in [Tsu20, Lemma 3.4].

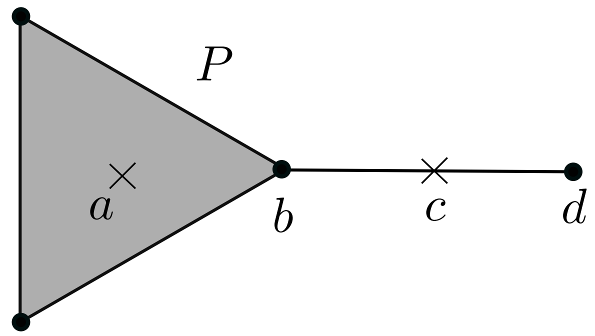

Figure 2. Here P has four vertexes (denoted by dots), four 1-dimensional simplexes and one 2-dimensional simplex. The points b and d are vertexes of P whereas a and c are not. We have $\dim'_a P = 2$, $\dim'_b P = 0$, $\dim'_c P = 1$ and $\dim'_d P = 0$. Recall $\dim_a P = \dim_b P = 2$ and $\dim_c P = \dim_d P = 1$.

Lemma 5.1.

$$\mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) \leq \mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) \leq \mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) + \mathrm{var}_\varepsilon(\varphi, \mathbf{d}).$$

The next lemma shows that we can use $\mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi)$ instead of $\mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi)$ in the definition of mean dimension with potential.

Lemma 5.2. *Let $T\colon \mathbb{Z}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{Z}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. Then*

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) = \lim_{\varepsilon \to 0} \left(\lim_{N \to \infty} \frac{\mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}_N, \varphi_N)}{N^d} \right). \tag{5·1}$$

Here the limit in the right-hand side exists as in §1.2.

Proof. By Lemma 5.1, for any natural number N, we have

$$\mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}_N, \varphi_N) \leq \mathrm{Widim}_\varepsilon(\mathcal{X}, \mathbf{d}_N, \varphi_N) \leq \mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}_N, \varphi_N) + \mathrm{var}_\varepsilon(\varphi_N, \mathbf{d}_N).$$

We have

$$\mathrm{var}_\varepsilon(\varphi_N, \mathbf{d}_N) \leq N^d \mathrm{var}_\varepsilon(\varphi, \mathbf{d}).$$

Then

$$\lim_{\varepsilon \to 0} \left(\limsup_{N \to \infty} \frac{\mathrm{var}_\varepsilon(\varphi_N, \mathbf{d}_N)}{N^d} \right) \leq \lim_{\varepsilon \to 0} \mathrm{var}_\varepsilon(\varphi, \mathbf{d}) = 0.$$

□

5.2. **Case of $\mathbb{Z}^d$-actions.** In this subsection we prove that, for $\mathbb{Z}^d$-actions, mean dimension with potential is bounded from above by lower mean Hausdorff dimension with potential. A key ingredient of the proof is the following result on metric geometry. This was proved in [Tsu20, Lemma 3.8].

Lemma 5.3. *Let $(\mathcal{X}, \mathbf{d})$ be a compact metric space and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. Let ε and L be positive numbers, and let s be a real number with $s > \max_{\mathcal{X}} \varphi$. Suppose there exists a map $f\colon \mathcal{X} \to [0,1]^N$ such that*

- *$\|f(x) - f(y)\|_\infty \le L\, \mathbf{d}(x,y)$ for all $x, y \in \mathcal{X}$,*
- *if $d(x,y) \ge \varepsilon$ then $\|f(x) - f(y)\|_\infty = 1$.*

Here $\|\cdot\|_\infty$ is the ℓ^∞-norm. Moreover we assume

$$4^N (L+1)^{1+s+\|\varphi\|_\infty} \mathcal{H}_1^s(\mathcal{X}, \mathbf{d}, \varphi) < 1,$$

where $\|\varphi\|_\infty = \max_{x\in\mathcal{X}} |\varphi(x)|$. Then we conclude that

$$\mathrm{Widim}'_\varepsilon(\mathcal{X}, \mathbf{d}, \varphi) \le s + 1.$$

The following theorem is the main result of this subsection.

Theorem 5.4. *Let $T\colon \mathbb{Z}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{Z}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. Then*

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi).$$

Proof. If $\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi)$ is infinite then the statement is trivial. So we assume that it is finite. Let s be an arbitrary number larger than $\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi)$. We prove that $\mathrm{mdim}(\mathcal{X}, T, \varphi) \le s$.

Let ε be a positive number. There is a Lipschitz map $f\colon \mathcal{X} \to [0,1]^M$ such that[4] if $\mathbf{d}(x,y) \ge \varepsilon$ then $\|f(x) - f(y)\|_\infty = 1$. Let L be a Lipschitz constant of f. Namely we have $\|f(x) - f(y)\|_\infty \le L\, \mathbf{d}(x,y)$. For each natural number N we define $f_N\colon \mathcal{X} \to \left([0,1]^M\right)^{[N]^d}$ by

$$f_N(x) = \left(f(T^u x)\right)_{u\in[N]^d}.$$

Then we have

- $\|f_N(x) - f_N(y)\|_\infty \le L\, \mathbf{d}_N(x,y)$ for all $x, y \in \mathcal{X}$,
- if $\mathbf{d}_N(x,y) \ge \varepsilon$ then $\|f_N(x) - f_N(y)\|_\infty = 1$.

Let τ be an arbitrary positive number. We choose a positive number $\delta < 1$ satisfying

$$4^M (L+1)^{1+s+\tau+\|\varphi\|_\infty} \delta^\tau < 1. \tag{5·2}$$

From $\underline{\mathrm{mdim}}_{\mathrm{H}}(\mathcal{X}, T, \mathbf{d}, \varphi) < s$, there is a sequence of natural numbers $N_1 < N_2 < N_3 < \dots$ such that

$$\dim_{\mathrm{H}}(\mathcal{X}, \mathbf{d}_{N_n}, \varphi_{N_n}, \delta) < s N_n^d \quad \text{for all } n \ge 1.$$

Then $\mathcal{H}_\delta^{sN_n^d}(\mathcal{X}, \mathbf{d}_{N_n}, \varphi_{N_n}) < 1$ and hence

$$\mathcal{H}_\delta^{(s+\tau)N_n^d}(\mathcal{X}, \mathbf{d}_{N_n}, \varphi_{N_n}) \le \delta^{\tau N_n^d} \mathcal{H}_\delta^{sN_n^d}(\mathcal{X}, \mathbf{d}_{N_n}, \varphi_{N_n}) < \delta^{\tau N_n^d}.$$

[4] The construction of f is as follows. Take a Lipschitz function $\psi\colon [0,\infty) \to [0,1]$ such that $\psi(t) = 1$ for $0 \le t \le \varepsilon/4$ and $\psi(t) = 0$ for $t \ge \varepsilon/2$. Let $\{x_1, \dots, x_M\}$ be a $(\varepsilon/4)$-spanning set of $\mathcal{X}$. Then we set $f(x) = (\psi(d(x,x_1)), \psi(d(x,x_2)), \dots, \psi(d(x,x_M)))$.

Therefore

$$4^{MN_n^d}(L+1)^{1+(s+\tau)N_n^d+\|\varphi_{N_n}\|_\infty}\,\mathcal{H}_1^{(s+\tau)N_n^d}\left(\mathcal{X},\mathbf{d}_{N_n},\varphi_{N_n}\right) < \left\{4^M(L+1)^{1+s+\tau+\|\varphi\|_\infty}\,\delta^\tau\right\}^{N_n^d}$$
$$< 1 \quad (\text{by } (5{\cdot}2)).$$

Here we have used $\|\varphi_{N_n}\|_\infty \le N_n^d\|\varphi\|_\infty$ in the first inequality. Now we can use Lemma 5.3 and conclude

$$\mathrm{Widim}'_\varepsilon\left(\mathcal{X},\mathbf{d}_{N_n},\varphi_{N_n}\right) \le (s+\tau)N_n^d + 1.$$

Therefore

$$\varliminf_{N\to\infty}\frac{\mathrm{Widim}'_\varepsilon\left(\mathcal{X},\mathbf{d}_N,\varphi_N\right)}{N^d} = \lim_{n\to\infty}\frac{\mathrm{Widim}'_\varepsilon\left(\mathcal{X},\mathbf{d}_{N_n},\varphi_{N_n}\right)}{N_n^d} \le s+\tau.$$

Since $\tau>0$ is arbitrary, we have

$$\varliminf_{N\to\infty}\frac{\mathrm{Widim}'_\varepsilon\left(\mathcal{X},\mathbf{d}_N,\varphi_N\right)}{N^d} \le s.$$

Thus, letting $\varepsilon\to 0$, we have $\mathrm{mdim}(\mathcal{X},T,\varphi)\le s$ by Lemma 5.2. $\square$

5.3. **Proof of Theorem 3.4.** Now we can prove Theorem 3.4. We write the statement again.

Theorem 5.5 (= Theorem 3.4)**.** *Let $T\colon \mathbb{R}^d\times\mathcal{X}\to\mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon\mathcal{X}\to\mathbb{R}$ a continuous function. Then we have*

$$\mathrm{mdim}(\mathcal{X},T,\varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X},T,\mathbf{d},\varphi\right).$$

Proof. We use the notations in Lemma 4.4. Namely, we denote by $T|_{\mathbb{Z}^d}\colon\mathbb{Z}^d\times\mathcal{X}\to\mathcal{X}$ the restriction of T to the subgroup $\mathbb{Z}^d$. We define a metric $\overline{\mathbf{d}}_1^{\mathbb{R}}$ and a function $\varphi_1^{\mathbb{R}}$ on $\mathcal{X}$ by

$$\overline{\mathbf{d}}_1^{\mathbb{R}}(x,y) = \int_{[0,1)^d}\mathbf{d}(T^u x, T^u y)\,du, \quad \varphi_1^{\mathbb{R}}(x) = \int_{[0,1)^d}\varphi(T^u x)\,du.$$

By Lemma 4.4

$$\mathrm{mdim}(\mathcal{X},T,\varphi) = \mathrm{mdim}\left(\mathcal{X},T|_{\mathbb{Z}^d},\varphi_1^{\mathbb{R}}\right)$$
$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X},T,\mathbf{d},\varphi\right) = \underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X},T|_{\mathbb{Z}^d},\overline{\mathbf{d}}_1^{\mathbb{R}},\varphi_1^{\mathbb{R}}\right).$$

By Lemma 4.2 there exists a metric $\mathbf{d}'$ on $\mathcal{X}$ such that $\mathbf{d}'(x,y)\le\overline{\mathbf{d}}_1^{\mathbb{R}}(x,y)$ for all $x,y\in\mathcal{X}$ and that $(\mathcal{X},\mathbf{d}')$ has the tame growth of covering numbers. By $\mathbf{d}'(x,y)\le\overline{\mathbf{d}}_1^{\mathbb{R}}(x,y)$ we have

$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X},T|_{\mathbb{Z}^d},\mathbf{d}',\varphi_1^{\mathbb{R}}\right) \le \underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X},T|_{\mathbb{Z}^d},\overline{\mathbf{d}}_1^{\mathbb{R}},\varphi_1^{\mathbb{R}}\right).$$

Since $(\mathcal{X},\mathbf{d}')$ has the tame growth of covering numbers, we can apply Proposition 4.3 to it and get

$$\underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X},T|_{\mathbb{Z}^d},\mathbf{d}',\varphi_1^{\mathbb{R}}\right) = \underline{\mathrm{mdim}}_{\mathrm{H}}\left(\mathcal{X},T|_{\mathbb{Z}^d},\mathbf{d}',\varphi_1^{\mathbb{R}}\right).$$

By Theorem 5.4

$$\mathrm{mdim}\left(\mathcal{X}, T|_{\mathbb{Z}^d}, \varphi_1^{\mathbb{R}}\right) \le \underline{\mathrm{mdim}}_{\mathrm{H}}\left(\mathcal{X}, T|_{\mathbb{Z}^d}, \mathbf{d}', \varphi_1^{\mathbb{R}}\right).$$

Combining all the above inequalities, we conclude

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H}, L^1}\left(\mathcal{X}, T, \mathbf{d}, \varphi\right).$$

$\square$

Remark 5.6. Since $\underline{\mathrm{mdim}}_{\mathrm{H},L^1}\left(\mathcal{X}, T, \mathbf{d}, \varphi\right) \le \underline{\mathrm{mdim}}_{\mathrm{H}}\left(\mathcal{X}, T, \mathbf{d}, \varphi\right)$, we also have

$$\mathrm{mdim}(\mathcal{X}, T, \varphi) \le \underline{\mathrm{mdim}}_{\mathrm{H}}\left(\mathcal{X}, T, \mathbf{d}, \varphi\right).$$

We can directly prove this inequality (for $\mathbb{R}^d$-actions) without using $\mathbb{Z}^d$-actions. The proof is almost the identical with the proof of Theorem 5.4. However we have not found a direct proof Theorem 3.4 (the L^1 version) so far.

6. Mean Hausdorff dimension is bounded by rate distortion dimension: proof of Theorem 3.7

In this section we prove Theorem 3.7 (dynamical Frostman's lemma). The proof is based on results on mutual information prepared in §2.2. Another key ingredient is the following version of Frostman's lemma. This was proved in [LT19, Corollary 4.4].

Lemma 6.1. *For any $0 < c < 1$ there exists $\delta_0 = \delta_0(c) \in (0,1)$ such that for any compact metric space $(\mathcal{X}, \mathbf{d})$ and any $0 < \delta \le \delta_0$ there exists a Borel probability measure ν on $\mathcal{X}$ satisfying*

$$\nu(E) \le (\mathrm{Diam}\, E)^{c\cdot \mathrm{dim}_{\mathrm{H}}(\mathcal{X}, \mathbf{d}, \delta)} \quad \textit{for all Borel sets } E \subset \mathcal{X} \textit{ with } \mathrm{Diam}\, E < \frac{\delta}{6}.$$

We also need the following elementary lemma. This was proved in [LT18, Appendix].

Lemma 6.2. *Let A be a finite set and $\{\mu_n\}$ a sequence of probability measures on A. Suppose that μ_n converges to some probability measure μ in the weak* topology (i.e. $\mu_n(a) \to \mu(a)$ for every $a \in A$). Then there exists a sequence of probability measures π_n on $A \times A$ such that*

- *π_n is a coupling between μ_n and μ, i.e. the first and second marginals of π_n are μ_n and μ respectively,*
- *π_n converges to $(\mathrm{id} \times \mathrm{id})_* \mu$ in the weak* topology, namely*

$$\pi_n(a,b) \to \begin{cases} \mu(a) & (\textit{if } a = b) \\ 0 & (\textit{if } a \ne b) \end{cases}.$$

We write the statement of Theorem 3.7 again.

Theorem 6.3 (= Theorem 3.7)**.** *Let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. Then we have*

$$\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) \leq \sup_{\mu \in \mathscr{M}^T(\mathcal{X})} \left(\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \right).$$

Here recall that $\mathscr{M}^T(\mathcal{X})$ is the set of all T-invariant Borel probability measures on $\mathcal{X}$.

Proof. Let c and s be arbitrary real numbers with $0 < c < 1$ and $s < \overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi)$. We will construct $\mu \in \mathscr{M}^T(\mathcal{X})$ satisfying

$$\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \geq cs - (1-c) \|\varphi\|_\infty . \tag{6·1}$$

If this is proved then we get the claim of the theorem by letting $c \to 1$ and $s \to \overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi)$.

Take $\eta > 0$ with $\overline{\mathrm{mdim}}_{\mathrm{H},L^1}(\mathcal{X}, T, \mathbf{d}, \varphi) > s + 2\eta$. Let $\delta_0 = \delta_0(c) \in (0,1)$ be the constant introduced in Lemma 6.1. There are $\delta \in (0, \delta_0)$ and a sequence of positive numbers $L_1 < L_2 < L_3 < \cdots \to \infty$ satisfying

$$\dim_{\mathrm{H}}\left(\mathcal{X}, \overline{\mathbf{d}}_{L_n}, \varphi_{L_n}, \delta\right) > (s+2\eta) L_n^d$$

for all $n \geq 1$.

For a real number t we set

$$\mathcal{X}_n(t) := \left(\frac{\varphi_{L_n}}{L_n^d}\right)^{-1} [t, t+\eta] = \left\{ x \in \mathcal{X} \,\middle|\, t \leq \frac{\varphi_{L_n}(x)}{L_n^d} \leq t + \eta \right\}.$$

Claim 6.4. *We can choose $t \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ such that for infinitely many n we have*

$$\dim_{\mathrm{H}}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}, \delta\right) \geq (s-t) L_n^d.$$

Notice that, in particular, this inequality implies that $\mathcal{X}_n(t)$ is not empty because we assumed that $\dim_{\mathrm{H}}(\cdot)$ is $-\infty$ for the empty set.

Proof. We have[5] $\mathcal{H}_\delta^{(s+2\eta)L_n^d}\left(\mathcal{X}, \overline{\mathbf{d}}_{L_n}, \varphi_{L_n}\right) \geq 1$. Set $m = \lceil 2 \|\varphi\|_\infty / \eta \rceil$. We have

$$\mathcal{X} = \bigcup_{\ell=0}^{m-1} \mathcal{X}_n\left(-\|\varphi\|_\infty + \ell\eta\right).$$

Then there exists $t \in \{-\|\varphi\|_\infty + \ell\eta \mid \ell = 0, 1, \dots, m-1\}$ such that

$$\mathcal{H}_\delta^{(s+2\eta)L_n^d}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}, \varphi_{L_n}\right) \geq \frac{1}{m} \quad \text{for infinitely many } n.$$

[5]The quantity $\mathcal{H}_\delta^{(s+2\eta)L_n^d}\left(\mathcal{X}, \overline{\mathbf{d}}_{L_n}, \varphi_{L_n}\right)$ is defined only when $(s+2\eta)L_n^d > \max \varphi_{L_n}$. Therefore the following argument is problematic if we have $(s+2\eta)L_n^d \leq \max \varphi_{L_n}$ for all but finitely many n. However, in this case, there is $t \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ such that $t \geq s + \eta$ and $\mathcal{X}_n(t) \neq \emptyset$ for infinitely many n. Then we have $\dim_{\mathrm{H}}(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}, \delta) \geq 0 > c(s-t)L_n^d$ for infinitely many n for this choice of t.

On the set $\mathcal{X}_n(t)$ we have

$$(s+2\eta)L_n^d - \varphi_{L_n} \geq (s+2\eta)L_n^d - (t+\eta)L_n^d = (s-t+\eta)L_n^d.$$

Hence

$$\begin{aligned}\mathcal{H}_\delta^{(s+2\eta)L_n^d}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}, \varphi_{L_n}\right) &\leq \mathcal{H}_\delta^{(s-t+\eta)L_n^d}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}\right) \\ &\leq \delta^{\eta L_n^d}\mathcal{H}_\delta^{(s-t)L_n^d}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}\right).\end{aligned}$$

Therefore for infinitely many n we have

$$\mathcal{H}_\delta^{(s-t)L_n^d}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}\right) \geq \frac{\delta^{-\eta L_n^d}}{m} \to \infty \quad (n \to \infty).$$

Thus $\dim_{\mathrm{H}}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}, \delta\right) \geq (s-t)L_n^d$ for infinitely many n. $\square$

We fix $t \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ satisfying the statement of this claim. By choosing a subsequence (also denoted by L_n) we can assume that

$$\dim_{\mathrm{H}}\left(\mathcal{X}_n(t), \overline{\mathbf{d}}_{L_n}, \delta\right) \geq (s-t)L_n^d \quad \text{for all } n.$$

By a version of Frostman's lemma (Lemma 6.1), there is a Borel probability measure ν_n on $\mathcal{X}_n(t)$ such that

(6·2)
$$\nu_n(E) \leq \left(\mathrm{Diam}\left(E, \overline{\mathbf{d}}_{L_n}\right)\right)^{c(s-t)L_n^d} \quad \text{for all Borel sets } E \subset \mathcal{X} \text{ with } \mathrm{Diam}\left(E, \overline{\mathbf{d}}_{L_n}\right) < \frac{\delta}{6}.$$

We define a Borel probability measure μ_n on $\mathcal{X}$ by

$$\mu_n = \frac{1}{L_n^d}\int_{[0,L_n)^d} T_*^u \nu_n \, du.$$

By choosing a subsequence (also denoted by μ_n) we can assume that μ_n converges to $\mu \in \mathscr{M}^T(\mathcal{X})$ in the weak* topology. We have

$$\int_{\mathcal{X}} \varphi \, d\mu_n = \int_{\mathcal{X}} \frac{\varphi_{L_n}}{L_n^d} \, d\nu_n = \int_{\mathcal{X}_n(t)} \frac{\varphi_{L_n}}{L_n^d} \, d\nu_n \geq t.$$

Here we have used that ν_n is supported in $\mathcal{X}_n(t)$ in the second inequality and that $\varphi_{L_n}/L_n^d \geq t$ on $\mathcal{X}_n(t)$ in the last inequality. Since $\mu_n \rightharpoonup \mu$, we have

$$\int_{\mathcal{X}} \varphi \, d\mu \geq t.$$

If $t \geq s$ then (6·1) trivially holds (recalling $0 < c < 1$):

$$\underline{\mathrm{rdim}}(\mathcal{X}, T, \mathbf{d}, \mu) + \int_{\mathcal{X}} \varphi \, d\mu \geq t \geq cs - (1-c)\|\varphi\|_\infty.$$

Therefore we assume $s > t$.

We will prove that for sufficiently small $\varepsilon > 0$

$$R(\mathbf{d}, \mu, \varepsilon) \geq c(s-t)\log(1/\varepsilon) - Kc(s-t), \tag{6·3}$$

where $R(\mathbf{d},\mu,\varepsilon)$ is the rate distortion function and K is the universal positive constant introduced in Proposition 2.13. Once this is proved, we have

$$\underline{\mathrm{rdim}}(\mathcal{X},T,\mathbf{d},\mu) = \liminf_{\varepsilon\to 0}\frac{R(\mathbf{d},\mu,\varepsilon)}{\log(1/\varepsilon)} \geq c(s-t).$$

Then we get (6·1) by

$$\underline{\mathrm{rdim}}(\mathcal{X},T,\mathbf{d},\mu) + \int_{\mathcal{X}}\varphi\, d\mu \geq c(s-t)+t = cs+(1-c)t \geq cs-(1-c)\left\|\varphi\right\|_\infty.$$

Here we have used $0<c<1$ and $t\geq -\left\|\varphi\right\|_\infty$. So the task is to prove (6·3).

Let ε be a positive number with $2\varepsilon\log(1/\varepsilon) < \delta/6$. Let M be a positive number, and let X and Y be random variables such that

- X takes values in $\mathcal{X}$ with $\mathrm{Law}X = \mu$,
- Y takes values in $L^1([0,M)^d,\mathcal{X})$ with $\mathbb{E}\left(\int_{[0,M)^d}\mathbf{d}(T^vX,Y_v)\,dv\right) < \varepsilon M^d$.

We want to prove

$$\frac{1}{M^d}I(X;Y) \geq c(s-t)\log(1/\varepsilon) - Kc(s-t).$$

If this is proved then we get (6·3) and the proof is done. We can assume that Y takes only finitely many values (Remark 2.15). We denote the set of values of Y by $\mathcal{Y}$. This is a finite subset of $L^1([0,M)^d,\mathcal{X})$.

Take a positive number τ satisfying $\mathbb{E}\left(\int_{[0,M)^d}\mathbf{d}(T^vX,Y_v)\,dv\right) < (\varepsilon-3\tau)M^d$. We take a measurable partition

$$\mathcal{X} = P_1\cup P_2\cup\cdots\cup P_\alpha \quad \text{(disjoint union)}$$

such that $\mathrm{Diam}(P_i,\overline{\mathbf{d}}_M)<\tau$ and $\mu(\partial P_i)=0$ for all $1\leq i\leq\alpha$. We pick a point $x_i\in P_i$ for each i and set $A=\{x_1,\dots,x_\alpha\}$. We define a map $\mathcal{P}\colon\mathcal{X}\to A$ by $\mathcal{P}(P_i)=\{x_i\}$. Then we have

$$\mathbb{E}\left(\frac{1}{M^d}\int_{[0,M)^d}\mathbf{d}\left(T^v\mathcal{P}(X),Y_v\right)\,dv\right) < \varepsilon-2\tau. \tag{6·4}$$

We consider the push-forward measures $\mathcal{P}_*\mu_n$ on A. They converge to $\mathcal{P}_*\mu$ as $n\to\infty$ in the weak* topology by $\mu(\partial P_i)=0$.

By Lemma 6.2, we can construct random variables $X(n)$ coupled to $\mathcal{P}(X)$ such that $X(n)$ take values in A with $\mathrm{Law}X(n)=\mathcal{P}_*\mu_n$ and

$$\mathbb{P}\left(X(n)=x_i,\mathcal{P}(X)=x_j\right)\to\delta_{ij}\mathbb{P}\left(\mathcal{P}(X)=x_j\right)\quad (n\to\infty).$$

Then $\mathbb{E}\overline{\mathbf{d}}_M\left(X(n),\mathcal{P}(X)\right)\to 0$ as $n\to\infty$. We consider[6] that $X(n)$ is coupled to Y with the conditional distribution

$$\mathbb{P}\left(X(n)=x_i, Y=y \middle| \mathcal{P}(X)=x_j\right)=\mathbb{P}\left(X(n)=x_i \middle| \mathcal{P}(X)=x_j\right)\cdot\mathbb{P}\left(Y=y \middle| \mathcal{P}(X)=x_j\right)$$

for $x_i,x_j\in A$ and $y\in\mathcal{Y}$. Namely $X(n)$ and Y are conditionally independent given $\mathcal{P}(X)$. Then

$$\begin{aligned}\mathbb{P}\left(X(n)=x_i, Y=y\right) &= \sum_{j=1}^{\alpha}\mathbb{P}\left(X(n)=x_i,\mathcal{P}(X)=x_j\right)\cdot\mathbb{P}\left(Y=y \middle| \mathcal{P}(X)=x_j\right)\\ &\to \mathbb{P}\left(\mathcal{P}(X)=x_i, Y=y\right) \quad (n\to\infty).\end{aligned}$$

By (6·4)

$$\mathbb{E}\left(\frac{1}{M^d}\int_{[0,M)^d}\mathbf{d}\left(T^uX(n),Y_u\right)du\right)<\varepsilon-2\tau \quad \text{for large } n. \tag{6·5}$$

Notice that $(X(n),Y)$ take values in a fixed finite set $A\times\mathcal{Y}$ and that their distributions converge to that of $(\mathcal{P}(X),Y)$. Hence by Lemma 2.6

$$I\left(X(n);Y\right)\to I\left(\mathcal{P}(X);Y\right) \quad (n\to\infty).$$

We want to estimate $I\left(X(n);Y\right)$ from below.

Fix a point $x_0\in\mathcal{X}$. We will also denote by x_0 any constant function whose value is x_0. For $x\in A$ and $y\in L^1([0,M)^d,\mathcal{X})$ we define a conditional probability mass function by

$$\rho_n(y|x)=\mathbb{P}\left(Y=y \middle| X(n)=x\right).$$

This is nonzero only for $y\in\mathcal{Y}$. (Here $\rho_n(\cdot|x)$ may be an arbitrary probability measure on $\mathcal{Y}$ if $\mathbb{P}\left(X(n)=x\right)=0$.)

We define $\Lambda\subset\mathbb{R}^d$ by

$$\Lambda=\left\{(Mm_1,Mm_2,\dots,Mm_d)\middle| m_k\in\mathbb{Z},\ 0\le m_k\le\frac{L_n}{M}-2\,(1\le k\le d)\right\}.$$

Let $v\in[0,M)^d$. We have

$$\bigcup_{\lambda\in\Lambda}\left(v+\lambda+[0,M)^d\right)\subset[0,L_n)^d.$$

[6]This sentence is not rigorous. Strictly speaking, we can construct random variables $X(n),X',Y'$ defined on a common probability space such that $\mathrm{Law}\,(X',Y')=\mathrm{Law}\,(\mathcal{P}(X),Y)$,

$$\mathbb{P}\left(X(n)=x_i,X'=x_j\right)\to\delta_{ij}\mathbb{P}\left(X'=x_j\right) \quad (n\to\infty),$$

and

$$\mathbb{P}\left(X(n)=x_i,Y=y \middle| X'=x_j\right)=\mathbb{P}\left(X(n)=x_i \middle| X'=x_j\right)\cdot\mathbb{P}\left(Y=y \middle| X'=x_j\right).$$

For simplicity we identify X' and Y' with $\mathcal{P}(X)$ and Y respectively.

Notice that the left-hand side is a disjoint union. Here $v+\lambda+[0,M)^d = \{v+\lambda+w \mid w \in [0,M)^d\}$. Set

$$E_v = [0,L_n)^d \setminus \bigcup_{\lambda\in\Lambda} \left(v+\lambda+[0,M)^d\right).$$

See Figure 3.

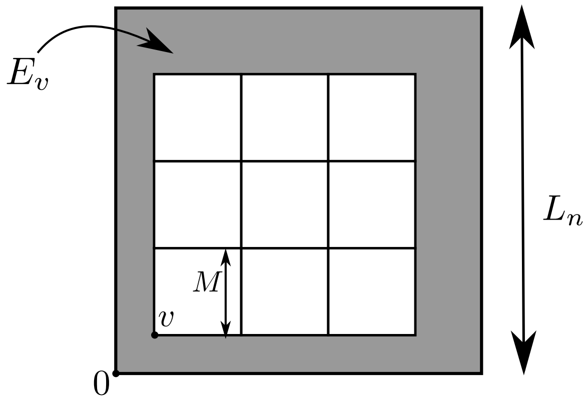

FIGURE 3. The big square is $[0,L_n)^d$ and small squares are $v+\lambda+[0,M)^d$ $(\lambda\in\Lambda)$. The shadowed region is E_v.

For $x\in\mathcal{X}$ and $f\in L^1\left([0,L_n)^d,\mathcal{X}\right)$ we define a conditional probability mass function $\sigma_{n,v}(f|x)$ by

$$\sigma_{n,v}(f|x) = \delta_{x_0}\left(f|_{E_v}\right)\cdot\prod_{\lambda\in\Lambda}\rho_n\left(f|_{v+\lambda+[0,M)^d}\middle|\,\mathcal{P}(T^{v+\lambda}x)\right).$$

Here $f|_{E_v}$ is the restriction of f to E_v and δ_{x_0} is the delta probability measure concentrated at the constant function $x_0\in L^1(E_v,\mathcal{X})$. We naturally consider $f|_{v+\lambda+[0,M)^d}$ (the restriction of f to $v+\lambda+[0,M)^d$) as an element of $L^1\left([0,M)^d,\mathcal{X}\right)$.

We define a transition probability σ_n on $\mathcal{X}\times L^1([0,L_n)^d,\mathcal{X})$ by

$$\sigma_n(B\mid x) = \frac{1}{M^d}\int_{[0,M)^d}\sigma_{n,v}(B\mid x)\,dv$$

for $x\in\mathcal{X}$ and Borel subsets $B\subset L^1([0,L_n)^d,\mathcal{X})$. Here $\sigma_{n,v}(B\mid x)=\sum_{f\in B}\sigma_{n,v}(f|x)$. (Notice that for each $x\in\mathcal{X}$ the function $\sigma_{n,v}(f|x)$ is nonzero only for finitely many f.)

Take random variables Z and W such that Z takes values in $\mathcal{X}$ with $\mathrm{Law}Z=\nu_n$ and that W takes values in $L^1([0,L_n)^d,\mathcal{X})$ with

$$\mathbb{P}\left(W\in B|\,Z=x\right) = \sigma_n(B\mid x)\quad (x\in\mathcal{X}, B\subset L^1([0,L_n)^d,\mathcal{X})).$$

Notice that Z and W depend on n. Rigorously speaking, we should denote them by $Z^{(n)}$ and $W^{(n)}$. However we suppress their dependence on n in the notations for simplicity.

We estimate $\mathbb{E}\left(\frac{1}{L_n^d}\int_{[0,L_n)^d}\mathbf{d}(T^uZ,W_u)\,du\right)$ and $I(Z;W)$.

Claim 6.5. *For all sufficiently large n we have*

$$\mathbb{E}\left(\frac{1}{L_n^d}\int_{[0,L_n)^d}\mathbf{d}(T^uZ,W_u)\,du\right)<\varepsilon.$$

Proof. For each $v \in [0, M)^d$ we take a random variable $W(v)$ coupled to Z such that $W(v)$ takes values in $L^1([0, L_n)^d, \mathcal{X})$ with $\mathbb{P}(W(v) = f \mid Z = x) = \sigma_{n,v}(f|x)$ for $x \in \mathcal{X}$ and $f \in L^1([0, L_n)^d, \mathcal{X})$. Then

$$\mathbb{E} \int_{[0,L_n)^d} \mathbf{d}(T^u Z, W_u)\, du = \frac{1}{M^d} \int_{[0,M)^d} \mathbb{E} \left(\int_{[0,L_n)^d} \mathbf{d}(T^u Z, W(v)_u)\, du \right) dv.$$

For each $v \in [0, M)^d$ we have

$$\begin{aligned}
&\mathbb{E} \int_{[0,L_n)^d} \mathbf{d}(T^u Z, W(v)_u)\, du \\
&= \mathbb{E} \int_{E_v} \mathbf{d}(T^u Z, W(v)_u)\, du + \sum_{\lambda \in \Lambda} \mathbb{E} \int_{v+\lambda+[0,M)^d} \mathbf{d}(T^u Z, W(v)_u)\, du \\
&\leq C L_n^{d-1} + \sum_{\lambda \in \Lambda} \mathbb{E} \int_{v+\lambda+[0,M)^d} \mathbf{d}(T^u Z, W(v)_u)\, du,
\end{aligned}$$

where C is a positive constant independent of v, L_n. In the last inequality we have used $\mathbf{m}(E_v) \leq \text{const} \cdot L_n^{d-1}$. Since $\overline{\mathbf{d}}_M(x, \mathcal{P}(x)) < \tau$ for every $x \in \mathcal{X}$, we have

$$\begin{aligned}
&\mathbb{E} \int_{v+\lambda+[0,M)^d} \mathbf{d}(T^u Z, W(v)_u)\, du \\
&\leq M^d \tau + \mathbb{E} \int_{[0,M)^d} \mathbf{d}\left(T^u \mathcal{P}(T^{v+\lambda} Z), W(v)_{v+\lambda+u}\right) du \\
&\leq M^d \tau + \sum_{f \in \mathcal{Y}} \int_{[0,M)^d} \left(\int_{\mathcal{X}} \mathbf{d}(T^u x, f_u) \rho_n(f|x)\, d\left(\mathcal{P}_* T_*^{v+\lambda} \nu_n(x)\right) \right) du.
\end{aligned}$$

We sum up these estimates over $\lambda \in \Lambda$. Noting $M^d |\Lambda| \leq L_n^d$, we have

$$\begin{aligned}
&\mathbb{E} \int_{[0,L_n)^d} \mathbf{d}(T^u Z, W(v)_u)\, du \\
&\leq C L_n^{d-1} + \tau L_n^d + \sum_{\substack{\lambda \in \Lambda \\ f \in \mathcal{Y}}} \int_{[0,M)^d} \left(\int_{\mathcal{X}} \mathbf{d}(T^u x, f_u) \rho_n(f|x) d\left(\mathcal{P}_* T_*^{v+\lambda} \nu_n(x)\right) \right) du.
\end{aligned}$$

We integrate this over $v \in [0, M)^d$. Note that[7]

$$\begin{aligned}
\int_{[0,M)^d} \left(\sum_{\lambda \in \Lambda} \mathcal{P}_* T_*^{v+\lambda} \nu_n \right) dv &= \sum_{\lambda \in \Lambda} \int_{\lambda+[0,M)^d} \mathcal{P}_* T_*^v \nu_n\, dv \\
&\leq \int_{[0,L_n)^d} \mathcal{P}_* T_*^v \nu_n\, dv = L_n^d \mathcal{P}_* \mu_n.
\end{aligned}$$

[7] For two measures $\mathbf{m}_1$ and $\mathbf{m}_2$ on $\mathcal{X}$ we write $\mathbf{m}_1 \leq \mathbf{m}_2$ if we have $\mathbf{m}_1(B) \leq \mathbf{m}_2(B)$ for all Borel subsets $B \subset \mathcal{X}$.

Hence we have

$$\begin{aligned}
&\frac{1}{M^d}\int_{[0,M)^d}\mathbb{E}\left(\int_{[0,L_n)^d}\mathbf{d}(T^u Z, W(v)_u)\,du\right)dv \\
&\le CL_n^{d-1}+\tau L_n^d+\frac{L_n^d}{M^d}\sum_{f\in\mathcal{Y}}\int_{[0,M)^d}\left(\int_{\mathcal{X}}\mathbf{d}(T^u x, f_u)\rho_n(f|x)d\mathcal{P}_*\mu_n(x)\right)du \\
&= CL_n^{d-1}+\tau L_n^d+\frac{L_n^d}{M^d}\mathbb{E}\left(\int_{[0,M)^d}\mathbf{d}\left(T^u X(n), Y_u\right)du\right).
\end{aligned}$$

In the last equality we have used $\mathrm{Law}X(n)=\mathcal{P}_*\mu_n$ and $\rho_n(f|x)=\mathbb{P}(Y=f\mid X(n)=x)$. Therefore

$$\mathbb{E}\left(\frac{1}{L_n^d}\int_{[0,L_n)^d}\mathbf{d}(T^u Z, W_u)\,du\right)\le\frac{C}{L_n}+\tau+\mathbb{E}\left(\frac{1}{M^d}\int_{[0,M)^d}\mathbf{d}\left(T^u X(n), Y_u\right)du\right).$$

The third term in the right-hand side is smaller than $\varepsilon-2\tau$ for large n by (6·5). Therefore we have

$$\mathbb{E}\left(\frac{1}{L_n^d}\int_{[0,L_n)^d}\mathbf{d}(T^u Z, W_u)\,du\right)<\varepsilon$$

for all sufficiently large n. $\square$

Claim 6.6.

$$\frac{1}{L_n^d}I(Z;W)\le\frac{1}{M^d}I\left(X(n);Y\right).$$

Proof. We have $I(Z;W)=I(\nu_n,\sigma_n)$. Since $\sigma_n=(1/M^d)\int_{[0,M)^d}\sigma_{n,v}\,dv$, we use Proposition 2.10 (2) (the convexity of mutual information in transition probability):

$$I(\nu_n,\sigma_n)\le\frac{1}{M^d}\int_{[0,M)^d}I(\nu_n,\sigma_{n,v})\,dv.$$

By Lemma 2.7 (subadditivity of mutual information)[8]

$$I(\nu_n,\sigma_{n,v})\le\sum_{\lambda\in\Lambda}I\left(\mathcal{P}_*T_*^{v+\lambda}\nu_n,\rho_n\right).$$

[8]Let $W(v)$ be the random variable introduced in Claim 6.5. Then $I(\nu_n,\sigma_{n,v})=I(Z;W(v))$. Consider the restrictions $W(v)|_{v+\lambda+[0,M)^d}$ $(\lambda\in\Lambda)$ and $W(v)|_{E_v}$. From the definition of the measure $\sigma_{n,v}$, they are conditionally independent given Z. By Lemma 2.7

$$I(Z;W)\le I\left(Z;W(v)|_{E_v}\right)+\sum_{\lambda\in\Lambda}I\left(Z;W(v)|_{v+\lambda+[0,M)^d}\right).$$

$I\left(Z;W(v)|_{E_v}\right)=0$ because $W(v)|_{E_v}$ is constantly equal to x_0. We have

$$I\left(Z;W(v)|_{v+\lambda+[0,M)^d}\right)=I\left(\mathcal{P}(T^{v+\lambda}Z);W(v)|_{v+\lambda+[0,M)^d}\right)=I\left(\mathcal{P}_*T_*^{v+\lambda}\nu_n,\rho_n\right).$$

Hence

$$\begin{aligned} I(\nu_n, \sigma_n) &\leq \frac{1}{M^d} \sum_{\lambda \in \Lambda} \int_{[0,M)^d} I\left(\mathcal{P}_* T_*^{\lambda+v} \nu_n, \rho_n\right) dv \\ &= \frac{1}{M^d} \int_{\cup_{\lambda\in\Lambda}\left(\lambda + [0,M)^d\right)} I\left(\mathcal{P}_* T_*^{v} \nu_n, \rho_n\right) dv \\ &\leq \frac{1}{M^d} \int_{[0,L_n)^d} I\left(\mathcal{P}_* T_*^{v} \nu_n, \rho_n\right) dv \end{aligned}$$

By Proposition 2.10 (1) (the concavity of mutual information in probability measure)

$$\begin{aligned} \frac{1}{L_n^d} \int_{[0,L_n)^d} I\left(\mathcal{P}_* T_*^{v} \nu_n, \rho_n\right) dv &\leq I\left(\frac{1}{L_n^d} \int_{[0,L_n)^d} \mathcal{P}_* T_*^{v} \nu_n \, dv, \rho_n\right) \\ &= I\left(\mathcal{P}_* \mu_n, \rho_n\right) \\ &= I(X(n); Y). \end{aligned}$$

Therefore we conclude

$$I(Z; W) = I(\nu_n, \sigma_n) \leq \frac{L_n^d}{M^d} I(X(n); Y).$$

$\square$

We define a metric D_n on $L^1\left([0, L_n)^d, \mathcal{X}\right)$ by

$$D_n(f, g) = \frac{1}{L_n^d} \int_{[0,L_n)^d} \mathbf{d}\left(f(u), g(u)\right) du.$$

Then the map

$$F_n \colon \left(\mathcal{X}, \overline{\mathbf{d}}_{L_n}\right) \ni x \mapsto \left(T^t x\right)_{t \in [0,L_n)^d} \in \left(L^1\left([0, L_n)^d, \mathcal{X}\right), D_n\right)$$

is an isometric embedding. Consider the push-forward measure $F_{n*}\nu_n$ on $L^1\left([0, L_n)^d, \mathcal{X}\right)$. It follows from (6·2) that

$$F_{n*}\nu_n\left(E\right) \leq \left(\mathrm{Diam}(E, D_n)\right)^{c(s-t)L_n^d}$$

for all Borel subsets $E \subset L^1\left([0, L_n)^d, \mathcal{X}\right)$ with $\mathrm{Diam}(E, D_n) < \delta/6$.

We have $\mathrm{Law} F_n(Z) = F_{n*}\nu_n$, and by Claim 6.5

$$\mathbb{E}\left(D_n(F_n(Z), W)\right) < \varepsilon \quad \text{for large } n.$$

Since $2\varepsilon \log(1/\varepsilon) < \delta/6$, we apply Proposition 2.13 (Kawabata–Dembo estimate) to $(F_n(Z), W)$ and get

$$I(Z; W) = I\left(F_n(Z); W\right) \geq c(s-t) L_n^d \log(1/\varepsilon) - K\left(c(s-t)L_n^d + 1\right)$$

for large n. Here K is a universal positive constant. By Claim 6.6

$$\frac{1}{M^d} I\left(X(n); Y\right) \geq \frac{1}{L_n^d} I\left(Z; W\right) \geq c(s-t)\log(1/\varepsilon) - K\left(c(s-t) + \frac{1}{L_n^d}\right)$$

for large n. Since $I(X(n);Y) \to I(\mathcal{P}(X);Y)$ as $n \to \infty$, we get

$$\frac{1}{M^d} I(\mathcal{P}(X);Y) \geq c(s-t)\log(1/\varepsilon) - Kc(s-t).$$

Then we have

$$\frac{1}{M^d} I(X;Y) \geq \frac{1}{M^d} I(\mathcal{P}(X);Y) \geq c(s-t)\log(1/\varepsilon) - Kc(s-t).$$

This is what we want to prove. $\square$

Now we have proved Theorems 3.4 and 3.7. These two theorems implies Theorem 1.3 (Main theorem) as we already explained in §3. Therefore we have completed the proof of Theorem 1.3.

7. Local nature of metric mean dimension with potential

This section is independent of the proof of Theorem 1.3 (Main Theorem). It can be read independently of Sections 2, 4, 5 and 6. Here we present a formula expressing metric mean dimension with potential by using a certain *local* quantity. We plan to use it in a future study of geometric examples of dynamical systems [Gro99, Tsu18].

7.1. A formula of metric mean dimension with potential.

Let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. For a subset $E \subset \mathcal{X}$ and $\varepsilon > 0$ we set

$$P_T(E, \mathbf{d}, \varphi, \varepsilon) = \liminf_{L\to\infty} \frac{\log \#\left(E, \mathbf{d}_L, \varphi_L, \varepsilon\right)}{L^d}.$$

Here recall that

$$\#\left(E, \mathbf{d}_L, \varphi_L, \varepsilon\right) = \inf\left\{ \sum_{i=1}^n (1/\varepsilon)^{\sup_{U_i}\varphi} \,\middle|\, \begin{array}{c} E \subset U_1\cup\cdots\cup U_n. \text{ Each } U_i \text{ is an open set} \\ \text{of } \mathcal{X} \text{ with } \mathrm{Diam}(U_i, \mathbf{d}_L) < \varepsilon. \end{array} \right\}.$$

Also recall that the upper and lower metric mean dimensions with potential are defined by

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \limsup_{\varepsilon\to 0} \frac{P_T(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon)}{\log(1/\varepsilon)},$$

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \liminf_{\varepsilon\to 0} \frac{P_T(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon)}{\log(1/\varepsilon)}.$$

For a (not necessarily bounded) subset A of $\mathbb{R}^d$ we define a metric $\mathbf{d}_A$ on $\mathcal{X}$ by

$$\mathbf{d}_A(x,y) = \sup_{a\in A} \mathbf{d}\left(T^a x, T^a y\right).$$

(If A is unbounded, this metric is not compatible with the given topology of $\mathcal{X}$ in general.) For $x \in \mathcal{X}$ and $\delta > 0$ we define $B_\delta(x, \mathbf{d}_{\mathbb{R}^d})$ as the closed δ-ball with respect to $\mathbf{d}_{\mathbb{R}^d}$:

$$B_\delta(x, \mathbf{d}_{\mathbb{R}^d}) = \{y \in \mathcal{X} \mid \mathbf{d}_{\mathbb{R}^d}(x,y) \leq \delta\} = \{y \in \mathcal{X} \mid \mathbf{d}(T^u x, T^u y) \leq \delta \ (\forall u \in \mathbb{R}^d)\}.$$

The following is the main result of this section.

Theorem 7.1. *For any $\delta > 0$ we have*

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \limsup_{\varepsilon \to 0} \frac{\sup_{x\in\mathcal{X}} P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)},$$

$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \liminf_{\varepsilon \to 0} \frac{\sup_{x\in\mathcal{X}} P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)}.$$

Notice that $B_\delta(x, \mathbf{d}_{\mathbb{R}^d})$ is not a neighborhood of x with respect to the original metric $\mathbf{d}$ in general. Nevertheless we can calculate the metric mean dimension with potential by gathering such information.

In the case that φ is identically zero, Theorem 7.1 was proved in [Tsu22]. The proof of Theorem 7.1 follows the argument of [Tsu22], which is in turn based on the paper of Bowen [Bow72].

7.2. **Tiling argument.** Here we prepare a technical lemma (Lemma 7.2 below). For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $\|x\|_\infty = \max_{1\le i\le d} |x_i|$. A **cube** of $\mathbb{R}^d$ is a set Λ of the form

$$\Lambda = u + [0, L]^d = \{u + v \mid v \in [0, L]^d\},$$

where $u \in \mathbb{R}^d$ and $L > 0$. We set $\ell(\Lambda) = L$. For $r > 0$ and $A \subset \mathbb{R}^d$ we define

$$\partial(A, r) = \left\{x \in \mathbb{R}^d \mid \exists y \in A, \exists z \in \mathbb{R}^d \setminus A : \|x - y\|_\infty \le r \text{ and } \|x - z\|_\infty \le r\right\},$$

$$B_r(A) = A \cup \partial(A, r) = \{x \in \mathbb{R}^d \mid \exists y \in A : \|x - y\|_\infty \le r\}.$$

For a finite set $\mathcal{C} = \{\Lambda_1, \dots, \Lambda_n\}$ of cubes of $\mathbb{R}^d$ we set

$$\ell_{\min}(\mathcal{C}) = \min_{1\le i\le n} \ell(\Lambda_i), \quad \ell_{\max}(\mathcal{C}) = \max_{1\le i\le n} \ell(\Lambda_i).$$

The following lemma was proved in [Tsu22, Proposition 3.4].

Lemma 7.2. *For any $\eta > 0$ there exists a natural number $k_0 = k_0(\eta) > 0$ for which the following statement holds. Let A be a bounded Borel subset of $\mathbb{R}^d$. Let $\mathcal{C}_k$ $(1 \le k \le k_0)$ be finite sets of cubes of $\mathbb{R}^d$ such that*

1. *$\ell_{\max}(\mathcal{C}_1) \ge 1$ and $\ell_{\min}(\mathcal{C}_{k+1}) \ge k_0 \cdot \ell_{\max}(\mathcal{C}_k)$ for all $1 \le k \le k_0 - 1$,*
2. *$\mathbf{m}\left(\partial(A, \ell_{\max}(\mathcal{C}_{k_0}))\right) < (\eta/3) \cdot \mathbf{m}(A)$,*
3. *$A \subset \bigcup_{\Lambda\in\mathcal{C}_k} \Lambda$ for every $1 \le k \le k_0$.*

Then there is a disjoint subfamily $\mathcal{C}' \subset \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{k_0}$ satisfying

$$\bigcup_{\Lambda\in\mathcal{C}'} \Lambda \subset A, \quad \mathbf{m}\left(B_1\left(A \setminus \bigcup_{\Lambda\in\mathcal{C}'} \Lambda\right)\right) < \eta \cdot \mathbf{m}(A).$$

Here "disjoint" means that for any two distinct $\Lambda_1, \Lambda_2 \in \mathcal{C}'$ we have $\Lambda_1 \cap \Lambda_2 = \emptyset$.

This is a rather technical statement. The assumption (1) means that some cube of $\mathcal{C}_1$ is not so small and that every cube in $\mathcal{C}_{k+1}$ is much larger than cubes in $\mathcal{C}_k$. The assumption (2) means that A is much larger than all the given cubes. The assumption (3) means that each $\mathcal{C}_k$ covers A. The conclusion means that we can find a disjoint subfamily $\mathcal{C}'$ which covers a substantial portion of A.

7.3. **The case that φ is nonnegative.** This subsection is also a preparation for the proof of Theorem 7.1. Let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. Throughout this subsection, we assume that φ is a nonnegative function.

Recall that for a bounded Borel subset $A \subset \mathbb{R}^d$ a new function $\varphi_A\colon \mathcal{X} \to \mathbb{R}$ is defined by

$$\varphi_A(x) = \int_A \varphi(T^u x)\, du.$$

Lemma 7.3. *Let $0 < \varepsilon < 1$ and $E \subset \mathcal{X}$. Let $A, A_1, A_2, \dots, A_n$ be bounded Borel subsets of $\mathbb{R}^d$. If $A \subset A_1 \cup A_2 \cup \dots \cup A_n$ then*

$$\#\left(E, \mathbf{d}_A, \varphi_A, \varepsilon\right) \le \prod_{k=1}^{n} \#\left(E, \mathbf{d}_{A_k}, \varphi_{A_k}, \varepsilon\right).$$

Proof. Suppose we are given an open cover $E \subset U_{k1} \cup \dots \cup U_{km_k}$ with $\mathrm{Diam}(U_{kj}, \mathbf{d}_{A_k}) < \varepsilon$ for each $1 \le k \le n$. Then

$$E \subset \bigcup \left\{U_{1j_1} \cap U_{2j_2} \cap \dots \cap U_{nj_n} \middle| 1 \le j_1 \le m_1, 1 \le j_2 \le m_2, \dots, 1 \le j_n \le m_n\right\}.$$

From $A \subset A_1 \cup \dots \cup A_n$, the diameter of $U_{1j_1} \cap U_{2j_2} \cap \dots \cap U_{nj_n}$ is smaller than ε with respect to the metric $\mathbf{d}_A$. Since φ is nonnegative (here we use this assumption), we have

$$\varphi_A \le \varphi_{A_1} + \varphi_{A_2} + \dots + \varphi_{A_n}$$

and hence

$$\sup_{U_{1j_1} \cap \dots \cap U_{nj_n}} \varphi_A \le \sum_{k=1}^{n} \sup_{U_{kj_k}} \varphi_{A_k}.$$

Therefore we have

$$\sum_{\substack{1 \le j_1 \le m_1 \\ \vdots \\ 1 \le j_n \le m_n}} \left(\frac{1}{\varepsilon}\right)^{\sup_{U_{1j_1} \cap \dots \cap U_{nj_n}} \varphi_A} \le \prod_{k=1}^{n} \left(\sum_{j=1}^{m_k} \left(\frac{1}{\varepsilon}\right)^{\sup_{U_{kj}} \varphi_{A_k}}\right).$$

This proves the claim of the lemma. $\square$

Lemma 7.4. *For $0 < \varepsilon < 1$ and a bounded Borel subset $A \subset \mathbb{R}^d$, we have*

$$\#\left(\mathcal{X}, \mathbf{d}_A, \varphi_A, \varepsilon\right) \le \left\{\#\left(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon\right)\right\}^{\mathbf{m}(B_1(A))}.$$

Notice that $\#\left(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon\right) \ge 1$ because $0 < \varepsilon < 1$ and φ is nonnegative.

Proof. Let Ω be the set of $u \in \mathbb{Z}^d$ with $\left(u + [0,1]^d\right) \cap A \neq \emptyset$. We have

$$A \subset \bigcup_{u \in \Omega} \left(u + [0,1]^d\right) \subset B_1(A).$$

In particular the cardinality of Ω is bounded from above by $\mathbf{m}(B_1(A))$. Then by Lemma 7.3

$$\begin{aligned}\#\left(\mathcal{X}, \mathbf{d}_A, \varphi_A, \varepsilon\right) &\leq \prod_{u\in\Omega} \#\left(\mathcal{X}, \mathbf{d}_{u+[0,1]^d}, \varphi_{u+[0,1]^d}, \varepsilon\right) \\ &= \prod_{u\in\Omega} \#\left(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon\right) \\ &\leq \left\{\#\left(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon\right)\right\}^{\mathbf{m}(B_1(A))}.\end{aligned}$$

□

The following is the main result of this subsection. This is a modification of a classical result of Bowen [Bow72, Proposition 2.2]. Here recall that we have assumed that φ is nonnegative.

Proposition 7.5. *For positive numbers δ, β and $0 < \varepsilon < 1$ there is a positive number $D = D(\delta, \beta, \varepsilon)$ for which the following statement holds. Set*

$$a = \frac{\sup_{x\in\mathcal{X}} P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)}.$$

Then for all sufficiently large L we have

$$\sup_{x\in\mathcal{X}} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right) \leq \left(\frac{1}{\varepsilon}\right)^{(a+\beta)L^d}.$$

Here $B_\delta(x, \mathbf{d}_{[-D,L+D]^d}) = \{y \in \mathcal{X} \mid \mathbf{d}_{[-D,L+D]^d}(x,y) \leq \delta\}$.

Proof. Choose a positive number η satisfying

$$(7{\cdot}1) \qquad \left(\#(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon)\right)^\eta < \left(\frac{1}{\varepsilon}\right)^{\frac{\beta}{2}}.$$

Let $k_0 = k_0(\eta)$ be the natural number introduced in Lemma 7.2.

We will construct the following data inductively on $k = 1, 2, \dots, k_0$.

- A finite set $Y_k \subset \mathcal{X}$.
- Positive numbers $L_k(y)$ and $M_k(y)$ for each $y \in Y_k$.
- Open neighborhoods $V_k(y)$ and $U_k(y)$ of y in $\mathcal{X}$ with $V_k(y) \subset U_k(y)$ for each $y \in Y_k$.

We assume the following conditions.

(1) $L_1(y) > 1$ for all $y \in Y_1$.
(2) $L_k(y) > k_0 L_{k-1}(z)$ for all $y \in Y_k$, $z \in Y_{k-1}$ and $2 \leq k \leq k_0$.
(3) $\#\left(U_k(y), \mathbf{d}_{L_k(y)}, \varphi_{L_k(y)}, \varepsilon\right) < (1/\varepsilon)^{(a+\frac{\beta}{2})L_k(y)^d}$ for all $y \in Y_k$.
(4) $B_\delta(v, \mathbf{d}_{[-M_k(y), M_k(y)]^d}) \subset U_k(y)$ for all $y \in Y_k$ and $v \in V_k(y)$.
(5) $X = \bigcup_{y\in Y_k} V_k(y)$ for every $1 \leq k \leq k_0$.

The construction of these data go as follows. Suppose that the data of $(k-1)$-th step (i.e. $Y_{k-1}, L_{k-1}(y), M_{k-1}(y), V_{k-1}(y), U_{k-1}(y)$) have been constructed. We consider the k-th step. (The case of $k=1$ is similar.)

Take an arbitrary $y \in \mathcal{X}$. Since we have

$$\frac{P_T\left(B_\delta(y, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)} \leq a < a + \frac{\beta}{2},$$

there is a positive number $L_k(y)$ larger than $k_0 \max_{z \in Y_{k-1}} L_{k-1}(z)$ (we assume $L_1(y) > 1$ in the case of $k=1$) satisfying

$$\frac{1}{L_k(y)^d} \log \#\left(B_\delta(y, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}_{L_k(y)}, \varphi_{L_k(y)}, \varepsilon\right) < \left(a + \frac{\beta}{2}\right) \log(1/\varepsilon).$$

Then there is an open set $U_k(y) \supset B_\delta(y, \mathbf{d}_{\mathbb{R}^d})$ such that

$$\frac{1}{L_k(y)^d} \log \#\left(U_k(y), \mathbf{d}_{L_k(y)}, \varphi_{L_k(y)}, \varepsilon\right) < \left(a + \frac{\beta}{2}\right) \log(1/\varepsilon).$$

Namely we have

$$\#\left(U_k(y), \mathbf{d}_{L_k(y)}, \varphi_{L_k(y)}, \varepsilon\right) < \left(\frac{1}{\varepsilon}\right)^{(a+\frac{\beta}{2})L_k(y)^d}.$$

Since $B_\delta(y, \mathbf{d}_{\mathbb{R}^d}) = \bigcap_{M=1}^\infty B_\delta(y, \mathbf{d}_{[-M,M]^d}) \subset U_k(y)$, there is a positive number $M_k(y)$ with $B_\delta(y, \mathbf{d}_{[-M_k(y),M_k(y)]^d}) \subset U_k(y)$. There is an open neighborhood $V_k(y)$ of y such that for every $v \in V_k(y)$ we have $B_\delta(v, \mathbf{d}_{[-M_k(y),M_k(y)]^d}) \subset U_k(y)$. Since $\mathcal{X}$ is compact we can find a finite set $Y_k \subset \mathcal{X}$ satisfying $\mathcal{X} = \bigcup_{y \in Y_k} V_k(y)$. The construction of the k-th step has been finished.

Take $D > 1$ with $D > \max\{M_k(y) \mid 1 \leq k \leq k_0, y \in Y_k\}$. Let L be a sufficiently large number so that the cube $A := [0, L]^d$ satisfies $\mathbf{m}\left(\partial(A, \max_{y \in Y_{k_0}} L_{k_0}(y))\right) < \frac{\eta}{3} L^d$. We will show that $\sup_{x \in \mathcal{X}} \#\left(B_\delta(x, \mathbf{d}_{[-D, L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right) \leq \left(\frac{1}{\varepsilon}\right)^{(a+\beta)L^d}$.

Take an arbitrary point $x \in \mathcal{X}$. For each $1 \leq k \leq k_0$ and $t \in A \cap \mathbb{Z}^d$ we pick $y \in Y_k$ with $T^t x \in V_k(y)$. Set $\Lambda_{k,t} = t + [0, L_k(y)]^d$. It follows from the choice of D that

$$T^t\left(B_\delta(x, \mathbf{d}_{[-D, L+D]^d})\right) \subset B_\delta(T^t x, \mathbf{d}_{[-M_k(y), M_k(y)]^d}) \subset U_k(y).$$

Hence

$$\#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_{\Lambda_{k,t}}, \varphi_{\Lambda_{k,t}}, \varepsilon\right) \leq \#\left(U_k(y), \mathbf{d}_{L_k(y)}, \varphi_{L_k(y)}, \varepsilon\right) < \left(\frac{1}{\varepsilon}\right)^{(a+\frac{\beta}{2})L_k(y)^d}.$$

Set $\mathcal{C}_k = \{\Lambda_{k,t} \mid t \in A \cap \mathbb{Z}^d\}$. This is a finite family of cubes covering $A = [0, L]^d$. Notice that $\mathcal{C}_k$ depends on the choice of x; we suppress its dependence on x in the notation for simplicity.

By Lemma 7.2 there is a disjoint subfamily $\mathcal{C}' \subset \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k_0}$ such that

$$\bigcup_{\Lambda \in \mathcal{C}'} \Lambda \subset A, \quad \mathbf{m}\left(B_1\left(A \setminus \bigcup_{\Lambda \in \mathcal{C}'} \Lambda\right)\right) < \eta\, \mathbf{m}(A).$$

Set

$$A' = A \setminus \bigcup_{\Lambda \in \mathcal{C}'} \Lambda.$$

For every $\Lambda \in \mathcal{C}'$ we have

$$\#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_\Lambda, \varphi_\Lambda, \varepsilon\right) < \left(\frac{1}{\varepsilon}\right)^{(a+\frac{\beta}{2})\mathbf{m}(\Lambda)}.$$

By Lemma 7.4

$$\begin{aligned}
\#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_{A'}, \varphi_{A'}, \varepsilon\right) &\le \#\left(\mathcal{X}, \mathbf{d}_{A'}, \varphi_{A'}, \varepsilon\right) \\
&\le \left\{\#\left(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon\right)\right\}^{\mathbf{m}(B_1(A'))} \\
&< \left\{\#\left(\mathcal{X}, \mathbf{d}_{[0,1]^d}, \varphi_{[0,1]^d}, \varepsilon\right)\right\}^{\eta\, \mathbf{m}(A)} \\
&< \left(\frac{1}{\varepsilon}\right)^{\frac{\beta}{2}\mathbf{m}(A)}.
\end{aligned}$$

In the last inequality we have used (7·1). From Lemma 7.3

$$\begin{aligned}
&\#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_A, \varphi_A, \varepsilon\right) \\
&\le \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_{A'}, \varphi_{A'}, \varepsilon\right) \prod_{\Lambda \in \mathcal{C}'} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_\Lambda, \varphi_\Lambda, \varepsilon\right) \\
&< \left(\frac{1}{\varepsilon}\right)^{(a+\beta)\mathbf{m}(A)}.
\end{aligned}$$

This holds for every point $x \in \mathcal{X}$. Thus we have proved the claim of the proposition. $\square$

7.4. **Proof of Theorem 7.1.** Here we prove Theorem 7.1. Let $T\colon \mathbb{R}^d \times \mathcal{X} \to \mathcal{X}$ be a continuous action of $\mathbb{R}^d$ on a compact metrizable space $\mathcal{X}$. Let $\mathbf{d}$ be a metric on $\mathcal{X}$ and $\varphi\colon \mathcal{X} \to \mathbb{R}$ a continuous function. We do not assume that φ is nonnegative.

The next proposition looks the same as Proposition 7.5. The point is that we do not assume the nonnegativity of φ here whereas we assumed it in Proposition 7.5.

Proposition 7.6. *Let $\delta, \beta, \varepsilon$ be positive numbers with $0 < \varepsilon < 1$. Set*

$$a = \sup_{x \in \mathcal{X}} \frac{P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)}.$$

Then for all sufficiently large L we have

$$\sup_{x \in \mathcal{X}} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right) \le \left(\frac{1}{\varepsilon}\right)^{(a+\beta)L^d}.$$

Here $D = D(\delta, \beta, \varepsilon)$ is the positive constant[9] *introduced in Proposition 7.5.*

[9]Strictly speaking, the constant D depends on not only $\delta, \beta, \varepsilon$ but also $(\mathcal{X}, T, \mathbf{d}, \psi)$ where $\psi := \varphi - \min_{\mathcal{X}} \varphi$.

Proof. Set $c = \min_{x\in\mathcal{X}} \varphi(x)$ and $\psi(x) = \varphi(x) - c$. We have $\psi(x) \geq 0$. For any positive number L we have

$$\psi_L(x) = \varphi_L(x) - cL^d.$$

For any subset $E \subset \mathcal{X}$

$$\#\left(E, \mathbf{d}_L, \psi_L, \varepsilon\right) = (1/\varepsilon)^{-cL^d} \#\left(E, \mathbf{d}_L, \varphi_L, \varepsilon\right).$$

Hence

$$P_T(E, \mathbf{d}, \psi, \varepsilon) = P_T(E, \mathbf{d}, \varphi, \varepsilon) - c\log(1/\varepsilon),$$

$$\sup_{x\in\mathcal{X}} \frac{P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \psi, \varepsilon\right)}{\log(1/\varepsilon)} = a - c.$$

Since ψ is a nonnegative function, we apply Proposition 7.5 and get

$$\sup_{x\in\mathcal{X}} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \psi_L, \varepsilon\right) \leq \left(\frac{1}{\varepsilon}\right)^{(a-c+\beta)L^d}$$

for sufficiently large L. We have

$$\#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \psi_L, \varepsilon\right) = \left(\frac{1}{\varepsilon}\right)^{-cL^d} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right).$$

Therefore

$$\sup_{x\in\mathcal{X}} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right) \leq \left(\frac{1}{\varepsilon}\right)^{(a+\beta)L^d}.$$

□

Now we prove Theorem 7.1. We write the statement again.

Theorem 7.7 (= Theorem 7.1)**.** *For any positive number δ*

$$\overline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \limsup_{\varepsilon\to 0} \frac{\sup_{x\in\mathcal{X}} P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)},$$
$$\underline{\mathrm{mdim}}_{\mathrm{M}}(\mathcal{X}, T, \mathbf{d}, \varphi) = \liminf_{\varepsilon\to 0} \frac{\sup_{x\in\mathcal{X}} P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)}. \tag{7·2}$$

Proof. It is obvious that the left-hand sides of (7·2) are greater than or equal to the right-hand sides. So it is enough to prove the reverse inequalities. Let β and ε be arbitrary positive numbers with $0 < \varepsilon < 1$. Let $D = D(\delta, \beta, \varepsilon)$ be the positive constant introduced in Proposition 7.5. Set

$$a = \sup_{x\in\mathcal{X}} \frac{P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right)}{\log(1/\varepsilon)}.$$

For any positive number L we can take points $x_1, \dots, x_M \in \mathcal{X}$ such that

$$\mathcal{X} = \bigcup_{m=1}^{M} B_\delta(x_m, \mathbf{d}_{[-D,L+D]^d}),$$

$$M \leq \#\left(\mathcal{X}, \mathbf{d}_{[-D,L+D]^d}, \delta\right) = \#\left(\mathcal{X}, \mathbf{d}_{[0,L+2D]^d}, \delta\right).$$

Then we have

$$
\begin{aligned}
\#\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right) &\leq \sum_{m=1}^{M} \#\left(B_\delta(x_m, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right) \\
&\leq M \sup_{x\in\mathcal{X}} \#\left(B_\delta(x, \mathbf{d}_{[-D,L+D]^d}), \mathbf{d}_L, \varphi_L, \varepsilon\right) \\
&\leq M \left(\frac{1}{\varepsilon}\right)^{(a+\beta)L^d}.
\end{aligned}
$$

The last inequality holds for all sufficiently large L by Proposition 7.6. Therefore

$$\log \#\left(\mathcal{X}, \mathbf{d}_L, \varphi_L, \varepsilon\right) \leq \log \#\left(\mathcal{X}, \mathbf{d}_{L+2D}, \delta\right) + (a+\beta)L^d \log(1/\varepsilon).$$

Dividing this by L^d and letting $L \to \infty$, we have

$$
\begin{aligned}
P_T(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon) &\leq \lim_{L\to\infty} \frac{\log \#\left(\mathcal{X}, \mathbf{d}_L, \delta\right)}{L^d} + (a+\beta)\log(1/\varepsilon) \\
&\leq \log \#\left(\mathcal{X}, \mathbf{d}_1, \delta\right) + (a+\beta)\log(1/\varepsilon).
\end{aligned}
$$

We can let $\beta \to 0$ and get

$$
\begin{aligned}
P_T(\mathcal{X}, \mathbf{d}, \varphi, \varepsilon) &\leq \log \#\left(\mathcal{X}, \mathbf{d}_1, \delta\right) + a\log(1/\varepsilon) \\
&= \log \#\left(\mathcal{X}, \mathbf{d}_1, \delta\right) + \sup_{x\in\mathcal{X}} P_T\left(B_\delta(x, \mathbf{d}_{\mathbb{R}^d}), \mathbf{d}, \varphi, \varepsilon\right).
\end{aligned}
$$

We divide this by $\log(1/\varepsilon)$ and let $\varepsilon \to 0$. Then we conclude that the left-hand sides of (7·2) are less than or equal to the right-hand sides. $\square$

References

[Bow72] R. Bowen, Entropy-expansive maps, Trans. Amer. Math. Soc. **164** (1972) 323-331.

[CT06] T. M. Cover, J. A. Thomas, Elements of information theory, second edition, Wiley, New York, 2006.

[Din70] E. I. Dinaburg, A correlation between topological entropy and metric entropy, Dokl. Akad. Nauk SSSR **190** (1970) 19-22.

[Dur10] R. Durrett, Probability: theory and examples, Fourth edition, Cambridge Series in Statistical and Probabilistic Mathematics, 31. Cambridge University Press, Cambridge, 2010.

[Goodm71] T. N. T. Goodman, Relating topological entropy and measure entropy, Bull. London Math. Soc. **3** (1971) 176-180.

[Goodw69] L. W. Goodwyn, Topological entropy bounds measure-theoretic entropy, Proc. Amer. Math. Soc. **23** (1969) 679-688.

[Gra09] R. M. Gray, Probability, random processes, and ergodic properties, second edition, Springer, Dordrecht, 2009.

[Gra11] R. M. Gray, Entropy and information theory, second edition, Springer, New York, 2011.

[Gro99] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, Math. Phys. Anal. Geom. **2** (1999) 323-415.

[GLT16] Y. Gutman, E. Lindenstrauss, M. Tsukamoto, Mean dimension of $\mathbb{Z}^k$-actions, Geom. Funct. Anal. **26** (2016) 778-817.

[GQT19] Y. Gutman, Y. Qiao, M. Tsukamoto, Application of signal analysis to the embedding problem of $\mathbb{Z}^k$-actions, Geom. Funct. Anal. **29** (2019) 1440-1502.

[GŚ20] Y. Gutman, A. Śpiewak, Metric mean dimension and analog compression, IEEE Trans. Inf. Theory, vol. 66, no. 11 (2020) 6977–6998.

[GT20] Y. Gutman, M. Tsukamoto, Embedding minimal dynamical systems into Hilbert cubes, Invent. Math. **221** (2020) 113-166.

[HY] Q. Huo, R. Yuan, Double variational principle for mean dimension of $\mathbb{Z}^K$-actions, arXiv:2310.03194.

[IW81] N. Ikeda, S. Watanabe, Stochastic differential equations and diffusion processes, second edition, North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1989.

[KD94] T. Kawabata and A. Dembo, The rate distortion dimension of sets and measures, IEEE Trans. Inf. Theory, vol. 40, no. 5, pp. 1564-1572, Sep. 1994.

[Lin99] E. Lindenstrauss, Mean dimension, small entropy factors and an embedding theorem, Inst. Hautes Études Sci. Publ. Math. vol. 89 pp. 227-262, 1999.

[LT18] E. Lindenstrauss, M. Tsukamoto, From rate distortion theory to metric mean dimension: variational principle, IEEE Trans. Inform. Theory **64** (2018) 3590-3609.

[LT19] E. Lindenstrauss, M. Tsukamoto, Double variational principle for mean dimension, Geom. Funct. Anal. **29** (2019) 1048–1109.

[LW00] E. Lindenstrauss, B. Weiss, Mean topological dimension, Israel J. Math. **115** (2000) 1-24.

[MT15] S. Matsuo, M. Tsukamoto, Brody curves and mean dimension, J. Amer. Math. Soc. **28** (2015) 159-182.

[Mat95] P. Mattila, Geometry of sets and measures in Euclidean spaces, Fractals and rectifiability, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.

[OS87] D. S. Ornstein, B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, J. Analyse Math. **48** (1987) 1-141.

[PG77] M. B. Pursley, R. M. Gray, Source coding theorems for stationary, continuous-time stochastic processes, Ann. Probab. **5** (1977) 966-986.

[Rue73] D. Ruelle, Statistical mechanics on a compact set with Z^ν action satisfying expansiveness and specification, Trans. Amer. Math. Soc. **185** (1973) 237-251.

[Sri98] S. M. Srivastava, A course on Borel sets, Grad. Texts in Math., 180, Springer-Verlag, New York, 1998

[Tsu18] M. Tsukamoto, Mean dimension of the dynamical systems of Brody curves, Invent. Math. **211** (2018) 935-968.

[Tsu20] M. Tsukamoto, Double variational principle for mean dimension with potential, Adv. Math. **361** (2020) 106935.

[Tsu22] M. Tsukamoto, Remark on the local nature of metric mean dimension, Kyushu J. Math. **76** (2022) 143-162.

[Wal75] P. Walters, A variational principle for the pressure of continuous transformations, Amer. J. Math. **17** (1975) 937-971.

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