MULTIPLE ZETA-STAR VALUES FOR INDICES OF INFINITE LENGTH

MINORU HIROSE, HIDEKI MURAHARA, AND TOMOKAZU ONOZUKA

Abstract. In this paper, we consider infinite-length versions of multiple zeta-star values. We give several explicit formulas for the infinite-length versions of multiple zeta-star values. We also discuss analytic properties of the map from indices to the infinite-length versions of multiple zeta-star values.

1. Main results

The multiple zeta-star value is the convergent series

$$\zeta^\star(k_1,\dots,k_r)=\sum_{n_1\ge\cdots\ge n_r\ge1}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}$$

for $(k_1,\dots,k_r)\in\mathbb{Z}_{\ge1}^r$ with $k_1\ge2$ and has been studied variously, along with the multiple zeta values. In this paper, we consider the infinite length version of the multiple zeta-star values.

Definition 1.1. For $(k_1,k_2,\dots)\in\mathbb{Z}_{\ge1}^\infty$ with $k_1\ge2$, we define multiple zeta-star values for indices of infinite length by

$$\zeta^\star(k_1,k_2,\dots)=\sum_{m_1\ge m_2\ge\cdots\ge1}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots}\qquad(=\lim_{r\to\infty}\zeta^\star(k_1,\dots,k_r)),$$

where the summation is over the all decreasing sequence $(m_j)_{j=1}^\infty$ of positive integers such that $\lim_{r\to\infty}m_r=1$.

We will see later that the above sum converges except for the case where $k_1=2$ and $k_j=1$ for all $j>1$ (see Section 3). First, we will show some formulas for the multiple zeta-star values for indices of infinite length. Let $\{k\}^r$ denote the r times repetition of the k, e.g., $(\{k\}^3)=(k,k,k)$. We use the summation symbol $\sum'$ in an extended meaning of $\sum$, i.e., ${\sum'}_{j=a}^{b-1}$ means $-\sum_{j=b}^{a-1}$ if $b<a$, 0 if $b=a$, $\sum_{j=a}^{b-1}$ if $b>a$.

Theorem 1.2. *We have the following equalities:*

(1) *For $k_1,\dots,k_r\in\mathbb{Z}_{\ge1}^r$ with $k_1\ge2$,*

$$\zeta^\star(k_1,\dots,k_{r-1},k_r+1,\{1\}^\infty)=\zeta^\star(k_1,\dots,k_r).$$

(2) *For $k_1,\dots,k_r\in\mathbb{Z}_{\ge1}^r$ with $k_1\ge2$,*

$$\begin{aligned}&\zeta^\star(k_1,\dots,k_r,\{2\}^\infty)\\&=(-1)^{k_1+\cdots+k_r}\left(2-2\sum_{s=1}^{r}{\sum_{j=2}^{k_s-1}}'(-1)^{k_1+\cdots+k_{s-1}+j}\zeta^\star(k_1,\dots,k_{s-1},j)\right).\end{aligned}$$

2020 *Mathematics Subject Classification.* Primary 11M32.
Key words and phrases. Multiple zeta values; Multiple zeta-star values; Indices of infinite length.

(3) *For* $k \geq 2$,
$$\zeta^\star(\{k\}^\infty) = \prod_{m=2}^{\infty} \left(\frac{m^k}{m^k - 1} \right) = \prod_{c^k=1} \Gamma(2-c).$$
(4) *For* $n \geq 2$,
$$\zeta^\star(\{2, \{1\}^{n-2}\}^\infty) = n.$$
(5) *For* $n \geq 1$,
$$\zeta^\star(\{\{2\}^n, 1\}^\infty) = 2 \prod_{c^{2n+1}=1} \frac{\Gamma(2-c)}{\Gamma(2+c)}.$$
(6) *For* $n \geq 0$,
$$\zeta^\star(\{\{2\}^n, 3, \{2\}^n, 1\}^\infty) = 2 \prod_{s \in \{\pm 1\}} \prod_{c^{2n+2}=s} \Gamma(2-c)^{-s} \Gamma\left(1 - \frac{c}{2}\right)^{2s}.$$

Example 1.3. We have the following equalities:
(1) $\zeta^\star(\{4\}^\infty) = \frac{8\pi}{e^\pi - e^{-\pi}}$,
(2) $\zeta^\star(3, \{2\}^\infty) = 2\zeta(2) - 2$,
(3) $\zeta^\star(\{3,1\}^\infty) = \frac{4(e^\pi+1)}{\pi(e^\pi-1)}$,
(4) $\zeta^\star(\{2\}^\infty) = 2$,
(5) $\zeta^\star(\{2,1\}^\infty) = 3$.

Second, let us define $Z^\star : [0,1] \to [1,\infty]$ by $Z^\star(0) = 1$ and
$$Z^\star\left(\sum_{j=1}^{\infty} \frac{1}{2^{k_1+\cdots+k_j}} \right) = \zeta^\star(k_1+1, k_2, k_3, \dots),$$
where $k_1, k_2, \dots \in \mathbb{Z}_{\geq 1}$. For example, we have
$$Z^\star\left(\frac{1}{2}\right) = Z^\star\left(\sum_{j=2}^{\infty} \frac{1}{2^j} \right) = \zeta^\star(3,1,1,\dots) = \zeta(2).$$
The function $Z^\star$ contains information of all multiple zeta-star values for indices of infinite length (see Figure 1 for the graph of $Z^\star$). Given two indices $(k_1, k_2, \dots)$ and $(l_1, l_2, \dots)$, we say that the former is lexicographically smaller than the latter if there exists j such that $k_i = l_i$ and $k_j < l_j$ ($i \in \{1, \dots, j-1\}$).

Theorem 1.4. *$Z^\star$ is a continuous and bijective function, or equivalently, the map*
$$(k_1, k_2, k_3, \dots) \mapsto \zeta^\star(k_1+1, k_2, k_3, \dots)$$
gives an order-reversing bijection between $(\mathbb{Z}_{\geq 1}^\infty, \prec)$ *and* $(1, \infty]$ *where* $\prec$ *is the lexicographic order.*

Remark 1.5. The order structure for the set of multiple zeta values is studied by Kumar [1].

Remark 1.6. Li, independently of our study, obtained the same results as Theorem 1.2 (1), (4), Theorem 1.4, and further studied related topics. For more details, see [2].

Theorem 1.7. *The function $Z^\star$ on $z \in [0,1)$ is not differentiable on some dense set. More precisely, we have the followings:*
(1) *The map $Z^\star$ is right-differentiable at z for $0 \leq z < 1$.*
(2) *The map $Z^\star$ is left-differentiable at z if $z \notin \{1 - \frac{1}{2^n} \mid n > 0\}$.*

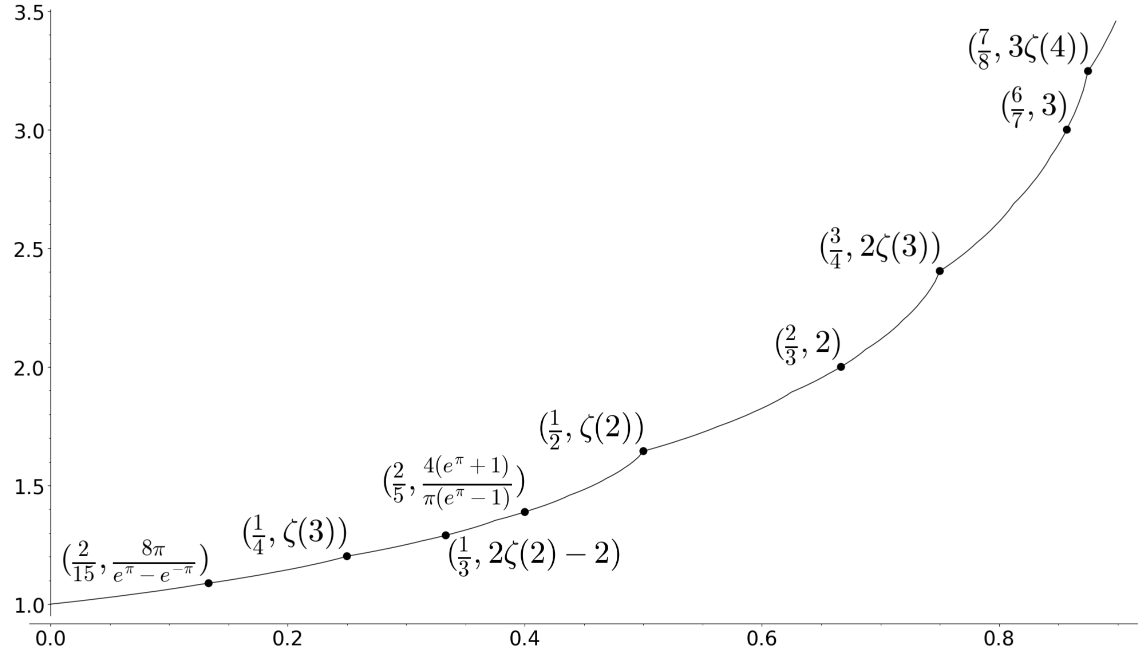

FIGURE 1. The graph and some special values of $Z^\star$.

(3) *The map* $Z^\star$ *is not left-differentiable at* z *if* $z \in \{1 - \frac{1}{2^n} \mid n > 0\}$.
(4) *The left-differential* $\partial_- Z^\star(z)$ *is equal to the right-differential* $\partial_+ Z^\star(z)$ *if* $z \in (0,1) \setminus \left\{\frac{a}{2^n} \mid 0 < a < 2^n, n > 0\right\}$.
(5) *The left-differential* $\partial_- Z^\star(z)$ *is greater than the right differential* $\partial_+ Z^\star(z)$ *if* $z \in \left\{\frac{a}{2^n} \mid 0 < a < 2^n - 1, n > 0\right\}$.

The proof of Theorem 1.7 will be given in Theorems 4.7, 4.8, 4.10, and 5.5 (see also Remark 4.9).

2. Special values

Lemma 2.1. *For* $k_1, \ldots, k_r, l \in \mathbb{Z}_{\geq 1}$ *with* $k_1 > 1$, *we have*

$$\zeta^\star(k_1, \ldots, k_r, \{l\}^\infty) = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \prod_{s=2}^{m_r} \frac{s^l}{s^l - 1}.$$

Proof. It follows from the following calculation:

$$\begin{aligned}
&\zeta^\star(k_1, \ldots, k_r, \{l\}^\infty) \\
&= \lim_{R \to \infty} \sum_{m_1 \geq \cdots \geq m_r \geq n_1 \geq \cdots \geq n_R \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^l \cdots n_R^l} \\
&= \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \lim_{R \to \infty} \sum_{m_r \geq n_1 \geq \cdots \geq n_R \geq 1} \frac{1}{n_1^l \cdots n_R^l} \\
&= \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \lim_{R \to \infty} \sum_{c_1 + \cdots + c_{m_r} = R} \prod_{s=1}^{m_r} \frac{1}{s^{lc_s}} \qquad (c_s := \#\{j : n_j = s\}) \\
&= \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \sum_{c_2, \ldots, c_{m_r} = 0}^{\infty} \prod_{s=2}^{m_r} \frac{1}{s^{lc_s}}
\end{aligned}$$

$$= \sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \prod_{s=2}^{m_r} \sum_{c=0}^{\infty} \frac{1}{s^{lc}}$$
$$= \sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \prod_{s=2}^{m_r} \frac{s^l}{s^l-1}. \qquad \square$$

Proof of Theorem 1.2 (1). By Lemma 2.1, we have

$$\begin{aligned}\zeta^\star(k_1,\dots,k_{r-1},k_r+1,\{1\}^\infty) &= \sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r+1}} \prod_{s=2}^{m_r} \frac{s}{s-1}\\ &= \sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\\ &= \zeta^\star(k_1,\dots,k_r). \qquad \square\end{aligned}$$

Proof of Theorem 1.2 (2). Let $L(k_1,\dots,k_r)$ (resp. $R(k_1,\dots,k_r)$) be the left (resp. right) hand side of the theorem. By Lemma 2.1, we have

$$\begin{aligned}L(k_1,\dots,k_r) &= \sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \prod_{m=2}^{m_r} \frac{m^2}{m^2-1}\\ &= 2\sum_{m_1\geq\cdots\geq m_r\geq 1} \frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} \cdot \frac{1}{m_r^{k_r-1}(m_r+1)}.\end{aligned}$$

Thus,

$$\begin{aligned}&L(k_1,\dots,k_r,a)+L(k_1,\dots,k_r,a+1)\\ &= 2\sum_{m_1\geq\cdots\geq m_r\geq n\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \cdot \frac{1}{n+1}\left(\frac{1}{n^{a-1}}+\frac{1}{n^a}\right)\\ &= 2\sum_{m_1\geq\cdots\geq m_r\geq n\geq 1} \frac{1}{m_1^{k_1}\cdots m_r^{k_r} n^a}\\ &= 2\zeta^\star(k_1,\dots,k_r,a).\end{aligned}$$

On the other hand, by definition,

$$R(k_1,\dots,k_r,a)+R(k_1,\dots,k_r,a+1) = 2\zeta^\star(k_1,\dots,k_r,a).$$

Thus we have

$$L(k_1,\dots,k_r,a)+L(k_1,\dots,k_r,a+1) = R(k_1,\dots,k_r,a)+R(k_1,\dots,k_r,a+1).$$

Furthermore, we have

$$\begin{aligned}L(k_1,\dots,k_r,2) &= L(k_1,\dots,k_r),\\ R(k_1,\dots,k_r,2) &= R(k_1,\dots,k_r),\end{aligned}$$

and $L(2)=2=R(2)$. Thus the claim follows by induction. $\square$

Proof of Theorem 1.2 (3). We have

$$\zeta^\star(\{k\}^\infty) = \sum_{m_1\geq m_2\geq\cdots\geq 1} \frac{1}{m_1^k m_2^k\cdots} = \prod_{m=2}^{\infty}\sum_{s=0}^{\infty} \frac{1}{m^{ks}} = \prod_{m=2}^{\infty} \frac{m^k}{m^k-1}.$$

Since

$$m^k - 1 = \prod_{c^k=1} (m-c),$$

we have

$$\begin{aligned}\prod_{m=2}^{\infty} \frac{m^k}{m^k-1} &= \lim_{n\to\infty} \prod_{m=2}^{n} \prod_{c^k=1} \frac{m}{m-c} \\ &= \prod_{c^k=1} \lim_{n\to\infty} \prod_{m=2}^{n} \frac{2+(m-2)}{2-c+(m-2)}.\end{aligned}$$

Using

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)},$$

we obtain

$$\prod_{m=2}^{\infty} \frac{m^k}{m^k-1} = \prod_{c^k=1} \frac{\Gamma(2-c)}{\Gamma(2)} = \prod_{c^k=1} \Gamma(2-c). \qquad \square$$

Proof of Theorem 1.2 (4). It is known that $\zeta^\star(\{2,\{1\}^{n-2}\}^a,1) = n\zeta(an+1)$ (see [3] and [5]). Thus, we have

$$\zeta^\star(\{2,\{1\}^{n-2}\}^\infty) = \lim_{a\to\infty} \zeta^\star(\{2,\{1\}^{n-2}\}^a,1) = n. \qquad \square$$

Proof of Theorem 1.2 (5). Using [4, Theorem 1.2], we have

$$\begin{aligned}\zeta^\star(\{\{2\}^n,1\}^d) &= \sum_{m_1\geq\cdots\geq m_d\geq 1} \frac{2^{\#\{m_1,\dots,m_d\}}}{m_1^{2n+1}\cdots m_d^{2n+1}} \\ &= \sum_{\substack{(c_1,c_2,\dots)\in\mathbb{Z}_{\geq 0}^\infty \\ \sum_{m=1}^\infty c_m = d}} \prod_{\substack{m\geq 1 \\ c_m\geq 1}} \frac{2}{m^{(2n+1)c_m}},\end{aligned}$$

where we put $c_m = \#\{j : m_j = m\}$. Then

$$\zeta^\star(\{\{2\}^n,1\}^d) = \sum_{\substack{(c_2,c_3,\dots)\in\mathbb{Z}_{\geq 0}^\infty \\ \sum_{m=2}^\infty c_m = d}} \prod_{\substack{m\geq 2 \\ c_m\geq 1}} \frac{2}{m^{(2n+1)c_m}} + 2 \sum_{\substack{(c_2,c_3,\dots)\in\mathbb{Z}_{\geq 0}^\infty \\ \sum_{m=2}^\infty c_m < d}} \prod_{\substack{m\geq 2 \\ c_m\geq 1}} \frac{2}{m^{(2n+1)c_m}}.$$

Since

$$\begin{aligned}\lim_{d\to\infty} \sum_{\substack{(c_2,c_3,\dots)\in\mathbb{Z}_{\geq 0}^\infty \\ \sum_{m=2}^\infty c_m = d}} \prod_{\substack{m\geq 2 \\ c_m\geq 1}} \frac{2}{m^{(2n+1)c_m}} &= \lim_{d\to\infty} \sum_{m_1\geq\cdots\geq m_d\geq 2} \frac{2^{\#\{m_1,\dots,m_d\}}}{m_1^{2n+1}\cdots m_d^{2n+1}} \\ &\leq \lim_{d\to\infty} \sum_{m_1\geq\cdots\geq m_d\geq 2} \frac{1}{m_1^{2n}\cdots m_d^{2n}} \\ &\leq \lim_{d\to\infty} \left(\sum_{m\geq 2} \frac{1}{m^{2n}}\right)^d = \lim_{d\to\infty} (\zeta(2n)-1)^d = 0\end{aligned}$$

and

$$\lim_{d\to\infty}\sum_{\substack{(c_2,c_3,\dots)\in\mathbb{Z}_{\ge 0}^{\infty}\\ \sum_{m=2}^{\infty}c_m<d}}\prod_{\substack{m\ge 2\\ c_m\ge 1}}\frac{2}{m^{(2n+1)c_m}}=\sum_{(c_2,c_3,\dots)\in\mathbb{Z}_{\ge 0}^{\infty}}\prod_{\substack{m\ge 2\\ c_m\ge 1}}\frac{2}{m^{(2n+1)c_m}}$$
$$=\prod_{m=2}^{\infty}\left(1+2\sum_{c=1}^{\infty}\frac{1}{m^{(2n+1)c}}\right),$$

we have

$$\zeta^\star(\{\{2\}^n,1\}^\infty)=2\prod_{m=2}^{\infty}\frac{m^{2n+1}+1}{m^{2n+1}-1}.$$

We obtain the result by a similar calculation as in the proof of Theorem 1.2 (3). □

Proof of Theorem 1.2 (6). Using the equation [4, Theorem 4.8 (2-c-2-1)] with $c_1=\dots=c_r=3$ and $a_1=\dots=a_r=b_1=\dots=b_r=n$, we have

$$\zeta^\star(\{\{2\}^n,3,\{2\}^n,1\}^d)=\sum_{m_1\ge\dots\ge m_{2d}\ge 1}\frac{(-1)^{m_1+\dots+m_{2d}}2^{\#\{m_1,\dots,m_{2d}\}}}{m_1^{2n+2}\cdots m_{2d}^{2n+2}}.$$

Using this equality, we have

$$\lim_{d\to\infty}\zeta^\star(\{\{2\}^n,3,\{2\}^n,1\}^d)=2\prod_{m=2}^{\infty}\left(1+2\left(\frac{(-1)^m}{m^{2n+2}}+\frac{(-1)^{2m}}{m^{2(2n+2)}}+\cdots\right)\right)$$
$$=2\prod_{m=2}^{\infty}\left(1+\frac{2}{(-1)^m m^{2n+2}-1}\right)$$
$$=2\prod_{m=2}^{\infty}\left(\frac{m^{2n+2}+(-1)^m}{m^{2n+2}-(-1)^m}\right).$$

Then we find

$$\lim_{d\to\infty}\zeta^\star(\{\{2\}^n,3,\{2\}^n,1\}^d)=2\prod_{m=2}^{\infty}\left(\frac{m^{2n+2}-1}{m^{2n+2}+1}\right)\times\prod_{m=2:\text{even}}^{\infty}\left(\frac{m^{2n+2}+1}{m^{2n+2}-1}\right)^2$$
$$=2\prod_{m=2}^{\infty}\left(\frac{m^{2n+2}-1}{m^{2n+2}+1}\right)\times\prod_{m=1}^{\infty}\left(\frac{m^{2n+2}+(1/2)^{2n+2}}{m^{2n+2}-(1/2)^{2n+2}}\right)^2.$$

By a similar calculation as in Proof of Theorem 1.2 (3), we get

$$\lim_{d\to\infty}\zeta^\star(\{\{2\}^n,3,\{2\}^n,1\}^d)=2\frac{\prod_{c^{2n+2}=1}\Gamma(2-c)^{-1}\Gamma(1-\frac{c}{2})^2}{\prod_{c^{2n+2}=-1}\Gamma(2-c)^{-1}\Gamma(1-\frac{c}{2})^2}$$
$$=2\prod_{s\in\{\pm1\}}\prod_{c^{2n+2}=s}\Gamma(2-c)^{-s}\Gamma\left(1-\frac{c}{2}\right)^{2s}.$$ □

3. Order property and continuity of the zeta-star map

In this section, we will give a proof of Theorem 1.4.

Lemma 3.1. *For positive integers a, b, A with $A \geq 2$, we have*

$$\sum_{m_1 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{m_1^2 m_2 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}} \leq \sum_{n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(n_1 - 1) n_1^{k_1} \cdots n_b^{k_b}},$$

$$\sum_{m_1 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(m_1 - 1) m_1^2 m_2 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}} \leq \left(\frac{A+1}{2A}\right)^a \sum_{n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(n_1 - 1) n_1^{k_1+1} n_2^{k_2} \cdots n_b^{k_b}}.$$

Proof. We have

$$\begin{aligned}
&\text{L.H.S. of the first equality} \\
&\leq \sum_{m_1 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(m_1 - 1) m_1 m_2 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}} \\
&= \sum_{m_1 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \left(\frac{1}{m_1 - 1} - \frac{1}{m_1}\right) \cdot \frac{1}{m_2 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}} \\
&= \sum_{m_2 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(m_2 - 1) m_2 m_3 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}}.
\end{aligned}$$

Repeating the similar calculations, we obtain the first result. As for the second equality, we have

$$\begin{aligned}
&\text{L.H.S. of the second equality} \\
&\leq \frac{A+1}{A} \sum_{m_1 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(m_1 - 1) m_1 (m_1 + 1) m_2 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}} \\
&= \frac{A+1}{A} \sum_{m_1 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{2} \left(\frac{1}{m_1(m_1 - 1)} - \frac{1}{m_1(m_1 + 1)}\right) \cdot \frac{1}{m_2 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}} \\
&= \frac{A+1}{2A} \sum_{m_2 \geq \cdots \geq m_a \geq n_1 \geq \cdots \geq n_b \geq A} \frac{1}{(m_2 - 1) m_2^2 m_3 \cdots m_a n_1^{k_1} \cdots n_b^{k_b}}.
\end{aligned}$$

By repeating the above procedure, we obtain the second result. □

Proof that the map $\zeta^\star$ is order-reversing. Let $\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{Z}_{\geq 1}^\infty$ and $\mathbf{k}' = (k_1', k_2', \dots) \in \mathbb{Z}_{\geq 1}^\infty$. Put $\mathbf{k}_+ = (k_1 + 1, k_2, k_3, \dots)$ for $\mathbf{k}$ and $\mathbf{k}'_+$ in the same manner. Assume that $\mathbf{k} \prec \mathbf{k}'$ by the lexicographic order. Then there exists $r \geq 1$ such that $k_i = k_i'$ for $1 \leq i < r$ and $k_r < k_r'$. Then

$$\begin{aligned}
\zeta^\star(\mathbf{k}_+) &= \sum_{m_1 \geq m_2 \geq \cdots} \frac{1}{m_1^{k_1+1} m_2^{k_2} m_3^{k_3} \cdots} \\
&> \sum_{m_1 \geq m_2 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1+1} m_2^{k_2} \cdots m_r^{k_r}} \\
&= \zeta^\star(k_1 + 1, k_2, \dots, k_r)
\end{aligned}$$

and

$$\begin{aligned}
\zeta^\star(\mathbf{k}'_+) &\leq \zeta^\star(k_1 + 1, k_2, \dots, k_{r-1}, k_r', \{1\}^\infty) \\
&= \zeta^\star(k_1 + 1, k_2, \dots, k_{r-1}, k_r' - 1) \leq \zeta^\star(k_1 + 1, k_2, \dots, k_r).
\end{aligned}$$

Thus, we have $\zeta^\star(\mathbf{k}_+) > \zeta^\star(\mathbf{k}'_+)$, i.e., $\zeta^\star$ is an order-reversing map. □

Proof that $\zeta^\star(\mathbf{k})$ is convergent for $\mathbf{k} \neq (2,\{1\}^\infty)$. Let $\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{Z}_{\geq 1}^\infty$ with $k_1 \geq 2$ and $\mathbf{k} \neq (2,\{1\}^\infty)$. Then there exists $n \geq 2$ such that $(\{2,\{1\}^{n-2}\}^\infty) \prec \mathbf{k}$. Thus, by Theorem 1.2 (4), $\zeta^\star(\mathbf{k}) \leq \zeta^\star(\{2,\{1\}^{n-2}\}^\infty) = n$, which implies the convergence of $\zeta^\star(\mathbf{k})$. □

Proof that the map $Z^\star$ is continuous. First, we show the continuity of $Z^\star(z)$ at $0 < z < 1$. Let $\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{Z}_{\geq 1}^\infty$ with $k_1 \geq 2$ and $\mathbf{k} \neq (2,\{1\}^\infty)$. We need to show that for any $\epsilon > 0$, there exists $\mathbf{l}$ and $\mathbf{l}'$ with $\mathbf{l}' \prec \mathbf{k} \prec \mathbf{l}$ such that

$$\begin{aligned} \mathbf{k} \prec \mathbf{k}' \prec \mathbf{l} &\implies \zeta^\star(\mathbf{k}) - \zeta^\star(\mathbf{k}') < \epsilon, \\ \mathbf{l}' \prec \mathbf{k}' \prec \mathbf{k} &\implies \zeta^\star(\mathbf{k}') - \zeta^\star(\mathbf{k}) < \epsilon. \end{aligned}$$

Since the sequence $(\zeta^\star(k_1, \dots, k_n))_{n=1}^\infty$ is bounded and monotone increasing, there exist $r \geq 1$ such that

$$\zeta^\star(k_1, \dots, k_r) > \zeta^\star(\mathbf{k}) - \epsilon.$$

Thus

$$\zeta^\star(k_1, \dots, k_{r-1}, k_r + 1, \{1\}^\infty) > \zeta^\star(\mathbf{k}) - \epsilon.$$

By taking $\mathbf{l} = (k_1, \dots, k_{r-1}, k_r + 1, \{1\}^\infty)$, we obtain the first line.

Now we will show the second line. We first show the claim for indices with a finite number of elements greater than or equal to 2. Let $\mathbf{k} = (k_1, \dots, k_r, \{1\}^\infty)$ with $k_r \geq 2$. Since

$$\begin{aligned} &\lim_{a\to\infty} \zeta^\star(k_1, \dots, k_{r-1}, k_r - 1, a + 1, \{1\}^\infty) \\ &= \lim_{a\to\infty} \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r - 1}} \sum_{s=1}^{m_r} \frac{1}{s^a} \\ &\leq \lim_{a\to\infty} \zeta(a) \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r - 1}} \\ &= \zeta^\star(k_1, \dots, k_{r-1}, k_r - 1) \\ &= \zeta^\star(\mathbf{k}), \end{aligned}$$

there exist $n \geq 1$ such that

$$\zeta^\star(k_1, \dots, k_{r-1}, k_r - 1, n + 1, \{1\}^\infty) < \zeta^\star(\mathbf{k}) + \epsilon.$$

Thus, the claim holds for indices with a finite number of elements greater than or equal to 2.

Assume that there exists infinitely many j such that $k_j > 1$.

$$\mathbf{k} = (a_1, \{1\}^{b_1}, a_2, \{1\}^{b_2}, \dots) \qquad (a_j \geq 2, b_j \geq 0).$$

We need to show the existence of s such that

$$\zeta^\star(a_1, \{1\}^{b_1}, a_2, \{1\}^{b_2}, \dots, a_{s-1}, \{1\}^{b_{s-1}}, a_s - 1) < \zeta^\star(\mathbf{k}) + \epsilon.$$

Note that

$$\begin{aligned}&\zeta^\star(a_1,\{1\}^{b_1},a_2,\{1\}^{b_2},\dots,a_{s-1},\{1\}^{b_{s-1}},a_s-1)-\zeta^\star(\mathbf{k})\\ &<\zeta^\star(a_1,\{1\}^{b_1},a_2,\{1\}^{b_2},\dots,a_{s-1},\{1\}^{b_{s-1}},a_s-1)\\ &\quad-\zeta^\star(a_1,\{1\}^{b_1},a_2,\{1\}^{b_2},\dots,a_{s-1},\{1\}^{b_{s-1}},a_s)\\ &\le\zeta^\star(2,\{1\}^{b},2,\{1\}^{r-2},1)-\zeta^\star(2,\{1\}^{b},2,\{1\}^{r-2},2),\end{aligned}$$

where $b=b_1$ and $r=b_2+\cdots+b_{s-1}+s-1$. Then we have

$$\begin{aligned}&\zeta^\star(a_1,\{1\}^{b_1},a_2,\{1\}^{b_2},\dots,a_{s-1},\{1\}^{b_{s-1}},a_s-1)-\zeta^\star(\mathbf{k})\\ &<\sum_{m\ge n_1\ge\cdots\ge n_{b+r}\ge 1}\frac{1}{m^2n_1\cdots n_bn_{b+1}^2n_{b+2}\cdots n_{b+r-1}}\left(\frac{1}{n_{b+r}}-\frac{1}{n_{b+r}^2}\right)\\ &=\sum_{m\ge n_1\ge\cdots\ge n_{b+r}\ge 2}\frac{1}{m^2n_1\cdots n_bn_{b+1}^2n_{b+2}\cdots n_{b+r-1}}\left(\frac{1}{n_{b+r}}-\frac{1}{n_{b+r}^2}\right)\\ &\le\sum_{m\ge n_1\ge\cdots\ge n_{b+r}\ge 2}\frac{1}{m^2n_1\cdots n_bn_{b+1}^2n_{b+2}\cdots n_{b+r}}\\ &\le\left(\frac{3}{4}\right)^{r-1}\sum_{n_{b+r}\ge 2}\frac{1}{(n_{b+r}-1)n_{b+r}^2}.\end{aligned}$$

Here, we used Lemma 3.1 for the last inequality. Thus, for any $\epsilon>0$, there exists s such that

$$\zeta^\star(a_1,\{1\}^{b_1},a_2,\{1\}^{b_2},\dots a_{s-1},\{1\}^{b_{s-1}},a_s-1)<\zeta^\star(\mathbf{k})+\epsilon.$$

Second, we show the continuity of $Z^\star(z)$ at $z=0$. Note that $Z^\star(2^{1-k})=\zeta^\star(k+1,\{1\}^\infty)=\zeta^\star(k)\to 1$ as $k\to\infty$ by Theorem 1.2 (1). Since $\zeta^\star$ is order-reversing, i.e., $Z^\star$ is order-preserving on $0<z\le 1$, we have $\lim_{z\to 0}Z^\star(z)=1$, which implies the continuity at 0.

Lastly, we show the continuity of $Z^\star(z)$ at $z=1$. By Theorem 1.2 (4) and the property of order-reversing, we find that for any $n\ge 2$,

$$\mathbf{k}\prec(\{2,\{1\}^{n-2}\}^\infty)\implies\zeta^\star(\mathbf{k})>n.$$

This finishes the proof. □

Proof that the map $Z^\star$ is bijective. Since the map $\zeta^\star$ is order-reversing, we see that the map $Z^\star$ is injective. Surjectivity of the map $Z^\star$ follows from the intermediate value theorem and the continuity of $Z^\star$. □

4. Analytic properties of the zeta-star map

This section investigates the differential of $Z^\star$. Hereinafter, we understand $0^0=1$.

Lemma 4.1. *For $z=\sum_{j=1}^\infty\frac{a_j}{2^j}$ with $a_j\in\{0,1\}$, we have*

$$Z^\star(z)=\sum_{m_1\ge m_2\ge\cdots\ge 1}\frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1^2m_2m_3\cdots}.$$

Proof. Note that $a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots$ vanishes except for the case $m_j=m_{j+1}$ for all j such that $a_j=0$. The case where there exists infinitely many j such that $a_j=1$ follows from the definition of $Z^\star$. The case $z = 0$ also follows from the definition of $Z^\star$. The other case follows from Theorem 1.2 (1), e.g., when $a_j = \delta_{j,1}$,

$$Z^\star\left(\frac{1}{2}+\frac{0}{4}+\frac{0}{8}+\cdots\right) = Z^\star\left(\frac{0}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right) = \zeta^\star(3,\{1\}^\infty)$$
$$= \zeta^\star(2) = \sum_{m_1\ge m_2\ge\cdots\ge 1} \frac{1^{m_1-m_2}0^{m_2-m_3}0^{m_3-m_4}\cdots}{m_1^2 m_2 m_3 m_4\cdots}. \qquad \square$$

Lemma 4.2. *For $z = \sum_{j=1}^\infty \frac{a_j}{2^j}$ with $a_j \in \{0,1\}$, we have*

$$Z^\star(z) = 1 + \frac{z}{2} + \sum_{d=1}^\infty a_d \left(\sum_{m_1\ge\cdots\ge m_d\ge 3} \frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d}\right) 2^d \left(z - \sum_{i=1}^{d-1}\frac{a_i}{2^i}\right).$$

Proof. It follows from the following calculation

$$\begin{aligned} Z^\star(z) &= \sum_{m_1\ge m_2\ge\cdots\ge 1} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1^2 m_2 m_3\cdots} \\ &= \sum_{1\le d\le e} \sum_{\substack{m_1\ge\cdots\ge m_d\ge 3\\ 2=m_{d+1}=\cdots=m_e\\ 1=m_{e+1}=m_{e+2}=\cdots}} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1^2 m_2 m_3\cdots} + \sum_{2\ge m_1\ge m_2\ge\cdots} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1^2 m_2 m_3\cdots}. \end{aligned}$$

Here,

$$\begin{aligned} &\sum_{1\le d\le e} \sum_{\substack{m_1\ge\cdots\ge m_d\ge 3\\ 2=m_{d+1}=\cdots=m_e\\ 1=m_{e+1}=m_{e+2}=\cdots}} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1^2 m_2 m_3\cdots} \\ &= \sum_{d=1}^\infty \left(\sum_{m_1\ge\cdots\ge m_d\ge 3} \frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d}\right) \sum_{e=d}^\infty \frac{a_d a_e}{2^{e-d}} \\ &= \sum_{d=1}^\infty a_d \left(\sum_{m_1\ge\cdots\ge m_d\ge 3} \frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d}\right) 2^d \left(z - \sum_{i=1}^{d-1}\frac{a_i}{2^i}\right) \end{aligned}$$

and

$$\begin{aligned} &\sum_{2\ge m_1\ge m_2\ge\cdots} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1^2 m_2 m_3\cdots} \\ &= 1 + \frac{1}{2} \sum_{\substack{2\ge m_1\ge m_2\ge\cdots\\ m_1\ne 1}} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1 m_2 m_3\cdots} \\ &= 1 + \frac{1}{2}\sum_{e=1}^\infty \sum_{\substack{2=m_1=\cdots=m_e\\ 1=m_{e+1}=m_{e+2}=\cdots}} \frac{a_1^{m_1-m_2}a_2^{m_2-m_3}\cdots}{m_1 m_2 m_3\cdots} \\ &= 1 + \frac{1}{2}\sum_{e=1}^\infty \frac{a_e}{2^e} \\ &= 1 + \frac{z}{2}. \qquad \square \end{aligned}$$

Lemma 4.3. *For $s \geq 1$, we have*

$$\sum_{m_1 \geq \cdots \geq m_s \geq 3} \frac{1}{m_1^4 m_2 \cdots m_s} = O\left(\frac{s}{3^s}\right).$$

Proof. For $x > 0$, put

$$F_s(x) = \sum_{m_1 \geq \cdots \geq m_s \geq 3} \frac{1}{m_2 \cdots m_s} \begin{cases} \frac{1}{m_1(m_1-1)(m_1-2)(m_1-3)} & \text{if } m_1 > 3, \\ x & \text{if } m_1 = 3. \end{cases}$$

Note that

$$\begin{aligned} \sum_{m=n}^{\infty} \frac{1}{m(m-1)(m-2)(m-3)} &= \frac{1}{3} \sum_{m=n}^{\infty} \left(\frac{1}{(m-1)(m-2)(m-3)} - \frac{1}{m(m-1)(m-2)} \right) \\ &= \frac{1}{3(n-1)(n-2)(n-3)}. \end{aligned}$$

Then

$$\begin{aligned} F_s(x) &= \sum_{\substack{m_2 \geq \cdots \geq m_s \geq 3 \\ m_2 > 3}} \frac{1}{m_2 \cdots m_s} \sum_{m_1 = m_2}^{\infty} \frac{1}{m_1(m_1-1)(m_1-2)(m_1-3)} \\ &\quad + \sum_{\substack{m_2 \geq \cdots \geq m_s \geq 3 \\ m_2 = 3}} \frac{1}{m_2 \cdots m_s} \left(\sum_{m_1=4}^{\infty} \frac{1}{m_1(m_1-1)(m_1-2)(m_1-3)} + x \right) \\ &= \frac{1}{3} \sum_{\substack{m_2 \geq \cdots \geq m_s \geq 3 \\ m_2 > 3}} \frac{1}{m_3 \cdots m_s} \cdot \frac{1}{m_2(m_2-1)(m_2-2)(m_2-3)} \\ &\quad + \frac{1}{3} \sum_{\substack{m_2 \geq \cdots \geq m_s \geq 3 \\ m_2 = 3}} \frac{1}{m_3 \cdots m_s} \left(\frac{1}{18} + x \right) \\ &= \frac{1}{3} F_{s-1}\left(x + \frac{1}{18} \right). \end{aligned}$$

Thus we have

$$F_s(x) = \frac{1}{3^{s-1}} F_1\left(x + \frac{s-1}{18} \right).$$

Then we get

$$\begin{aligned} \sum_{m_1 \geq \cdots \geq m_s \geq 3} \frac{1}{m_1^4 m_2 \cdots m_s} &\leq F_s\left(\frac{1}{18}\right) \\ &= \frac{1}{3^{s-1}} F_1\left(\frac{s}{18}\right) \\ &= \frac{1}{3^{s-1}} \left(\frac{s}{18} + \sum_{m>3} \frac{1}{m(m-1)(m-2)(m-3)} \right) \\ &= O\left(\frac{s}{3^s}\right). \end{aligned}$$

This finishes the proof. ∎

Lemma 4.4. *There exists a constant C such that for all $d > t > 1$ and $a_1, \dots, a_{d-1} \in \{0,1\}$ with $\sum_{i=1}^{t}(1-a_i) \geq 2$, we have*

$$\sum_{m_1 \geq \cdots \geq m_d \geq 3} \frac{a_1^{m_1-m_2} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d} \leq C \cdot \left(\frac{d-t}{3^{d-t}} \right).$$

Proof. From the assumption, let $a_u = a_v = 0$ $(u < v \leq t)$ and $a_1 = \cdots = a_{u-1} = a_{u+1} = \cdots = a_{v-1} = 1$. Then

$$\begin{aligned}
\text{L.H.S.} &= \sum_{m_1 \geq \cdots \geq m_d \geq 3} \frac{a_u^{m_u-m_{u+1}} a_v^{m_v-m_{v+1}} a_{v+1}^{m_{v+1}-m_{v+2}} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d} \\
&= \sum_{m_1 \geq \cdots \geq m_u = m_{u+1} \geq \cdots \geq m_v = m_{v+1} \geq \cdots \geq m_d \geq 3} \frac{a_{v+1}^{m_{v+1}-m_{v+2}} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d}.
\end{aligned}$$

Using Lemma 3.1, we have

$$\text{L.H.S.} \leq \frac{3}{2} \sum_{m_{v+1} \geq \cdots \geq m_d \geq 3} \frac{a_{v+1}^{m_{v+1}-m_{v+2}} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_{v+1}^4 m_{v+2} \cdots m_d}.$$

By Lemma 4.3, we obtain the result. □

Lemma 4.5. *Let $z = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$ with $a_j \in \{0,1\}$ and assume that $\sum_{i=1}^{t}(1-a_i) \geq 2$. Then we have*

$$Z^\star(z) = 1 + \frac{z}{2} + \sum_{d=1}^{r} a_d \left(\sum_{m_1 \geq \cdots \geq m_d \geq 3} \frac{a_1^{m_1-m_2} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d} \right) 2^d \left(z - \sum_{i=1}^{d-1} \frac{a_i}{2^i} \right) + O\left(\frac{r-t}{3^{r-t}} \right).$$

Proof. Since

$$z - \sum_{i=1}^{d-1} \frac{a_i}{2^i} \leq \sum_{i=d}^{\infty} \frac{1}{2^i} = 2^{1-d},$$

we have

$$\begin{aligned}
&\sum_{d=r+1}^{\infty} a_d \left(\sum_{m_1 \geq \cdots \geq m_d \geq 3} \frac{a_1^{m_1-m_2} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d} \right) 2^d \left(z - \sum_{i=1}^{d-1} \frac{a_i}{2^i} \right) \\
&\leq 2 \sum_{d=r+1}^{\infty} \left(\sum_{m_1 \geq \cdots \geq m_d \geq 3} \frac{a_1^{m_1-m_2} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d} \right) \\
&\leq 2 \sum_{d=r+1}^{\infty} O\left(\frac{d-t}{3^{d-t}} \right)
\end{aligned}$$

by Lemma 4.4. Note that in the above equality, the O-constants are independent of d. By Lemma 4.2, we get the result. □

Lemma 4.6. *Let $x = \sum_{j=1}^{\infty} a_j/2^j$ with $a_j \in \{0,1\}$ and $y = \sum_{j=1}^{\infty} b_j/2^j$ with $b_j \in \{0,1\}$. Assume that $a_j = b_j$ for $j = 1, \dots, r$. Furthermore, assume that $\sum_{i=1}^{t}(1-a_i) \geq 2$ with $t \leq r$. Then*

$$Z^\star(x) - Z^\star(y) = (x-y) \left(\frac{1}{2} + \sum_{d=1}^{r} a_d \left(\sum_{m_1 \geq \cdots \geq m_d \geq 3} \frac{a_1^{m_1-m_2} \cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2 \cdots m_d} \right) 2^d \right) + O\left(\frac{r-t}{3^{r-t}} \right).$$

Proof. By Lemma 4.5, the proof is clear. □

Theorem 4.7. *Let* $z = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$ *with* $a_j \in \{0,1\}$. *Assume that* $\sum_{j=1}^{\infty} a_j = \infty$ *and* $\sum_{j=1}^{t}(1-a_j) \geq 2$ *for some* t. *Then*

$$\partial_{-}Z^{\star}(z) := \lim_{x\to z-0} \frac{Z^{\star}(z)-Z^{\star}(x)}{z-x} = \frac{1}{2} + \sum_{d=1}^{\infty} a_d \left(\sum_{m_1\geq\cdots\geq m_d\geq 3} \frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d} \right) 2^d.$$

Thus, $Z^{\star}$ *is left-differentiable at* z *if* $z \notin \{1-\frac{1}{2^n} \mid n>0\}$.

Proof. Take $0 < x < z$ and put $x = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$. Let $p = p(x)$ be the minimal integer such that $(a_p, b_p) = (1,0)$. Put

$$y = \sum_{j=1}^{p} \frac{a_j}{2^j} = \sum_{j=1}^{p-1} \frac{a_j}{2^j} + \frac{1}{2^{p+1}} + \frac{1}{2^{p+2}} + \cdots .$$

Furthermore, let r be the maximal integer such that

$$a_{p+1} = a_{p+2} = \cdots = a_r = 0$$

and

$$b_{p+1} = b_{p+2} = \cdots = b_r = 1.$$

Then by Lemma 4.6, we have

$$Z^{\star}(z)-Z^{\star}(y) = (z-y)\left(\frac{1}{2} + \sum_{d=1}^{r} a_d \left(\sum_{m_1\geq\cdots\geq m_d\geq 3} \frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d} \right) 2^d\right) + O\left(\frac{r-t}{3^{r-t}}\right)$$

and

$$Z^{\star}(y)-Z^{\star}(x) = (y-x)\left(\frac{1}{2} + \sum_{d=1}^{r} b_d \left(\sum_{m_1\geq\cdots\geq m_d\geq 3} \frac{b_1^{m_1-m_2}\cdots b_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d} \right) 2^d\right) + O\left(\frac{r-t}{3^{r-t}}\right).$$

Note that, for any A, B and x, y, z with $z-y \geq 0$ and $y-x \geq 0$, there exists C between A and B such

$$(z-y)A + (y-x)B = (z-x)C.$$

Hence there exists a real number $u(x)$ between

$$\frac{1}{2} + \sum_{d=1}^{r} a_d \left(\sum_{m_1\geq\cdots\geq m_d\geq 3} \frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d} \right) 2^d$$

and

$$\frac{1}{2} + \sum_{d=1}^{r} b_d \left(\sum_{m_1\geq\cdots\geq m_d\geq 3} \frac{b_1^{m_1-m_2}\cdots b_{d-1}^{m_{d-1}-m_d}}{m_1^2 m_2\cdots m_d} \right) 2^d$$

such that

$$Z^{\star}(z) - Z^{\star}(x) = (z-x)u(x) + O\left(\frac{r-t}{3^{r-t}}\right).$$

Since

$$\begin{aligned} z - x &\geq \frac{1}{2^p} - \frac{1}{2^{p+1}} - \cdots - \frac{1}{2^r} - \sum_{j=r+2}^{\infty} \frac{1}{2^j} \\ &= \frac{1}{2^{r+1}}, \end{aligned}$$

we have

$$\frac{Z^\star(z)-Z^\star(x)}{z-x}=u(x)+O\left(r(2/3)^r3^t\right).$$

By the condition $\sum_{j=1}^\infty a_j=\infty$, we have $\lim_{x\to z-0}p(x)=\infty$ and thus

$$\lim_{x\to z-0}u(x)=\frac{1}{2}+\sum_{d=1}^\infty a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d,$$

which completes the proof. □

Theorem 4.8. *Let* $z=\sum_{j=1}^\infty\frac{a_j}{2^j}$ *with* $a_j\in\{0,1\}$ *and assume that* $\sum_{j=1}^\infty(1-a_j)=\infty$. *Then*

$$\partial_+Z^\star(z):=\lim_{x\to z+0}\frac{Z^\star(z)-Z^\star(x)}{z-x}=\frac{1}{2}+\sum_{d=1}^\infty a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d.$$

Thus, $Z^\star$ *is right-differentiable at* z *for any* $0\leq z<1$.

Proof. Take $x\in(z,1)$ and put $x=\sum_{j=1}^\infty\frac{b_j}{2^j}$. Let $p=p(x)$ be the minimal integer such that $(a_p,b_p)=(0,1)$. Put

$$y=\sum_{j=1}^p\frac{b_j}{2^j}=\sum_{j=1}^{p-1}\frac{b_j}{2^j}+\frac{1}{2^{p+1}}+\frac{1}{2^{p+2}}+\cdots.$$

Furthermore, let r be the maximal integer such that

$$a_{p+1}=a_{p+2}=\cdots=a_r=1$$

and

$$b_{p+1}=b_{p+2}=\cdots=b_r=0.$$

Then by Lemma 4.6, we have

$$Z^\star(x)-Z^\star(y)=(x-y)\left(\frac{1}{2}+\sum_{d=1}^r b_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{b_1^{m_1-m_2}\cdots b_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d\right)+O\left(\frac{r-t}{3^{r-t}}\right)$$

and

$$Z^\star(y)-Z^\star(z)=(y-z)\left(\frac{1}{2}+\sum_{d=1}^r a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d\right)+O\left(\frac{r-t}{3^{r-t}}\right).$$

Hence, there exists a real number $u(x)$ between

$$\frac{1}{2}+\sum_{d=1}^r a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d$$

and

$$\frac{1}{2}+\sum_{d=1}^r b_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{b_1^{m_1-m_2}\cdots b_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d$$

such that

$$Z^\star(z)-Z^\star(x)=(z-x)u(x)+O\left(\frac{r-t}{3^{r-t}}\right).$$

Since

$$\begin{aligned}|z-x| &\geq \frac{1}{2^p}-\frac{1}{2^{p+1}}-\cdots-\frac{1}{2^r}-\sum_{j=r+2}^{\infty}\frac{1}{2^j}\\ &= \frac{1}{2^{r+1}},\end{aligned}$$

we have

$$\frac{Z^\star(z)-Z^\star(x)}{z-x}=u(x)+O\left(r(2/3)^r 3^t\right).$$

By the condition $\sum_{j=1}^{\infty}(1-a_j)=\infty$, we have $\lim_{x\to z+0}p(x)=\infty$ and thus

$$\lim_{x\to z+0}u(x)=\frac{1}{2}+\sum_{d=1}^{\infty}a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d,$$

which completes the proof. □

Remark 4.9. By Theorems 4.7 and 4.8, if z admits a 2-adic expansion $\sum_{j=1}^{\infty}\frac{a_j}{2^j}$ satisfying $\sum_{j=1}^{\infty}a_j=\sum_{j=1}^{\infty}(1-a_j)=\infty$, then $\partial_+Z^\star(z)=\partial_-Z^\star(z)$. This implies Theorem 1.7 (4).

Theorem 4.10. *For $z=\sum_{j=1}^{r}\frac{a_j}{2^j}$ with $a_j\in\{0,1\}$ and $a_r=1$, we have*

$$\partial_+Z^\star(z)=\frac{1}{2}+\sum_{d=1}^{r}a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d$$

and

$$\partial_-Z^\star(z)=\partial_+Z^\star(z)+\sum_{m_1\geq\cdots\geq m_r\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{r-1}^{m_{r-1}-m_r}}{m_1^2m_2\cdots m_r}2^r(m_r-2)\qquad (z\neq 1-1/2^r).$$

Proof. The first statement is just a special case of Theorem 4.8. Since

$$z=\sum_{j=1}^{r-1}\frac{a_j}{2^j}+\frac{1}{2^{r+1}}+\frac{1}{2^{r+2}}+\cdots,$$

we have

$$\begin{aligned}\partial_-Z^\star(z)=\frac{1}{2}&+\sum_{d=1}^{r-1}a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d\\ &+\sum_{d=r+1}^{\infty}\left(\sum_{m_1\geq\cdots\geq m_r\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{r-1}^{m_{r-1}-m_r}}{m_1^2m_2\cdots m_r}\sum_{m_r=m_{r+1}\geq\cdots\geq m_d\geq 3}\frac{1}{m_{r+1}\cdots m_d}\right)2^d\end{aligned}$$

by Lemma 4.7. Since

$$\frac{2^{r+1}}{m_r}\prod_{n=3}^{m_r}\sum_{c=0}^{\infty}\left(\frac{2}{n}\right)^c=\frac{2^{r+1}}{m_r}\prod_{n=3}^{m_r}\frac{n}{n-2}=2^r(m_r-1),$$

if we put $p=d-r-1$, the third term equals

$$\sum_{p=0}^{\infty}\left(\sum_{m_1\geq\cdots\geq m_r\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{r-1}^{m_{r-1}-m_r}}{m_1^2m_2\cdots m_r}\frac{1}{m_r}\sum_{m_r\geq n_1\geq\cdots\geq m_p\geq 3}\frac{1}{n_1\cdots n_p}\right)2^{p+r+1}$$
$$=\sum_{m_1\geq\cdots\geq m_r\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{r-1}^{m_{r-1}-m_r}}{m_1^2m_2\cdots m_r}\cdot\frac{2^{r+1}}{m_r}\prod_{n=3}^{m_r}\sum_{c=0}^{\infty}\left(\frac{2}{n}\right)^c$$
$$=\sum_{d=r}^{r}a_d\left(\sum_{m_1\geq\cdots\geq m_d\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{d-1}^{m_{d-1}-m_d}}{m_1^2m_2\cdots m_d}\right)2^d+\sum_{m_1\geq\cdots\geq m_r\geq 3}\frac{a_1^{m_1-m_2}\cdots a_{r-1}^{m_{r-1}-m_r}}{m_1^2m_2\cdots m_r}2^r(m_r-2).$$

Thus the theorem is proved. □

5. Divergence of left-differential

In this section, we give a proof of Theorem 1.7 (3).

Lemma 5.1. *Fix $r\geq 1$. Then we have*

$$\sum_{m_1\geq\cdots\geq m_s\geq n}\frac{1}{m_1^{r+1}m_2\cdots m_s}\leq\frac{1}{(n-1)\cdots(n-r)r^s}$$

for all $s\geq 1$ and $n\geq r+1$. Furthermore, there exists $C_r\in\mathbb{R}_{>0}$ such that

$$\sum_{m_1\geq\cdots\geq m_s\geq n}\frac{1}{m_1^{r+1}m_2\cdots m_s}\geq\frac{1}{(n-1)\cdots(n-r)r^s}-\frac{C_r}{(n-1)\cdots(n-r)(n-r-1)(r+1)^s}$$

for all $s\geq 1$ and $n\geq r+2$.

Proof. The first claim follows from

$$\sum_{m_1\geq\cdots\geq m_s\geq n}\frac{1}{m_1^{r+1}m_2\cdots m_s}$$
$$\leq\sum_{m_1\geq\cdots\geq m_s\geq n}\frac{1}{m_1(m_1-1)\cdots(m_1-r)m_2\cdots m_s}$$
$$=\sum_{m_1\geq\cdots\geq m_s\geq n}\frac{1}{r}\left(\frac{1}{(m_1-1)\cdots(m_1-r)}-\frac{1}{m_1(m_1-1)\cdots(m_1-r+1)}\right)\frac{1}{m_2\cdots m_s}$$
$$=\sum_{m_2\geq\cdots\geq m_s\geq n}\frac{1}{r}\cdot\frac{1}{(m_2-1)\cdots(m_2-r)}\cdot\frac{1}{m_2\cdots m_s}$$
$$=\cdots$$
$$=\frac{1}{(n-1)\cdots(n-r)r^s}.$$

There exists $C_r\in\mathbb{R}_{>0}$ such that

$$\frac{1}{m^{r+1}}\geq\frac{1}{m(m-1)\cdots(m-r)}-\frac{C_r}{m(m-1)\cdots(m-r-1)}$$

for all $m \geq r+2$. Then, the second claim follows from

$$\begin{aligned}
&\sum_{m_1\geq\cdots\geq m_s\geq n} \frac{1}{m_1^{r+1}m_2\cdots m_s} \\
&\geq \sum_{m_1\geq\cdots\geq m_s\geq n} \frac{1}{m_1(m_1-1)\cdots(m_1-r)m_2\cdots m_s} \\
&\quad - C_r \sum_{m_1\geq\cdots\geq m_s\geq n} \frac{1}{m_1(m_1-1)\cdots(m_1-r-1)m_2\cdots m_s} \\
&= \frac{1}{(n-1)\cdots(n-r)r^s} - \frac{C_r}{(n-1)\cdots(n-r)(n-r-1)(r+1)^s}.
\end{aligned}$$

□

Lemma 5.2. *Fix $r \geq 1$. Then there exists $D_r \in \mathbb{R}_{>0}$ such that*

$$\frac{1}{r!r^s} - \frac{sD_r}{(r+1)^s} \leq \sum_{m_1\geq\cdots\geq m_s\geq r+1} \frac{1}{m_1^{r+1}m_2\cdots m_s} \leq \frac{1}{r!r^s}$$

for all $s \geq 1$.

Proof. We have

$$\begin{aligned}
\sum_{m_1\geq\cdots\geq m_s\geq r+1} \frac{1}{m_1^{r+1}m_2\cdots m_s} &= \sum_{i=0}^{s} \sum_{\substack{m_1\geq\cdots\geq m_i\geq r+2\\ m_{i+1}=\cdots=m_s=r+1}} \frac{1}{m_1^{r+1}m_2\cdots m_s} \\
&\geq \sum_{i=1}^{s} \frac{1}{(r+1)^{s-i}} \sum_{m_1\geq\cdots\geq m_i\geq r+2} \frac{1}{m_1^{r+1}m_2\cdots m_i},
\end{aligned}$$

and by the second claim of the previous lemma,

$$\begin{aligned}
&\geq \sum_{i=1}^{s} \frac{1}{(r+1)^{s-i}} \left(\frac{1}{(r+1)!r^i} - \frac{C_r}{(r+1)!(r+1)^i} \right) \\
&= \frac{1}{r!r^s} - \frac{1}{(r+1)!(r+1)^{s-1}} - \frac{sC_r}{(r+1)!(r+1)^s}
\end{aligned}$$

This proves the lower bound for the inequality, and the upper bound follows from the first claim of the previous lemma. □

Lemma 5.3. *Fix $r \geq 1$. Then there exists $E_r \in \mathbb{R}_{>0}$ such that*

$$\frac{s-E_r}{r!r^s} \leq \sum_{m_1\geq\cdots\geq m_s\geq r} \frac{1}{m_1^{r+1}m_2\cdots m_s} - \frac{1}{r^{r+s}} \leq \frac{s}{r!r^s}$$

for all $s \geq 1$.

Proof. We have

$$\begin{aligned}
\sum_{m_1\geq\cdots\geq m_s\geq r} \frac{1}{m_1^{r+1}m_2\cdots m_s} &= \frac{1}{r^{r+s}} + \sum_{i=1}^{s} \sum_{\substack{m_1\geq\cdots\geq m_i\geq r+1\\ m_{i+1}=\cdots=m_s=r}} \frac{1}{m_1^{r+1}m_2\cdots m_s} \\
&= \frac{1}{r^{r+s}} + \sum_{i=1}^{s} \frac{1}{r^{s-i}} \sum_{m_1\geq\cdots\geq m_i\geq r+1} \frac{1}{m_1^{r+1}m_2\cdots m_i}.
\end{aligned}$$

Here, by the previous lemma,

$$\frac{1}{r!r^i} - \frac{iD_r}{(r+1)^i} \le \sum_{m_1 \ge \cdots \ge m_i \ge r+1} \frac{1}{m_1^{r+1} m_2 \cdots m_i} \le \frac{1}{r!r^i}.$$

Furthermore,

$$\sum_{i=1}^{s} \frac{1}{r^{s-i}} \left(\frac{1}{r!r^i} \right) = \frac{s}{r!r^s}$$

and

$$\begin{aligned} \sum_{i=1}^{s} \frac{1}{r^{s-i}} \left(\frac{iD_r}{(r+1)^i} \right) &= D_r r \left(r^{1-s} + r^{-s} - r(r+1)^{-s} - (s+1)(r+1)^{-s} \right) \\ &\le D_r r \left(\frac{r+1}{r^s} \right). \end{aligned}$$

Putting $E_r = D_r r(r+1)r!$, we obtain the lemma. □

By Lemmas 5.1 and 5.3, we have

$$\sum_{m_1 \ge \cdots \ge m_s \ge n} \frac{1}{m_1^{r+1} m_2 \cdots m_s} \asymp_r \begin{cases} \frac{1}{(n-1)\cdots(n-r)r^s} & n > r \\ \frac{s}{r^s} & n = r. \end{cases}$$

Thus we have the following:

Lemma 5.4. *Fix $r \ge 1$. We have*

$$\sum_{n=r}^{\infty} \sum_{m_1 \ge \cdots \ge m_s \ge n} \frac{1}{m_1^{r+1} m_2 \cdots m_s} \asymp \frac{s}{r^s}$$

for all $s \ge 1$.

Theorem 5.5. *Fix $p > 0$ and put $z = 1 - \frac{1}{2^p}$. Then for $h = \frac{1}{2^q}$ with $q > p$, we have*

$$\frac{Z^\star(z) - Z^\star(z-h)}{h} \asymp q.$$

Thus, $Z^\star$ is not left-differentiable at $1 - \frac{1}{2^p}$.

Proof. We only consider the case $q > p+1$. Note that

$$z = \sum_{\substack{k=1 \\ k \ne p}}^{\infty} \frac{1}{2^k} \quad \text{and} \quad z - h = \sum_{\substack{k=1 \\ k \ne p,q}}^{\infty} \frac{1}{2^k}.$$

By definition, we have

$$Z^\star(z) = \sum_{\substack{m_1 \ge m_2 \ge \cdots \ge 1 \\ m_p = m_{p+1}}} \frac{1}{m_1^2 m_2 m_3 \cdots}$$

and

$$Z^\star(z-h) = \sum_{\substack{m_1 \ge m_2 \ge \cdots \ge 1 \\ m_p = m_{p+1}, m_q = m_{q+1}}} \frac{1}{m_1^2 m_2 m_3 \cdots}.$$

Then we have

$$\begin{aligned}
Z^\star(z) - Z^\star(z-h) &= \sum_{\substack{m_1 \ge m_2 \ge \cdots \ge 1 \\ m_p = m_{p+1}, m_q > m_{q+1}}} \frac{1}{m_1^2 m_2 m_3 \cdots} \\
&\asymp \sum_{\substack{m_1 \ge m_2 \ge \cdots \ge 1 \\ m_p = m_{p+1}, m_q > m_{q+1}}} \frac{1}{m_1(m_1-1) m_2 m_3 \cdots} \\
&= \sum_{\substack{m_p \ge m_{p+1} \ge \cdots \ge 1 \\ m_p = m_{p+1}, m_q > m_{q+1}}} \frac{1}{m_p(m_p-1) m_{p+1} m_{p+2} \cdots} \\
&= \sum_{\substack{m_{p+1} \ge \cdots \ge 1 \\ m_q > m_{q+1}}} \frac{1}{m_{p+1}^2 (m_{p+1}-1) m_{p+2} m_{p+3} \cdots}.
\end{aligned}$$

Similar to the proof of Lemma 2.1, we have

$$\begin{aligned}
&Z^\star(z) - Z^\star(z-h) \\
&\asymp \sum_{m_{p+1} \ge \cdots \ge m_q \ge 2} \frac{1}{m_{p+1}^2 (m_{p+1}-1) m_{p+2} m_{p+3} \cdots m_q} \sum_{m_q > m_{q+1} \ge m_{q+2} \cdots} \frac{1}{m_{q+1} m_{q+2} \cdots} \\
&= \sum_{m_{p+1} \ge \cdots \ge m_q \ge 2} \frac{1}{m_{p+1}^2 (m_{p+1}-1) m_{p+2} m_{p+3} \cdots m_q} \prod_{m=2}^{m_q - 1} \frac{m}{m-1} \\
&= \sum_{m_{p+1} \ge \cdots \ge m_q \ge 2} \frac{m_q - 1}{m_{p+1}^2 (m_{p+1}-1) m_{p+2} m_{p+3} \cdots m_q} \\
&\asymp \sum_{m_{p+1} \ge \cdots \ge m_q \ge 2} \frac{1}{m_{p+1}^3 m_{p+2} m_{p+3} \cdots m_{q-1}}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
Z^\star(z) - Z^\star(z-h) &\asymp \sum_{n=2}^{\infty} \sum_{m_{p+1} \ge \cdots \ge m_{q-1} \ge n} \frac{1}{m_{p+1}^3 m_{p+2} m_{p+3} \cdots m_{q-1}} \\
&= \sum_{n=2}^{\infty} \sum_{m_1 \ge \cdots \ge m_s \ge n} \frac{1}{m_1^3 m_2 m_3 \cdots m_s},
\end{aligned}$$

where $s = q - p - 1 > 0$. By Lemma 5.4, we find

$$Z^\star(z) - Z^\star(z-h) \asymp \frac{s}{2^s} \asymp qh,$$

which completes the proof. □

Acknowledgments

This work was supported by JSPS KAKENHI Grant Numbers JP18K13392, JP19K14511, JP22K03244, and JP22K13897. We thank the referee for carefully reading the manuscript and providing appropriate comments.

References

[1] K. Kumar, *Order Structure and Topological Properties of the Set of Multiple Zeta Values*, Int. Math. Res. Notices 2016, 1541–1562.

[2] J. Li, *The topology of the set of multiple zeta-star values*, arXiv:2309.07569.

[3] Y. Ohno and N. Wakabayashi, *Cyclic sum of multiple zeta values*, Acta Arithmetica **123** (2006), 289–295.

[4] J. Zhao, *Identity families of multiple harmonic sums and multiple zeta (star) families*, J. Math. Soc. Japan **68** (2016), 1669–1694.

[5] S.A. Zlobin, *Generating functions for the values of a multiple zeta function*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. no. 2 (2005), 55–59; Moscow Univ. Math. Bull. **60** (2005), 44–48 (English transl.).

(Minoru Hirose) Institute for Advanced Research, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan

Email address: minoru.hirose@math.nagoya-u.ac.jp

(Hideki Murahara) The University of Kitakyushu, 4-2-1 Kitagata, Kokuraminami-ku, Kitakyushu, Fukuoka, 802-8577, Japan

Email address: hmurahara@mathformula.page

(Tomokazu Onozuka) Institute of Mathematics for Industry, Kyushu University 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

Email address: t-onozuka@imi.kyushu-u.ac.jp