SEMICLASSICAL ASYMPTOTICS OF THE BLOCH–TORREY OPERATOR IN TWO DIMENSIONS

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ABSTRACT. The Bloch–Torrey operator $-h^2\Delta + e^{i\alpha}x_1$ on a bounded smooth planar domain, subject to Dirichlet boundary conditions, is analyzed. Assuming $\alpha \in \left[0, \frac{3\pi}{5}\right)$ and a non-degeneracy assumption on the left-hand side of the domain, asymptotics of eigenvalues in the limit $h \to 0$ are derived. The strategy is a backward complex scaling and the reduction to a tensorized operator involving a real Airy operator and a complex harmonic oscillator.

1. INTRODUCTION

Let Ω be a smooth bounded open connected set in $\mathbb{R}^2$. Given a small positive parameter h and a fixed real constant $\alpha \in [0, \pi]$, we consider the operator

$$\mathscr{L}_{h,\alpha} = -h^2\Delta + e^{i\alpha}x_1 \tag{1.1}$$

in $L^2(\Omega)$, subject to Dirichlet boundary conditions. On its natural domain $\mathrm{Dom}(\mathscr{L}_{h,\alpha}) = H^2(\Omega) \cap H^1_0(\Omega)$, the operator is closed, has non-empty resolvent set and compact resolvent. Consequently, the spectrum is purely discrete and can be written as an infinite sequence of complex numbers tending to $+\infty$ in modulus or as a (possibly empty) finite sequence. The latter cannot be *a priori* excluded because $\mathscr{L}_{h,\alpha}$ is non-selfadjoint unless $\alpha \in \{0, \pi\}$. Our goal is to show the existence of "low-lying" eigenvalues and derive their asymptotics in the semiclassical limit $h \to 0$.

1.1. **Motivations.** There are two sources of motivation for this work. First, the selfadjoint situation $\alpha = 0$ has been recently analysed in [11] in the context of semiconductor devices exposed to a strong uniform electric field. Indeed, $h^{-2}\mathscr{L}_{h,0}$ is the Hamiltonian of an electron confined to a nanostructure of shape Ω, subject to singularly scaled electric potential $h^{-2}x_1$. The following geometric hypothesis is adopted in [11]:

Assumption 1. The minimum $\min\{x_1 : x \in \overline{\Omega}\}$ is uniquely attained at a point A_0, assumed to be $(0,0)$ (without loss of generality). Moreover, the (signed) curvature κ_0 of $\partial\Omega$ (computed with respect to the inner normal of Ω) at $A_0 = (0,0)$ is positive.

Let $(\lambda_n(h))_{n\geqslant 1}$ denote the non-decreasing sequence of the eigenvalues of $\mathscr{L}_{h,0}$, where each eigenvalue is repeated according to its multiplicity. The following asymptotic estimate of each individual eigenvalue was established in [11]:

Theorem 1.1 ([11]). *Assume $\alpha = 0$ and Assumption 1. Then, for all $n \geqslant 1$,*

$$\lambda_n(h) = z_1 h^{\frac{2}{3}} + (2n-1)h\sqrt{\frac{\kappa_0}{2}} + o(h) \tag{1.2}$$

as $h \to 0$, where z_1 is the absolute value of the smallest zero of the Airy function Ai.

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The eigenvalue splitting given by the second term containing the curvature is experimentally spectacular, for it enables one to determine the shape of a convex nanostructure by imposing uniform electric fields in various directions [17].

Second, there have been an intensive study of the operator (1.1) for the purely imaginary choice $\alpha = \frac{\pi}{2}$ in various geometric settings (and even for more general electric potentials) [1, 6, 5, 14, 7, 12, 3, 2, 4, 13, 18]. Among the variety of physical motivations mentioned in these references, let us point out the Bloch–Torrey equation describing the diffusion-precession of spin-bearing particles in nuclear magnetic resonance experiments.

In particular, in [12, Theorem 1.1], quasimodes are constructed and allow to conjecture the behavior of the eigenvalues with the smallest real part. Motivated by these constructions, the behavior of the real part of the left-most spectrum has then been analyzed, see [3, Theorem 1.6]. For our linear electric potential, we can apply, for instance, the more general results [14, Theorem 4.1.1] and [7, Theorem 1.1], and we get the following typical estimate[1].

Theorem 1.2 ([7])**.** *Assume $\alpha = \frac{\pi}{2}$ and Assumption 1. Then*

$$\inf \operatorname{Re} \operatorname{sp}(\mathscr{L}_{h,\alpha}) = \frac{z_1 h^{\frac{2}{3}}}{2} + o(h^{\frac{2}{3}}) \tag{1.3}$$

as $h \to 0$.

As observed in [7, Introduction], the lower bound in Theorem 1.2 can be proved without Assumption 1 (see also [1, 14]).

In this article we explain the transition between $\alpha = 0$ (Theorem 1.1) and $\alpha = \frac{\pi}{2}$ (Theorem 1.2). First of all, we show how α enters the constant coefficient in the first term of the asymptotic expansions (1.2) and (1.3). We also aim at providing the reader with an accurate description of the spectrum by exhibiting spectral gaps in the left-most part of the spectrum (similarly to Theorem 1.1 in the case when $\alpha = 0$). This question is all the more interesting that, when $\alpha \in (0, \pi)$, the operator $\mathscr{L}_{h,\alpha}$ is not selfadjoint and therefore, classical tools and strategies such as the min-max and spectral theorems (used, for instance, in [11]) have to be replaced by unconventional arguments. Throughout this paper, we use the nickname Bloch–Torrey operator for (1.1) even if we also consider the case $\alpha \neq \frac{\pi}{2}$.

1.2. **Heuristics.** Before stating our main results, let us explain the intuitive origin of Theorems 1.1 and 1.2. This is also the opportunity to discuss the heuristics of our main theorem, which is stated in Section 1.3.

When $\alpha = 0$ and under Assumptions 1, due to the Agmon estimates, we can check that the eigenfunctions associated with the lowest eigenvalues are localized near A_0. We will see that such a localization behavior persists in some sense for certain eigenfunctions when $\alpha \in [0, \pi]$, especially for those associated with the left-most eigenvalues when $\alpha \in \left[0, \frac{\pi}{2}\right)$. Anyway, this naively suggests to use the classical tubular coordinates near the (outer) boundary defined through the map

$$\Gamma(s,t) = \gamma(s) - t\mathbf{n}(s) = (\Gamma_1(s,t), \Gamma_2(s,t)), \tag{1.4}$$

where γ is the arc-length parametrization of the outer boundary of Ω, denoted by $\partial\Omega_0$, and $\mathbf{n}$ is the outward pointing normal of Ω. Let $L > 0$ be the half-length of $\partial\Omega_0$ and consider the torus $\mathbb{T}_{2L} = \mathbb{R}/(2L\mathbb{Z})$. The map Γ induces a smooth diffeomorphism from

[1]With our linear electric potential, in the case $\alpha = \frac{\pi}{2}$, the analysis in [7] shows that the left-most spectrum is determined by the left-most *and* right-most points of Ω.

$B_{\delta_0} = \mathbb{T}_{2L} \times (0, \delta_0)$ to the tubular neighborhood T_{δ_0} of width $\delta_0 > 0$ of $\partial\Omega_0$ lying inside Ω. In the coordinates (s,t), the operator (1.1) becomes

$$-h^2(1-t\kappa(s))^{-1}\partial_t(1-t\kappa(s))\partial_t - h^2(1-t\kappa(s))^{-1}\partial_s(1-t\kappa(s))^{-1}\partial_s + e^{i\alpha}\Gamma_1(s,t)\,,$$

acting in the Hilbert space $L^2(B_{\delta_0}, (1-t\kappa(s))\mathrm{d}s\mathrm{d}t)$. Here the curvature function κ is defined via the Frenet formula $\mathbf{n}' = \kappa\gamma'$.

According to Assumption 1 (which involves $\Gamma(0,0) = A_0$), we have $\Gamma_1(s,t) = t + \frac{\kappa_0}{2}s^2 + \mathscr{O}(ts^2 + |s|^3)$. Since $1 - t\kappa(s) \simeq 1$ when t is small, this suggests to consider the operator

$$\mathscr{P}_{h,\alpha} = -h^2\partial_s^2 - h^2\partial_t^2 + e^{i\alpha}\left(\frac{\kappa_0}{2}s^2 + t\right)\,, \tag{1.5}$$

acting on $L^2(\mathbb{R}^2_+, \mathrm{d}s\mathrm{d}t)$, subject to Dirichlet boundary condition at $t = 0$.

Taking profit of the analyticity (since it is linear) in the variable t, we make the formal dilation $t = ue^{-i\alpha/3}$. The model operator $\mathscr{P}_{h,\alpha}$ then becomes

$$\mathscr{N}_{h,\alpha} = e^{\frac{2i\alpha}{3}}(h^2D_u^2 + u) + h^2D_s^2 + e^{i\alpha}\frac{\kappa_0 s^2}{2}\,, \tag{1.6}$$

which is, up to multiplications by complex constants, the sum of a real Airy operator and a complex harmonic oscillator, whose resolvent and spectra are rather well-known. *Heuristically*, this allows us to describe the spectrum of $\mathscr{L}_{h,\alpha}$ accurately in appropriate regions of the complex plane.

1.3. **The main result.** The main result of this article is the following theorem, which can be guessed from the heuristics of the previous section.

Theorem 1.3. *Consider $\alpha \in \left[0, \frac{3\pi}{5}\right)$ and $R > 0$ with $R \notin (2\mathbb{N}-1)\sqrt{\frac{\kappa_0}{2}}$. Under Assumption 1, there exist $h_0 > 0$ such that for all $h \in (0, h_0)$ the following holds. The spectrum of $\mathscr{L}_{h,\alpha}$ lying in the disk $D(h^{\frac{2}{3}}e^{2i\alpha/3}z_1, Rh)$ is made of exactly $N = \lfloor \frac{R}{\sqrt{2\kappa_0}} + \frac{1}{2} \rfloor$ eigenvalues of algebraic multiplicity 1 and they satisfy, for all $n \in \{1, \ldots, N\}$,*

$$\lambda_n(\alpha, h) = h^{\frac{2}{3}}e^{2i\alpha/3}z_1 + (2n-1)he^{i\alpha/2}\sqrt{\frac{\kappa_0}{2}} + o(h) \tag{1.7}$$

as $h \to 0$. Moreover, for all $\alpha \in [0, \frac{\pi}{2})$, there exist $C, h_0 > 0$ such that, for all $h \in (0, h_0)$, we have

$$\inf \operatorname{Re} \operatorname{sp}(\mathscr{L}_{h,\alpha}) \geqslant z_1 h^{\frac{2}{3}} \cos\left(\frac{2\alpha}{3}\right) - Ch^{\frac{4}{3}}\,. \tag{1.8}$$

In particular,

$$\inf \operatorname{Re} \operatorname{sp}(\mathscr{L}_{h,\alpha}) = z_1 h^{\frac{2}{3}} \cos\left(\frac{2\alpha}{3}\right) + o(h^{\frac{2}{3}})$$

as $h \to 0$.

Theorem 1.3 is illustrated on Figure 1: there is exactly one eigenvalue (with algebraic multiplicity) in each small circle (which has radius $o(h)$) and there is no spectrum in the gray region when $\alpha \in \left[0, \frac{\pi}{2}\right)$.

Remark 1.4.

(i) Theorem 1.3 gives an accurate description of the spectrum in large balls of size h when $\alpha \in \left[0, \frac{3\pi}{5}\right)$, but it only states the one-term asymptotics of the eigenvalue with the smallest real part when $\alpha \in \left[0, \frac{\pi}{2}\right)$. When $\alpha \in \left[0, \frac{\pi}{2}\right)$, we will see that elliptic estimates using the real part of an operator (which is isospectral to $\mathscr{L}_{h,\alpha}$) are enough to establish the semiclassical localization near $(0,0)$ (in the Agmon sense) of the eigenfunctions associated with eigenvalues having a real part less than $Mh^{\frac{2}{3}}$. This localization is the key to get the lower bound (1.8). When $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{5}\right)$, these considerations must be slightly adapted by introducing a parameter β and by multiplying the operator by $e^{-i\beta}$. This rotation is the reason why the control of infimum of the real part is lost with our method. This aspect is discussed in more detail in Section 1.4.

(ii) Our assumptions allow us to deal with the case $\alpha = \frac{\pi}{2}$ and to get the asymptotic estimate (1.7). For more general potentials, see [7, Theorem 1.1], only the existence of one eigenvalue in the disk is ensured (the one corresponding to $n = 1$). Not only our theorem gives the existence of more eigenvalues, it also states that they are algebraically simple and that they are the only ones in the disk. The proof of this simplicity involves rather subtle and tedious elliptic estimates, especially to exclude the existence of Jordan blocks.

(iii) As we explain in Section 1.5, the analysis used to establish Theorem 1.3 strongly relies on the analyticity of $V(x) = x_1$. However, it seems that arguments such as analytic dilations have not yet been used to investigate the spectrum of such Bloch–Torrey operators on domains. We believe that our method is of independent interest. It can easily be extended to more general analytic potentials V (still satisfying the generic assumptions in [7, 3]) and we may even think that it could be used to deal with smooth V by means of almost analytic extensions.

(iv) Unfortunately, our strategy does not allow us to recover Theorem 1.2, even though, at a formal level, (1.8) would give the appropriate lower bound when $\alpha = \frac{\pi}{2}$. In this case, our analysis shows that the real parts of two networks of eigenvalues cross, see Section 1.4.

(v) Our theorem does not say anything about the eigenfunctions (even if one could prove that they are localized near A_0 when $\alpha \in \left[0, \frac{\pi}{2}\right)$). Their accurate localization properties (in the Agmon sense) would be quite natural to investigate.

1.4. **Consequences and extensions.** The analysis in this article can be used to get an *a priori* location of the spectrum in the case when $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{5}\right)$.

Proposition 1.5. *Consider* $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{5}\right)$. *There exists*[2] $\beta \in \left(0, \frac{\pi}{10}\right]$ *with* $\frac{2\alpha}{3} \in \left(\beta - \frac{\pi}{2}, \beta + \frac{\pi}{2}\right)$, $C > 0$, *and* $h \in (0, h_0)$ *such that, for all* $h \in (0, h_0)$,

$$\inf \operatorname{Re} e^{-i\beta} \operatorname{sp}(\mathscr{L}_{h,\alpha}) \geqslant z_1 h^{\frac{2}{3}} \cos\left(\frac{2\alpha}{3} - \beta\right) - Ch^{\frac{4}{3}}.$$

In others terms, the eigenvalues λ *of* $\mathscr{L}_{h,\alpha}$ *belong to the half-plane given by*

$$\cos\beta \operatorname{Re}\lambda + \sin\beta \operatorname{Im}\lambda \geqslant z_1 h^{\frac{2}{3}} \cos\left(\frac{2\alpha}{3} - \beta\right) - Ch^{\frac{4}{3}}.$$

Proposition 1.5 is illustrated by Figure 2: there is no spectrum on the left of the dashed oblique line. In fact, in this case, there exist eigenvalues with a smaller real part (as one can see on the same figure). They are related to the right-most part of the domain.

Assumption 2. The maximum $\max\{x_1 : x \in \overline{\Omega}\} = x_{1,\max}$ is uniquely attained at a point A_1. Moreover, the curvature κ_1 of $\partial\Omega$ at A_1 is positive.

[2] In fact, all β such that $(\alpha, \beta) \in \mathcal{T}$ works, where $\mathcal{T}$ is given in Lemma 3.3.

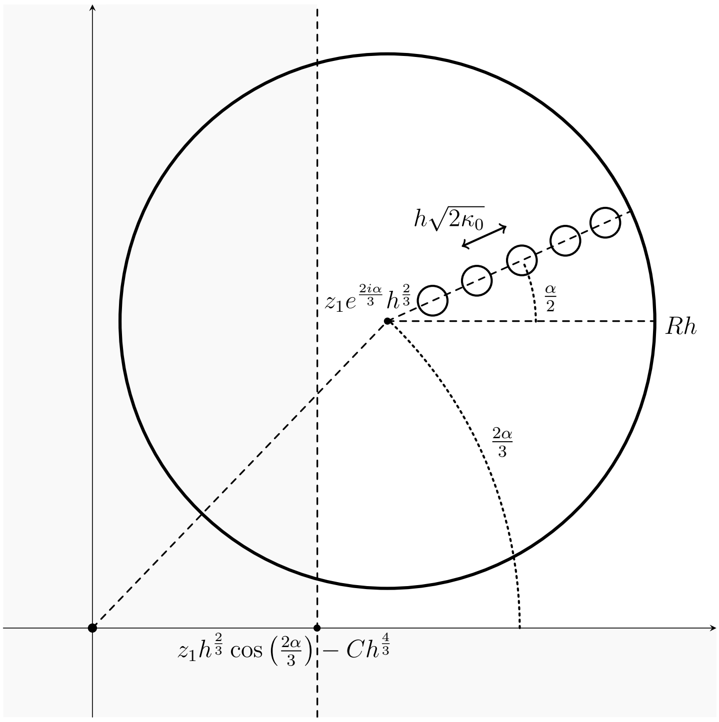

FIGURE 1. The spectrum in the disk of center $z_1 e^{\frac{2i\alpha}{3}} h^{\frac{2}{3}}$ and radius Rh, when $\alpha \in \left[0, \frac{3\pi}{5}\right)$.

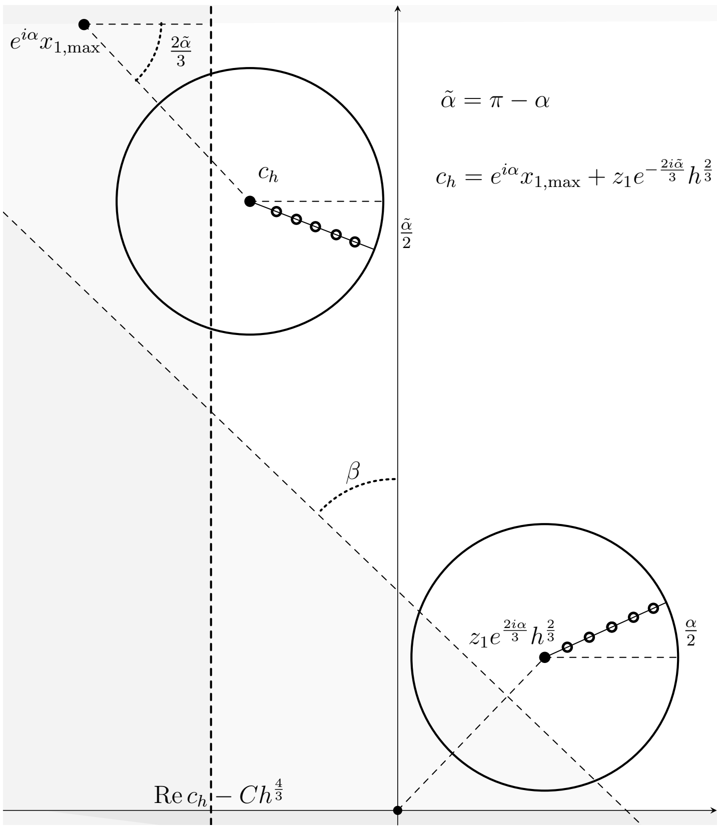

FIGURE 2. Case when $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{5}\right)$.

We let $\tilde{\alpha} = \pi - \alpha$ and we consider $\alpha \in \left(\frac{2\pi}{5}, \pi\right]$ so that $\tilde{\alpha} \in \left[0, \frac{3\pi}{5}\right)$. Then, the affine change of variable $y = F(x) = (-x_1 + x_{1,\max}, x_2)$ transforms $\mathscr{L}_{h,\alpha}^*$ into the unitarily equivalent operator

$$U^* \mathscr{L}_{h,\alpha}^* U = \mathscr{L}_{h,\tilde{\alpha}} + e^{-i\alpha} x_{1,\max}\,, \quad \mathrm{Dom}(\mathscr{L}_{h,\tilde{\alpha}}) = H^2(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega})\,, \quad \tilde{\Omega} = F(\Omega)\,.$$

Therefore, under Assumption 2, we can apply Theorem 1.3 to $\mathscr{L}_{h,\tilde{\alpha}}$ and we get the following.

Corollary 1.6. *Consider $\alpha \in \left(\frac{2\pi}{5}, \pi\right]$ and $R > 0$ with $R \notin (2\mathbb{N} - 1)\sqrt{\frac{\kappa_1}{2}}$. Then, the spectrum of $\mathscr{L}_{h,\alpha}$ lying in the disk $D(e^{i\alpha} x_{1,\max} + h^{\frac{2}{3}} e^{-2i\tilde{\alpha}/3} z_1, Rh)$ is made of N eigenvalues of algebraic multiplicity 1 and satisfying, for all $n \in \{1, \ldots, N\}$,*

$$\tilde{\lambda}_n(\alpha, h) = e^{i\alpha} x_{1,\max} + h^{\frac{2}{3}} e^{-2i\tilde{\alpha}/3} z_1 + (2n-1) h e^{-i\tilde{\alpha}/2} \sqrt{\frac{\kappa_1}{2}} + o(h)$$

as $h \to 0$. Moreover, when $\alpha \in \left(\frac{\pi}{2}, \pi\right]$,

$$\inf \mathrm{Re}\, \mathrm{sp}(\mathscr{L}_{h,\alpha}) = \cos(\alpha) x_{1,\max} + z_1 h^{\frac{2}{3}} \cos\left(\frac{2\tilde{\alpha}}{3}\right) + o(h^{\frac{2}{3}})$$

as $h \to 0$.

When $\alpha \in \left(\frac{2\pi}{5}, \frac{3\pi}{5}\right)$, under Assumptions 1 and 2, Theorem 1.3 and Corollary 1.6 apply. Therefore, we have coexistence of (at least) two networks of eigenvalues. This phenomenon is illustrated on Figure 2 when $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{5}\right)$: we see that the right-most part of Ω determines the left-most part of the spectrum.

1.5. **Organization and strategy.** The article is organized as follows. In Section 2, we make the analytic dilation argument rigorous. We introduce an analytic family of operators $(\mathscr{L}_{h,\alpha,\theta})_{\theta \in \Theta}$ in the sense of Kato. To do so, we use a real dilation with respect to the distance to the outer boundary t (acting only near the boundary). This is where we take advantage of the fact that Γ (see (1.4)) is always analytic in t. When $\theta \in \mathbb{R}$, $\mathscr{L}_{h,\alpha,\theta}$ is isospectral to $\mathscr{L}_{h,\alpha}$ (see Lemma 2.5). By the Kato theory, it is also isospectral to $\mathscr{M}_{h,\alpha} = \mathscr{L}_{h,\alpha,-i\frac{\alpha}{3}}$ (see Corollary 2.8).

In Section 3, we see that this special choice of complex parameter is particularly convenient since we can rather easily prove that the eigenfunctions of $\mathscr{M}_{h,\alpha}$ associated with eigenvalues located in a half-plane of the form $\mathrm{Re}\,(e^{-i\beta}\lambda) \leqslant M h^{\frac{2}{3}}$ are exponentially localized near A_0, at the scale $h^{\frac{2}{3}}$ near the boundary and $h^{\frac{1}{3}}$ along the boundary, see Proposition 3.5. Let us underline that, at this stage, the analytic dilation is not crucial to prove this localisation near A_0. One could prove it for the eigenfunctions of our original operator. The main interest of the analytic dilation will only appear in Section 4 when we establish an optimal tangential localization estimate near A_0 (the scale $h^{\frac{1}{3}}$ is *not* optimal). The introduction of the parameter β and the constraint on $\alpha \in \left[0, \frac{3\pi}{5}\right)$ originate from these localization arguments, which are based on ellipticity/coercivity estimates, induced by the complex electric potential (after the change of coordinates and the complex dilation), see Lemma 3.4.

Section 4 is devoted to the spectral analysis of $\mathscr{M}_{h,\alpha}$. When $\alpha \in \left[0, \frac{\pi}{2}\right)$, the asymptotic estimate of the infimum part of the spectrum is obtained, see Proposition 4.1 and its proof given in Section 4.1. This proves (1.8) (see also Remark 4.4, which proves Proposition 1.5). When $\alpha \in \left[0, \frac{3\pi}{5}\right)$, we first prove that the spectrum in the disk mentioned in Theorem 1.3 is necessarily close (essentially at a distance of order $h^{\frac{3}{2}}$) to the eigenvalue of a model operator $\mu_n(h, \alpha) = h^{\frac{2}{3}} e^{2i\alpha/3} z_1 + (2n-1) h e^{i\alpha/2} \sqrt{\frac{\kappa_0}{2}}$, see Proposition 4.2. Then, we prove that there is exactly one eigenvalue (with algebraic muliplicity 1) in these small discs (see Proposition 4.3 and Figure 1), and we deduce Theorem 1.3.

The proof of Proposition 4.2 relies on three important ingredients. First, it requires resolvent estimates of $\mathscr{N}_{h,\alpha}$ (we recall that $\mathscr{N}_{h,\alpha}$ is given in (1.6)), see Proposition 4.5. The fact that we performed an analytic dilation in t is a crucial help to get the control of the resolvent (by simply estimating the real part of the operator and by avoiding semi-groups estimates). The second ingredient is to show that the eigenfunctions associated with the eigenvalues in our disc are good quasimodes for $\mathscr{N}_{h,\alpha}$, see Proposition 4.6. To do so, we need to prove optimal localization estimates with respect to the curvilinear abscissa s (see Proposition 4.12) – at the scale $h^{\frac{1}{2}}$, and not only $h^{\frac{1}{3}}$ as given by the Agmon estimates of Section 3 – in order to estimate the remainders of order s^3 when Taylor expanding the electric potential. We stress that estimating the real part of $\mathscr{M}_{h,\alpha}$ is a key to get such estimates (and that this argument succeeds thanks to the analytic dilation). Proposition 4.6 and the resolvent estimate are then enough to locate the spectrum in the small discs.

The fact that the rank of the Riesz projector is at most one requires more work. This is where the third ingredient comes into play. We assume that this rank is at least two and even that we have a Jordan block (in the worst scenario) and we prove that a generalized eigenfunction also satisfies accurate localization estimates, see Section 4.4 and especially Proposition 4.13. This part of the proof is technically more involved and it is somewhat reminiscent of Caccioppoli estimates, see Proposition 4.14. There remains to estimate the Riesz projectors to get a contradiction, see Section 4.5.1. To prove that the projectors are non-zero, we consider a quasimode built from the Airy and Hermite functions, see Section 4.5.2.

2. The analytic dilation

2.1. **The sesquilinear form.** Before introducing the main idea of this paper, let us stress that the operator $\mathscr{L}_{h,\alpha}$ from (1.1) is rigorously introduced via its sesquilinear form defined on $H^1_0(\Omega)$ by

$$L_{h,\alpha}(\varphi,\psi) = \int_\Omega \nabla\varphi\,\overline{\nabla\psi}\,\mathrm{d}x + e^{i\alpha}\int_\Omega x_1\varphi\,\overline{\psi}\,\mathrm{d}x\,.$$

Notice that

$$\operatorname{Re} L_{h,\alpha}(\psi,\psi) \geqslant \|\nabla\psi\|^2 - \sup_{x\in\Omega}|x_1|\,\|\psi\|^2\,,$$

which enables to apply the standard Lax–Milgram theorem. An elementary argument shows that

$$\operatorname{sp}(\mathscr{L}_{h,\alpha}) \subset \{\lambda\in\mathbb{C} : 0 \leqslant \operatorname{Im}\lambda \leqslant (\sin\alpha)\sup_\Omega x_1\}\,.$$

2.2. **An isospectral operator.** Following the intuition described in Section 1.2, we would like to perform a complex scaling in the normal variable to the outer boundary. By doing that, we will preserve the spectrum as soon as we have a family of type (B) in the sense of Kato [16, Chap. VII]. This will reveal some hidden elliptic properties of the new operator.

Let $\delta\in(0,\delta_0)$ where we recall that δ_0 is defined just after (1.4). The heuristic considerations of Section 1.2 lead to introduce the following unitary transform $\mathscr{U}_\theta$, depending on the real parameter θ. For all $\varphi\in L^2(\Omega)$, we let

$$\mathscr{U}_\theta\varphi = (\varphi_{|\Omega\setminus T_\delta}, \varphi_{|T_\delta}\circ\Gamma(s,J_\theta(u)))\,,$$

with J_θ given by

$$t = J_\theta(u) = ue^{\theta\chi(u)}\,,$$

where χ is non-increasing smooth function from $[0,\delta]$ to $[0,1]$ such that $\chi=1$ near 0 and $\chi=0$ near δ. For all $\epsilon>0$, we can choose χ so that

$$\|\chi'\|_\infty \leqslant \frac{1+\epsilon}{\delta}\,. \tag{2.1}$$

Note that $t=J_\theta(u)=ue^\theta$ near 0 and that $t=J_\theta(u)=u$ at a distance larger than δ of the outer boundary and that the change of variable is smooth in between. There exists $\theta_0>0$ such that, for all $\theta\in(-\infty,\theta_0)$, the map $J_\theta:(0,\delta)\to(0,\delta)$ is smooth diffeomorphism and, for all $u\in(0,\delta)$,

$$J'_\theta(u)=(1+\theta u\chi'(u))e^{\theta\chi(u)}>0\,. \tag{2.2}$$

We let

$$m_\theta(s,u)=1-J_\theta(u)\kappa(s)\,, \tag{2.3}$$

where $\kappa(s)$ is the curvature at the point of curvilinear coordinate s. Thanks to a change of variables, we have the following.

Lemma 2.1. *For all $\theta\in(-\infty,\theta_0)$, $\mathscr{U}_\theta$ is an isometry from $L^2(\Omega)$ to the product $E_\theta := L^2(\Omega\setminus T_\delta)\times L^2(B_\delta, m_\theta(s,u)J'_\theta(u)\mathrm{d}s\mathrm{d}u)$. As vector spaces, we have $E_\theta=E_0=L^2(\Omega\setminus T_\delta)\times L^2(B_\delta)$.*

Then, let us describe the effect of $\mathscr{U}_\theta$ on the form domain of the operator $\mathscr{L}_{h,\alpha}$, which is $H^1_0(\Omega)$.

Lemma 2.2. *We have*

$$\begin{aligned}\mathscr{U}_\theta(H^1_0(\Omega)) &= \{(\varphi_1,\varphi_2)\in H^1(\Omega\setminus T_\delta)\times H^1(B_\delta) : \varphi_2(s,0)=0 \;\&\; \varphi_1(\Gamma(s,\delta))=\varphi_2(s,\delta)\}\\ &= \mathscr{U}_0(H^1_0(\Omega))\,.\end{aligned}$$

Proof. Let us first notice that, by standard trace theorems, the functions ϕ_2 is well defined a.e. on $\{t=\delta\}$ (where coordinates t and u coincide), as well as $\phi_1\circ\Gamma$ on ∂B_δ. For a smooth function ϕ, we have $\varphi_1(\Gamma(s,\delta))=\varphi_2(s,\delta)$ and the result follows by density and the fact that $\mathscr{U}_\theta$ is an isometry. ☐

Let us now consider the quadratic form induced by $\mathscr{U}_\theta$ from $L_{h,\alpha}$.

Proposition 2.3. *Letting, for all $\varphi\in\mathscr{U}_0(H^1_0(\Omega))$,*

$$\begin{aligned}\ell_{h,\alpha,\theta}(\varphi,\varphi) &= \int_{\Omega\setminus T_\delta}\left(|h\nabla\varphi_1|^2+e^{i\alpha}x_1|\varphi_1|^2\right)\mathrm{d}x\\ &\quad+\int_{B_\delta}(m_\theta^{-2}|h\partial_s\varphi_2|^2+[J'_\theta]^{-2}|h\partial_u\varphi_2|^2+e^{i\alpha}\Gamma_1(s,J_\theta(u))|\varphi_2|^2)m_\theta J'_\theta(u)\mathrm{d}s\mathrm{d}u\,,\end{aligned}$$

we have

$$L_{h,\alpha}(\mathscr{U}_\theta^{-1}\varphi,\mathscr{U}_\theta^{-1}\varphi)=\ell_{h,\alpha,\theta}(\varphi,\varphi)\,.$$

Proof. For all $\varphi\in\mathscr{U}_0(H^1_0(\Omega))$, we let $\psi=\mathscr{U}_\theta^{-1}\varphi\in H^1_0(\Omega)$. Let us first describe the kinetic part:

$$\begin{aligned}\int_\Omega|h\nabla\psi|^2\,\mathrm{d}x &= \int_{\Omega\setminus T_\delta}|h\nabla\psi|^2\,\mathrm{d}x+\int_{T_\delta}|h\nabla\psi|^2\,\mathrm{d}x\\ &= \int_{\Omega\setminus T_\delta}|h\nabla\varphi_1|^2\,\mathrm{d}x+\int_{B_\delta}(m_\theta^{-2}|h\partial_s\varphi_2|^2+[J'_\theta]^{-2}|h\partial_u\varphi_2|^2)m_\theta(s,u)J'_\theta(u)\mathrm{d}s\mathrm{d}u\,.\end{aligned}$$

Using the changes of variable $x\mapsto(s,u)$ on B_δ for the non-kinetic part completes the proof. ☐

From Proposition 2.3, we see that $\mathscr{U}_\theta \mathscr{L}_{h,\alpha} \mathscr{U}_\theta^{-1}$ is the operator associated with $\ell_{h,\alpha,\theta}$ in the ambient space E_θ (with the weighted scalar product, which depends on θ). To avoid the θ-dependence of the ambient space through its scalar product, we can consider the isometry

$$\mathscr{V}_\theta : \varphi \mapsto (\varphi_1, \underbrace{m_\theta^{\frac{1}{2}} (J_\theta')^{\frac{1}{2}} \varphi_2}_{=\phi_2}) ,$$

from $L^2(\Omega \setminus T_\delta) \times L^2(B_\delta, m_\theta(s,u) J_\theta'(u) \mathrm{d}s\mathrm{d}u)$ to $L^2(\Omega \setminus T_\delta) \times L^2(B_\delta, \mathrm{d}s\mathrm{d}u)$.

Lemma 2.4. *Let $\varphi \in \mathscr{U}_0(H_0^1(\Omega))$ and $\phi = \mathscr{V}_\theta \varphi = (\varphi_1, \phi_2)$. We have*

$$\int_{B_\delta} (m_\theta^{-2} |h\partial_s \varphi_2|^2 + [J_\theta']^{-2} |h\partial_u \varphi_2|^2) m_\theta(s,u) J_\theta'(u) \mathrm{d}s\mathrm{d}u$$
$$= \int_{B_\delta} (m_\theta^{-2} |h\partial_s \phi_2|^2 + [J_\theta']^{-2} |h\partial_u \phi_2|^2 + h^2 V_\theta(s,u) |\phi_2|^2) \mathrm{d}s\mathrm{d}u + h^2 \int_{-L}^{L} W_\theta(s) |\phi_2(s,\delta)|^2 \mathrm{d}s ,$$

where, letting $X = m_\theta^{-1/2} (J_\theta')^{-1/2}$ we have

$$V_\theta = m_\theta^{-2} (\partial_s X)^2 + (J_\theta')^{-2} (\partial_u X)^2 - \partial_s \big((m_\theta^{-2}) X \partial_s X \big) - \partial_u \big((J_\theta')^{-2} X \partial_u X \big)$$

and

$$W_\theta(s) = (J_\theta')(s,\delta) X(s,\delta) (\partial_u X)(s,\delta) .$$

Proof. This follows from two integrations by parts and from the fact that $\phi_2(s,0) = 0$ *a.e.* and that J_θ' is constant near ∂B_δ. As we shall see later, the exact values of V_θ and W_θ are unimportant; we only note that they are smooth. ☐

These considerations lead to define the following quadratic form, in the ambient Hilbert space $L^2(\Omega \setminus T_\delta) \times L^2(B_\delta, \mathrm{d}s\mathrm{d}u)$, for all $\varphi \in \mathscr{U}_0(H_0^1(\Omega))$,

$$\begin{aligned} L_{h,\alpha,\theta}(\varphi,\varphi) &= \int_{\Omega \setminus T_\delta} \left(|\nabla \varphi_1|^2 + e^{i\alpha} x_1 |\varphi_1|^2 \right) \mathrm{d}x \\ &+ \int_{B_\delta} (m_\theta^{-2} |h\partial_s \phi_2|^2 + [J_\theta']^{-2} |h\partial_u \phi_2|^2 + (e^{i\alpha} \Gamma_1(s, J_\theta(u)) + h^2 V_\theta(s,u)) |\phi_2|^2) \mathrm{d}s\mathrm{d}u \\ &+ h^2 \int_{-L}^{L} W_\theta(s) |\phi_2(s,\delta)|^2 \mathrm{d}s , \end{aligned} \tag{2.4}$$

where we recall that $\phi = \mathscr{V}_\theta \varphi = (\varphi_1, \phi_2)$.

We get the following lemma.

Lemma 2.5. *The operator associated with $L_{h,\alpha,\theta}$ is $\mathscr{L}_{h,\alpha,\theta} = \mathscr{V}_\theta \mathscr{U}_\theta \mathscr{L}_{h,\alpha} \mathscr{U}_\theta^{-1} \mathscr{V}_\theta^{-1}$. In particular, the spectrum of the operator $\mathscr{L}_{h,\alpha,\theta}$ is the same as that of $\mathscr{L}_{h,\alpha}$.*

2.3. **Complex deformation parameters.** According to our heuristic discussion, we would like to consider complex θ (as in the seminal paper [9]). More precisely, we would like the family $(\mathscr{L}_{h,\alpha,\theta})_{\theta \in \Theta}$ to be analytic of type (B) in the sense of Kato, where Θ is a connected open set containing $\theta = 0$ and $\theta = -i\frac{\alpha}{3}$. First, we notice that the form domain $\mathscr{U}_0(H_0^1(\Omega))$ is independent of θ and that, for all $\varphi \in \mathscr{U}_0(H_0^1(\Omega))$,

$$\Theta \ni \theta \mapsto L_{h,\alpha,\theta}(\varphi,\varphi) \in \mathbb{C}$$

is analytic. Then, it is sufficient to check that the form $L_{h,\alpha,\theta}$ is sectorial and closed on $\mathscr{U}_0(H_0^1(\Omega))$ for $\theta \in \Theta$.

Lemma 2.6. *Let $\theta_0 > 0$ and $\beta_0 \in (0, \frac{\pi}{4})$. For $\eta > 0$ let us consider the rectangle $\Theta_\eta = (-\theta_0, \eta) + i(-\beta_0, \eta)$. Then, if η and δ are small enough, there exists $c > 0$ such that, for all $\theta \in \Theta_\eta$, and all $u \in (0, \delta)$,*

$$\mathrm{Re}\, J_\theta'^{-2}(u) \geqslant c > 0\,.$$

Proof. Writing $\theta = \theta_1 + i\theta_2$ with $\theta_1, \theta_2 \in \mathbb{R}$ and taking $u \in (0, \delta)$, we notice that

$$J_\theta'^{-2}(u) = |J_\theta'|^{-4} J_{\overline{\theta}}'^2 = |J_\theta'|^{-4}(1 + \theta_1 u\chi' - i\theta_2 u\chi')^2 e^{2\theta_1\chi} e^{-2i\theta_2\chi}\,,$$

so that, by using that δ is small, we can write

$$J_\theta'^{-2}(u) = |J_\theta'|^{-4} J_{\overline{\theta}}'^2 = e^{2\theta_1\chi}|J_\theta'|^{-4} e^{-2i\arctan\left(\frac{\theta_2 u\chi'}{1+\theta_1 u\chi'}\right) - 2i\theta_2\chi} = e^{2\theta_1\chi}|J_\theta'|^{-4} e^{-iA(u,\theta)}\,, \tag{2.5}$$

where the argument $A(u, \theta)$ is given by

$$A(u, \theta) = 2\arctan\left(\frac{\theta_2 u\chi'}{1 + \theta_1 u\chi'}\right) + 2\theta_2\chi. \tag{2.6}$$

When θ_2 is positive, we notice that, for all $a > 0$, by choosing η small enough, and by using (2.1), we have, for all $\theta_2 \in (0, \eta)$, $|A(u, \theta)| \leqslant a$.

When θ_2 is non-positive, namely $\theta_2 \in (-\beta_0, 0]$, we have, by using again (2.1), that

$$-2\beta_0 \leqslant 2\theta_2 \leqslant 2\theta_2\chi \leqslant A(u, \theta) \leqslant \frac{2\theta_2 u\chi'}{1 + \theta_1 u\chi'} \leqslant 2(1 + 2\eta)|\theta_2||u\chi'|\,.$$

Thus,

$$-\frac{\pi}{2} < -2\beta_0 \leqslant A(u, \theta) \leqslant (1 + 4\eta)2\beta_0 < \frac{\pi}{2}\,.$$

We therefore get the result by (2.5). $\square$

Proposition 2.7. *There exist $c, C > 0$ such that, for all $\varphi \in \mathscr{U}_0(H_0^1(\Omega))$,*

$$\mathrm{Re}\, L_{h,\alpha,\theta}(\varphi, \varphi) \geqslant c\|h\nabla\varphi\|_{E_0}^2 - C\|\varphi\|_{E_0}^2\,.$$

Proof. Thanks to (2.4), there exists $C > 0$ such that the following holds: for all $\delta, \epsilon > 0$, there exists $C_{\delta,\epsilon} > 0$ such that, for all $\varphi \in \mathscr{U}_0(H_0^1(\Omega))$,

$$\begin{aligned}\mathrm{Re}\, L_{h,\alpha,\theta}(\varphi, \varphi) \geqslant &\int_{B_\delta} |h\partial_s\phi_2|^2 \mathrm{d}s\mathrm{d}u + \int_{B_\delta} \mathrm{Re}\, J_\theta'^{-2}(u)|h\partial_u\phi_2|^2 \mathrm{d}s\mathrm{d}u \\ &+ \|h\nabla\varphi_1\|_{L^2(\Omega\setminus T_\delta)}^2 - C\delta\|h\nabla\phi_2\|_{L^2(B_\delta)}^2 - C\|\varphi\|_{E_0}^2 - C(\epsilon\|h\nabla\phi_2\|^2 + C_{\delta,\epsilon}h^2\|\phi_2\|^2)\,,\end{aligned}$$

where we used the classical estimate

$$\int_{-L}^{L} |\phi_2(s, \delta)|^2 \mathrm{d}s \leqslant \epsilon\|\nabla\phi_2\|^2 + C_{\delta,\epsilon}\|\phi_2\|^2\,.$$

This concludes the proof. $\square$

Corollary 2.8. *The operators $\mathscr{L}_{h,\alpha}$ and $\mathscr{L}_{h,\alpha,-i\alpha/3}$ are isospectral.*

Proof. It is a consequence of the analytic pertubation theory, upon observing that $-i\frac{\alpha}{3} \in \Theta_\eta$ since $-\beta < -\frac{\alpha}{3}$ is equivalent to $\alpha < 3\beta$ which is satisfied for all $\alpha \in [0, \frac{3\pi}{4})$ as soon as β is close enough to $\frac{\pi}{4}$. $\square$

3. Localization estimates

In virtue of Corollary 2.8, we now focus on the spectral analysis of $\mathscr{L}_{h,\alpha,-i\alpha/3}$, for which ellipticity properties are established in the present section.

Definition 3.1. We denote $\mathscr{M}_{h,\alpha} = \mathscr{L}_{h,\alpha,-i\alpha/3}$ and by $M_{h,\alpha}$ the associate quadratic form on E_0, see (2.4) and Lemma 2.1.

In the following series of lemmas, we show bounds from below for the potential part of $e^{-i\beta}\mathscr{M}_{h,\alpha}$, where β is introduced to correct a lack of coercivity of the real part when α is larger than $\frac{\pi}{2}$. These lemmas lead to Proposition 3.5, which provides us with a precise semiclassical localization of the eigenfunctions of $\mathscr{M}_{h,\alpha}$.

The first lemma shows that the result of Lemma 2.6 remains true if we insert $e^{-i\beta}$.

Lemma 3.2. *Consider $\alpha \geqslant 0$ and $\beta \in \mathbb{R}$ such that*

$$\beta - \frac{2\alpha}{3} > -\frac{\pi}{2}, \quad \beta + \frac{2\alpha}{3} < \frac{\pi}{2}.$$

Then, there exists $C(\alpha,\beta) > 0$ such that, for all $u \in [0,\delta]$,

$$\mathrm{Re}\,(e^{-i\beta} J_\theta'^{-2}(u)) \geqslant C(\alpha,\beta).$$

Proof. We have

$$\begin{aligned}\mathrm{Re}\,(e^{-i\beta}J_\theta'^{-2}) &= \mathrm{Re}\,\left[e^{-i\beta+\frac{2i\alpha}{3}\chi}(1-i\frac{\alpha}{3}u\chi')^{-2}\right]\\ &= \left(1+\frac{\alpha^2}{9}(u\chi')^2\right)^{-1}\mathrm{Re}\,\left[e^{-i\beta+\frac{2i\alpha}{3}\chi-2i\arctan(\frac{\alpha}{3}|u\chi'|)}\right]\\ &= \mathrm{Re}\,\left[e^{-i\beta-iA(u,-i\alpha/3)}\right],\end{aligned}$$

where we recall $A(u,\theta)$ is defined in (2.6). Note that, by using (2.1), we have, for $\epsilon > 0$ small enough,

$$-\frac{\pi}{2} < \beta - \frac{2\alpha}{3} \leqslant \beta - \frac{2\alpha}{3}\chi + 2\arctan(\frac{\alpha}{3}|u\chi'|) \leqslant \beta + \frac{2\alpha}{3}(1+\epsilon) < \frac{\pi}{2},$$

so that $\mathrm{Re}\,(e^{-i\beta-iA(u,-i\alpha/3)})$ is uniformly bounded from below by a positive constant. This gives the result. □

The following lemma is a preparation lemma in order to get the ellipticity of the electric potential in $\mathscr{M}_{h,\alpha}$.

Lemma 3.3. *We let*

$$\mathcal{T} = \left\{(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R} : \beta - \frac{2\alpha}{3} > -\frac{\pi}{2}, \quad \beta + \frac{2\alpha}{3} < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \alpha - \beta < \frac{\pi}{2}\right\}.$$

Then $\sup\{\alpha : (\alpha,\beta) \in \mathcal{T}\} = \frac{3\pi}{5}$. Moreover, for all $\alpha \in \left[0, \frac{3\pi}{5}\right)$, we have $(\alpha, \frac{\pi}{10}) \in \mathcal{T}$. For all $\alpha \in [0, \frac{\pi}{2})$, we have $(\alpha, 0) \in \mathcal{T}$. We also have that $\left(\frac{\pi}{2}, \beta\right) \in \mathcal{T}$ for β positive and small enough.

Proof. These estimates follow from straightforward computations, which are conveniently supported by drawing a picture. We leave the details to the reader. □

Lemma 3.4. *Assume that $\alpha \in \left[0, \frac{3\pi}{5}\right)$ and consider β such that $(\alpha,\beta) \in \mathcal{T}$. There exist $s_0, \delta_0, c > 0$ such that the following holds. For all $\delta \in (0,\delta_0)$, for all $(s,u) \in [-L,L) \times (0,\delta)$, if $|s| \geqslant s_0$, then*

$$\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) \geqslant c,$$

and, if $|s| \leqslant s_0$,

$$\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) \geqslant c(u+s^2)\,.$$

Proof. Consider (α,β) as in the statement. We have

$$\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) = \mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, ue^{-i\frac{\alpha}{3}\chi}))\,.$$

From (1.4), we have

$$\Gamma_1(s, e^{-i\frac{\alpha}{3}\chi}u) = \gamma_1(s) - ue^{-i\frac{\alpha}{3}\chi}n_1(s)\,,$$

and thus, by using the Taylor expansion with respect to s near 0 and Assumption 1, we get

$$\Gamma_1(s, e^{-i\frac{\alpha}{3}\chi}u) = ue^{-i\frac{\alpha}{3}\chi} + \frac{\kappa_0}{2}s^2 + \mathscr{O}(|s|^3 + us^2)\,.$$

Therefore,

$$\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) = u\cos\left(\alpha-\beta-\frac{\alpha}{3}\chi\right) + \frac{\kappa_0}{2}s^2\cos(\alpha-\beta) + \mathscr{O}(|s|^3 + us^2)\,.$$

Since $(\alpha,\beta)\in\mathcal{T}$,

$$-\frac{\pi}{2} < \frac{2\alpha}{3} - \beta \leqslant \alpha-\beta-\frac{\alpha}{3}\chi < \frac{\pi}{2} \quad \text{and } -\frac{\pi}{2} < \alpha-\beta < \frac{\pi}{2}\,.$$

Therefore, there exist $s_0, \delta_0 > 0$ such that, for all $\delta\in(0,\delta_0)$, there exists $c>0$ such that, for all s such that $|s|\leqslant s_0$ and all $u\in(0,\delta)$, we have

$$\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) \geqslant c(u+s^2)\,.$$

Let us now study the case when $|s|\geqslant s_0$. We first introduce

$$B(s,u) = \arctan\left(\frac{u\sin(\frac{\alpha}{3}\chi(u))n_1(s)}{\gamma_1(s) - u\cos(\frac{\alpha}{3}\chi(u))n_1(s)}\right)\,,$$

so that

$$\Gamma_1(s, e^{-i\frac{\alpha}{3}\chi}u) = |\gamma_1(s) - ue^{-i\frac{\alpha}{3}\chi}n_1(s)|e^{iB(s,u)}.$$

We notice that when $|s|\geqslant s_0$, and choosing δ small enough, we have $\gamma_1(s) - u\cos(\frac{\alpha}{3}\chi)n_1(s)\geqslant c_0>0$ uniformly. This implies that

$$\begin{aligned}\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) &\geqslant |\gamma_1(s) - ue^{-i\frac{\alpha}{3}\chi}n_1(s)|\cos(\alpha-\beta+B(s,u))\\ &\geqslant c_0\cos(\alpha-\beta+B(s,u))\,.\end{aligned}$$

From the expression of $B(s,u)$, we deduce that, with a possibly smaller c, we have, for all $|s|\geqslant s_0$,

$$\mathrm{Re}\,(e^{i(\alpha-\beta)}\Gamma_1(s, J_\theta(u))) \geqslant c > 0\,.$$

This completes the proof. □

The following proposition gives Agmon type localization estimates for some eigenfunctions of $\mathscr{M}_{h,\alpha}$.

Proposition 3.5. *Let* $\alpha\in[0,\frac{3\pi}{5})$ *and consider suitable parameters* β, δ_0 *introduced in Lemma 3.4. Then for any* $M>0$ *and* $0<\delta<\delta_0$, *there exists* $h_0, C>0$ *such that for all* $h\in(0,h_0)$, *all eigenvalue* λ *(of* $\mathscr{M}_{h,\alpha}$*) such that* $\mathrm{Re}\,(e^{-i\beta}\lambda)\leqslant Mh^{\frac{2}{3}}$ *and all associated eigenfunction* $\varphi = (\varphi_1,\phi_2)$, *we have*

$$\int_{\Omega\setminus T_\delta} e^{2|x|/h^{\frac{2}{3}}}|\varphi_1|^2\mathrm{d}x + \int_{B_\delta} e^{2|\Gamma(s,u)|/h^{\frac{2}{3}}}|\phi_2|^2\mathrm{d}s\mathrm{d}u \leqslant C\|\varphi\|^2_{E_0}\,, \tag{3.1}$$

and

$$\int_{\Omega\setminus T_\delta} e^{2|x|/h^{\frac{2}{3}}}|h\nabla\varphi_1|^2\mathrm{d}x + \int_{B_\delta} e^{2|\Gamma(s,u)|/h^{\frac{2}{3}}}|h\nabla_{s,u}\phi_2|^2\mathrm{d}s\mathrm{d}u \leqslant Ch^{\frac{2}{3}}\|\varphi\|^2_{E_0}\,.$$

Proof. The proof essentially follows from the classical Agmon estimates. Considering $\tilde\varphi = (e^{2|x|/h^{2/3}}\varphi_1, e^{2|\Gamma(s,u)|/h^{2/3}}\phi_2)$, we see that $\tilde\varphi \in \mathscr{U}_0(H^1_0(\Omega))$. We have then

$$\langle \mathscr{M}_{h,\alpha}\varphi, \tilde\varphi\rangle = \lambda\left(\|e^{|x|/h^{2/3}}\varphi_1\|^2_{\Omega\setminus T_\delta} + \|e^{|\Gamma(s,u)|/h^{2/3}}\phi_2\|^2_{T_\delta}\right), \tag{3.2}$$

and, recalling Definition 3.1, we can write that

$$e^{-i\beta}M_{h,\alpha}(\varphi,\tilde\varphi) = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{3.3}$$

with

$$\mathrm{I} = e^{-i\beta}h^2\langle\nabla\varphi_1, \nabla(e^{2|x|/h^{2/3}}\varphi_1)\rangle_{\Omega\setminus T_\delta} + e^{i(\alpha-\beta)}\int_{\Omega\setminus T_\delta} x_1|e^{|x|/h^{2/3}}\varphi_1|^2\mathrm{d}x\,,$$

$$\begin{aligned}\mathrm{II} = \int_{B_\delta}\big[e^{-i\beta}m_\theta^{-2}h^2\partial_s\phi_2\partial_s(e^{2|\Gamma|/h^{2/3}}\overline{\phi}_2) + e^{-i\beta}[J_\theta']^{-2}h^2\partial_u\phi_2\partial_u(e^{2|\Gamma|/h^{2/3}}\overline{\phi}_2)\\ + e^{i(\alpha-\beta)}\Gamma_1(s,J_\theta(u))|e^{|\Gamma|/h^{2/3}}\phi_2|^2\big]\mathrm{d}s\mathrm{d}u\end{aligned}$$

and

$$\mathrm{III} = h^2\int_{B_\delta}V_\theta(s,u)|e^{|\Gamma|/h^{2/3}}\phi_2|^2\mathrm{d}s\mathrm{d}u + h^2\int_{-L}^{L}W_\theta(s)|e^{|\Gamma|/h^{2/3}}\phi_2(s,\delta)|^2\mathrm{d}s\,.$$

Let us now bound the real part of I and II from below. We have

$$\mathrm{Re}\,\mathrm{I} = \mathrm{Re}\,e^{-i\beta}h^2\langle\nabla\varphi_1, \nabla(e^{2|x|/h^{2/3}}\varphi_1)\rangle_{\Omega\setminus T_\delta} + \cos(\alpha-\beta)\|\sqrt{x_1}e^{|x|/h^{2/3}}\varphi_1\|^2\,.$$

Then, with the chain rule, we get

$$\begin{aligned}&h^2\langle\nabla\varphi_1, \nabla(e^{2|x|/h^{2/3}}\varphi_1)\rangle_{\Omega\setminus T_\delta}\\ &= h^2\langle e^{|x|/h^{2/3}}\nabla\varphi_1, \nabla(e^{|x|/h^{2/3}}\varphi_1)\rangle_{\Omega\setminus T_\delta} + h^{\frac43}\langle\nabla\varphi_1, (\nabla|x|)(e^{2|x|/h^{2/3}}\varphi_1)\rangle_{\Omega\setminus T_\delta}\\ &= h^2\|\nabla(e^{|x|/h^{2/3}}\varphi_1)\|^2_{\Omega\setminus T_\delta} + \mathscr{O}(h^{\frac43})\|e^{|x|/h^{2/3}}\varphi_1\|\|\nabla(e^{|x|/h^{2/3}}\varphi_1)\|\\ &\qquad + h^{\frac43}\langle\nabla\varphi_1, \nabla|x|(e^{2|x|/h^{2/3}}\varphi_1)\rangle_{\Omega\setminus T_\delta}\\ &= h^2\|\nabla(e^{|x|/h^{2/3}}\varphi_1)\|^2_{\Omega\setminus T_\delta} + \mathscr{O}(h^{\frac43})\|e^{|x|/h^{2/3}}\varphi_1\|\|\nabla(e^{|x|/h^{2/3}}\varphi_1)\| + \mathscr{O}(h^{\frac23})\|e^{|x|/h^{2/3}}\varphi_1\|^2\,.\end{aligned}$$

Thus, with the Young inequality, we deduce that

$$\begin{aligned}\mathrm{Re}\,\mathrm{I} \geqslant c(\alpha,\beta)\left(h^2\|\nabla(e^{|x|/h^{2/3}}\varphi_1)\|^2_{\Omega\setminus T_\delta} + \|\sqrt{x_1}e^{|x|/h^{2/3}}\varphi_1\|^2_{\Omega\setminus T_\delta}\right)\\ - Ch^{\frac23}\|e^{|x|/h^{2/3}}\varphi_1\|^2_{\Omega\setminus T_\delta}\,.\end{aligned} \tag{3.4}$$

Note that we did not use any integration by parts in the last computation, so that no boundary term appears. We proceed rather similarly to see that

$$\begin{aligned}&\int_{B_\delta}m_\theta^{-2}h^2\partial_s\phi_2\partial_s(e^{2|\Gamma|/h^{2/3}}\overline{\phi}_2)\mathrm{d}s\mathrm{d}u\\ &= \int_{B_\delta}m_\theta^{-2}h^2|\partial_s(e^{|\Gamma|/h^{2/3}}\phi_2)|^2\mathrm{d}s\mathrm{d}u + \mathscr{O}(h^{1/3})\|e^{|\Gamma|/h^{2/3}}\phi_2\|\|h\partial_s(e^{|\Gamma|/h^{2/3}}\phi_2)\|\\ &\qquad + \mathscr{O}(h^{\frac23})\|e^{|\Gamma|/h^{2/3}}\phi_2\|^2\\ &= \int_{B_\delta}h^2|\partial_s(e^{|\Gamma|/h^{2/3}}\phi_2)|^2\mathrm{d}s\mathrm{d}u + \mathscr{O}(h^{1/3})\|e^{|\Gamma|/h^{2/3}}\phi_2\|\|h\partial_s(e^{|\Gamma|/h^{2/3}}\phi_2)\|\\ &\qquad + \mathscr{O}(\delta)\|h\partial_s(e^{|\Gamma|/h^{2/3}}\phi_2)\|^2_{B_\delta} + \mathscr{O}(h^{\frac23})\|e^{|\Gamma|/h^{2/3}}\phi_2\|^2\,,\end{aligned}$$

where we used $m_\theta = 1 + \mathscr{O}(\delta)$ from (2.3). In the same way, we get

$$\begin{aligned}
&\int_{B_\delta} [J'_\theta]^{-2} h^2 \partial_u \phi_2 \partial_u (e^{2|\Gamma|/h^{2/3}} \overline{\phi}_2) \\
&= \int_{B_\delta} [J'_\theta(u)]^{-2} h^2 |\partial_u (e^{|\Gamma|/h^{2/3}} \phi_2)|^2 \mathrm{d}s\mathrm{d}u + \mathscr{O}(h^{4/3}) \|e^{|\Gamma|/h^{2/3}} \phi_2\| \|\partial_u (e^{|\Gamma|/h^{2/3}} \phi_2)\| \\
&\qquad + \mathscr{O}(\delta) h^2 \|\partial_u (e^{|\Gamma|/h^{2/3}} \phi_2)\|^2_{B_\delta} + \mathscr{O}(h^{\frac{2}{3}}) \|e^{|\Gamma|/h^{2/3}} \phi_2\|^2 .
\end{aligned}$$

We deduce that

$$\begin{aligned}
\mathrm{Re}\, \mathrm{II} \geqslant &\int_{B_\delta} \cos(\alpha - \beta) h^2 |\partial_s (e^{|\Gamma|/h^{2/3}} \phi_2)|^2 \mathrm{d}s\mathrm{d}u \\
&+ \int_{B_\delta} \mathrm{Re}\, (e^{-i\beta} J'_\theta(u)^{-2}) h^2 |\partial_u (e^{|\Gamma|/h^{2/3}} \phi_2)|^2 \mathrm{d}s\mathrm{d}u \\
&+ \int_{B_\delta} \mathrm{Re}\, (e^{i(\alpha-\beta)} \Gamma_1(s, J_\theta(u))) |e^{|\Gamma|/h^{2/3}} \phi_2|^2 \mathrm{d}s\mathrm{d}u \\
&+ \mathscr{O}(h^{4/3}) \|e^{|\Gamma|/h^{2/3}} \phi_2\| \|\nabla (e^{|\Gamma|/h^{2/3}} \phi_2)\| + \mathscr{O}(\delta) h^2 \|\nabla (e^{|\Gamma|/h^{2/3}} \phi_2)\|^2_{B_\delta} \\
&+ \mathscr{O}(h^{\frac{2}{3}}) \|e^{|\Gamma|/h^{2/3}} \phi_2\|^2 .
\end{aligned}$$

With Lemma 3.2, we get, for some $c(\alpha, \beta) > 0$,

$$\begin{aligned}
\mathrm{Re}\, \mathrm{II} \geqslant &c(\alpha, \beta) \int_{B_\delta} h^2 |\nabla (e^{|\Gamma|/h^{2/3}} \phi_2)|^2 \mathrm{d}s\mathrm{d}u \\
&+ \int_{B_\delta} \mathrm{Re}\, (e^{i(\alpha-\beta)} \Gamma_1(s, J_\theta(u))) |e^{|\Gamma|/h^{2/3}} \phi_2|^2 \mathrm{d}s\mathrm{d}u \\
&+ \mathscr{O}(h^{4/3}) \|e^{|\Gamma|/h^{2/3}} \phi_2\| \|\nabla (e^{|\Gamma|/h^{2/3}} \phi_2)\| + \mathscr{O}(\delta) h^2 \|\nabla (e^{|\Gamma|/h^{2/3}} \phi_2)\|^2_{B_\delta} \\
&+ \mathscr{O}(h^{\frac{2}{3}}) \|e^{|\Gamma|/h^{2/3}} \phi_2\|^2 ,
\end{aligned}$$

and thus, perhaps after changing the value of $c(\alpha, \beta) > 0$,

$$\begin{aligned}
\mathrm{Re}\, \mathrm{II} \geqslant &c(\alpha, \beta) \int_{B_\delta} |h\nabla (e^{|\Gamma|/h^{2/3}} \phi_2)|^2 \mathrm{d}s\mathrm{d}u - C h^{\frac{2}{3}} \|e^{|\Gamma|/h^{2/3}} \phi_2\|^2 \\
&+ \int_{B_\delta} \mathrm{Re}\, (e^{i(\alpha-\beta)} \Gamma_1(s, J_\theta(u))) |e^{|\Gamma|/h^{2/3}} \phi_2|^2 \mathrm{d}s\mathrm{d}u .
\end{aligned} \tag{3.5}$$

Moreover, by using a $H^{\frac{1}{2}}$-trace theorem, we get that, for all $\epsilon \in (0,1)$, there exists C_ϵ such that

$$\mathrm{Re}\, \mathrm{III} \geqslant -C_\epsilon h^2 \|e^{|\Gamma|/h^{2/3}} \phi_2\|^2 - \epsilon \left\| h\nabla \left(e^{|\Gamma|/h^{2/3}}\right) \phi_2 \right\|^2_{B_\delta} . \tag{3.6}$$

Now ,we can come back to (3.2) and (3.3) and we deduce, from (3.4), (3.5), (3.6), and $\mathrm{Re}\, (e^{-i\beta}\lambda) \leqslant M h^{\frac{2}{3}}$, that

$$\begin{aligned}
c\|\sqrt{x_1} e^{|x|/h^{2/3}} \varphi_1\|^2_{\Omega\setminus T_\delta} + \int_{B_\delta} \mathrm{Re}\, (e^{i(\alpha-\beta)} \Gamma_1(s, J_\theta(u))) |e^{|\Gamma|/h^{2/3}} \phi_2|^2 \mathrm{d}s\mathrm{d}u& \\
\leqslant M h^{\frac{2}{3}} \left(\|e^{|x|/h^{2/3}} \varphi_1\|^2_{\Omega\setminus T_\delta} + \|e^{|\Gamma(s,u)|/h^{2/3}} \phi_2\|^2_{T_\delta} \right)& .
\end{aligned}$$

To conclude, we now split the integral into two parts according to the decomposition

$$B_\delta = \{(s,u) \in B_\delta : u + s^2 \leqslant R h^{\frac{2}{3}}\} \cup B_\delta^{\mathrm{far}} .$$

where

$$B_\delta^{\rm far}\{(s,u)\in B_\delta : u+s^2>Rh^{\frac{2}{3}}\}\,.$$

The rest of the proof follows from the usual manipulations *à la* Agmon. Indeed, thanks to Lemma 3.4, there exists C_M such that

$$(cRh^{\frac{2}{3}}-C_Mh^{\frac{2}{3}})\big(\|e^{|x|/h^{2/3}}\varphi_1\|_{(\Omega\setminus T_\delta)\times B_\delta^{\rm far}}+\|e^{|\Gamma(s,u)|/h^{\frac{2}{3}}}\phi_2\|^2_{(\Omega\setminus T_\delta)\times B_\delta^{\rm far}}\big)\leqslant \tilde{C}h^{\frac{2}{3}}\|\varphi\|^2\,,$$

where we used that the exponential is bounded on $\{(s,u)\in B_\delta : u+s^2\leqslant Rh^{\frac{2}{3}}\}$. The estimate (3.1) follows by choosing R large enough. Then, the estimate of the gradient follows by gathering (3.2), (3.3), (3.4), (3.5), (3.6) and by using (3.1).

□

4. Spectral analysis

The aim of this section is to prove the following three propositions, which imply Theorem 1.1 (since $\mathscr{M}_{h,\alpha}$ is isospectral to $\mathscr{L}_{h,\alpha}$, see Lemma 2.8).

Proposition 4.1 (Rough localization of the spectrum)**.** *Let $\alpha\in[0,\frac{\pi}{2})$ and consider $M>0$. There exist $C,h_0>0$ such that, for all $h\in(0,h_0)$,*

$$\begin{aligned}(4.1)\quad \mathrm{sp}(\mathscr{L}_{h,\alpha})\cap\{z\in\mathbb{C}\ :\ \mathrm{Re}\,(z)<Mh^{\frac{2}{3}}\}\\ \subset\big\{z\in\mathbb{C}\ :\ \mathrm{Re}\,z\geqslant z_1h^{\frac{2}{3}}\cos(2\alpha/3)-Ch^{\frac{4}{3}}\big),\\ 0\leqslant\mathrm{Im}\,z\leqslant(\sin\alpha)\max\{x_1 : x\in\overline{\Omega}\big\}\,.\end{aligned}$$

Proposition 4.2 (Refined localization of the spectrum)**.** *Consider $\alpha\in\big[0,\frac{3\pi}{5}\big)$ and $R>0$ with $R\notin(2\mathbb{N}-1)\sqrt{\frac{\kappa_0}{2}}$. Then there exist $h_0>0$ and $N\in\mathbb{N}$ such that, for all $h\in(0,h_0)$,*

$$\mathrm{sp}(\mathscr{M}_{h,\alpha})\cap D(z_1e^{\frac{2i\alpha}{3}}h^{\frac{2}{3}},Rh)\subset\bigcup_{n=1}^{N}D(\mu_n(h,\alpha),h^{\frac{3}{2}-2\eta})\,,\tag{4.2}$$

with

$$\mu_n(h,\alpha)=h^{\frac{2}{3}}e^{2i\alpha/3}z_1+(2n-1)he^{i\alpha/2}\sqrt{\frac{\kappa_0}{2}}\,.\tag{4.3}$$

Moreover, for all $n\in\{1,\dots,N\}$, the Riesz projector

$$\Pi_{n,h}:=\frac{1}{2i\pi}\int_{\mathscr{C}_{n,h}}(z-\mathscr{M}_{h,\alpha})^{-1}\mathrm{d}z\,,\qquad \textit{where } \mathscr{C}_{n,h}=\partial D(\mu_n(h,\alpha),h^{\frac{3}{2}-2\eta})\,,$$

is of rank at most one.

Proposition 4.3 (Existence of the spectrum)**.** *Consider $\alpha\in\big[0,\frac{3\pi}{5}\big)$. There exists $h_0>0$ such that, for all $h\in(0,h_0)$, the rank of $\Pi_{n,h}$ is exactly one.*

Proposition 4.1 is proved in Section 4.1. Propositions 4.2 and 4.3 are proved in Section 4.5. Proposition 4.2 is a consequence of the analysis in Sections 4.2 and 4.3 where, by inserting an appropriate quasimode in the Riesz projector, we can prove that $\Pi_{n,h}$ is not zero.

4.1. **Rough localization of the spectrum.** In this section, we prove Proposition 4.1.

Let us first note that, in a first naive approach, when $\alpha \in [0, \frac{\pi}{2})$,

$$\mathrm{Re}\,\langle \mathscr{L}_{h,\alpha}\psi, \psi\rangle \geqslant 0\,, \quad \mathrm{Im}\,\langle \mathscr{L}_{h,\alpha}\psi, \psi\rangle = \sin\alpha \int_\Omega x_1|\psi|^2\mathrm{d}x\,.$$

This gives the estimate on the imaginary part of the spectrum, and it remains to refine the estimate on the real part. For this, we consider a smooth cutoff function χ_h in the form

$$\chi_h(s,u) = \chi(h^{-\frac{1}{3}+\eta}s, h^{-\frac{2}{3}+\eta}u)\,,$$

where $\eta > 0$ and we let

$$\phi_2^{\mathrm{cut}} = \chi_h\phi_2\,. \tag{4.4}$$

Consider now an eigenfunction φ of $\mathscr{M}_{h,\alpha}$ associated with an eigenvalue λ such that $\mathrm{Re}\,\lambda \leqslant Mh^{\frac{2}{3}}$. From (2.4), we get

$$\begin{aligned}\mathrm{Re}\,\lambda\|\varphi\|^2 &= \mathrm{Re}\,L_{h,\alpha,\theta}(\varphi,\varphi)\\ &\geqslant \int_{B_\delta}\Big(\mathrm{Re}\,(m_\theta^{-2})|h\partial_s\phi_2|^2 + \mathrm{Re}\,([J_\theta']^{-2})|h\partial_u\phi_2|^2\\ &\qquad + \mathrm{Re}\,\left(e^{i\alpha}\Gamma_1(s,J_\theta(u))|\phi_2|^2\right)\Big)\mathrm{d}s\mathrm{d}u - Ch^2\|\phi_2\|^2\,,\end{aligned}$$

where we used the trace theorem and Proposition 3.5 to control the boundary term. Then, with a Taylor expansion in the expressions (2.3) of m_θ and (2.2) of J_θ' and with Proposition 3.5, we get

$$\begin{aligned}\mathrm{Re}\,\lambda\|\varphi\|^2 &= \mathrm{Re}\,L_{h,\alpha,\theta}(\varphi,\varphi)\\ &\geqslant \int_{B_\delta}|h\partial_s\phi_2^{\mathrm{cut}}|^2 + \cos\left(\frac{2\alpha}{3}\right)\left(|h\partial_u\phi_2^{\mathrm{cut}}|^2 + u|\phi_2^{\mathrm{cut}}|^2\right)\mathrm{d}s\mathrm{d}u - Ch^{\frac{4}{3}}\|\phi_2\|^2\,.\end{aligned}$$

From the min-max theorem applied to the real Airy operator, we deduce that

$$\mathrm{Re}\,\lambda\|\varphi\|^2 \geqslant z_1h^{\frac{2}{3}}\cos\left(\frac{2\alpha}{3}\right)\|\phi_2^{\mathrm{cut}}\|^2 - Ch^{\frac{4}{3}}\|\phi_2\|^2\,.$$

By using again Proposition 3.5, we infer that

$$\mathrm{Re}\,\lambda \geqslant z_1h^{\frac{2}{3}}\cos\left(\frac{2\alpha}{3}\right) - Ch^{\frac{4}{3}}\,.$$

This proves Proposition 4.1.

Remark 4.4. Let us mention what happens in the case $\alpha \in (\frac{\pi}{2}, \frac{3\pi}{5})$. In that case we do not even *a priori* have $\mathrm{Re}\,\langle \mathscr{M}_{h,\alpha}\psi,\psi\rangle \geqslant 0$ because of the electric potential. Choosing β such that $(\alpha,\beta) \in \mathcal{T}$ where $\mathcal{T}$ is defined in Lemma 3.3, we can still use the localization properties proved in Section 3. In particular, working with $e^{-i\beta}\mathscr{M}_{h,\alpha}$ instead of $\mathscr{M}_{h,\alpha}$ and using Proposition 3.5, as well as Taylor expansions of m_θ and of J_θ', we get that, for any eigenvalue λ of $\mathcal{L}_{h,\alpha}$, we have

$$\mathrm{Re}\,e^{-i\beta}\lambda \geqslant z_1h^{\frac{2}{3}}\cos\left(\frac{2\alpha}{3} - \beta\right) - Ch^{\frac{4}{3}}\,.$$

4.2. **Resolvent estimates.** To prepare the proof of Proposition 4.2, let us describe the spectrum and resolvent of our model operator, acting in $L^2(\mathbb{R}_+ \times \mathbb{R})$,

$$\mathscr{N}_{h,\alpha} = e^{\frac{2i\alpha}{3}}(h^2 D_u^2 + u) + h^2 D_s^2 + e^{i\alpha}\frac{\kappa_0 s^2}{2}\,, \tag{4.5}$$

where $D = -i\partial$. As we will see below, the following description is rather easy to establish since $h^2 D_u^2 + u$ is selfadjoint so that we can use an orthogonal spectral decomposition to reduce our resovent estimates to those of the complex harmonic oscillator, which is well-known.

Proposition 4.5. *Let $R > 0$ with $R \notin (2\mathbb{N}-1)\sqrt{\frac{\kappa_0}{2}}$. There exist $C, h_0 > 0$ and $N \in \mathbb{N}$ such that the following holds. The spectrum of $\mathscr{N}_{h,\alpha}$ in $D(z_1 e^{\frac{2i\alpha}{3}} h^{\frac{2}{3}}, Rh)$ is made of N eigenvalues of algebraic multiplicity one, which are the $(\mu_n(h,\alpha))_{1\leqslant n\leqslant N}$ as given in (4.3). Moreover, for all $z \in D(z_1 e^{\frac{2i\alpha}{3}} h^{\frac{2}{3}}, Rh)$ such that $z \notin \{\mu_n(h,\alpha)\,, n \in \{1,\ldots,N\}\}$, we have*

$$\|(z - \mathscr{N}_{h,\alpha})^{-1}\| \leqslant Ch^{-\frac{2}{3}} + \frac{C}{\operatorname{dist}(z, \operatorname{sp}(\mathscr{N}_{h,\alpha}))}\,.$$

Proof. Note that $\mathscr{N}_{h,\alpha}$ is the sum of two decoupled operators, an Airy operator and a harmonic oscillator. It has compact resolvent and its spectrum and eigenfunctions are completely known. Consider

$$\Psi_{m,n,h}(s,u) = h^{-\frac{1}{3}-\frac{1}{4}}\mathsf{Ai}(h^{-\frac{2}{3}}u - z_m) f_n\Big(h^{-\frac{1}{2}} e^{i\alpha/2}\sqrt{\frac{\kappa_0}{2}}s\Big)\,, \tag{4.6}$$

where Ai is the usual Airy function (and z_m its m-th positive zero) and f_n the n-th normalized Hermite function. We have

$$\mathscr{N}_{h,\alpha}\Psi_{m,n,h} = \left(z_m h^{\frac{2}{3}} e^{\frac{2i\alpha}{3}} + (2n-1)h\sqrt{\frac{\kappa_0}{2}}\right)\Psi_{m,n,h}\,.$$

Moreover, there are no other eigenvalues and they are all of multiplicity one. Indeed, by analytic dilation, we see that $\mathscr{N}_{h,\alpha}$ is isospectral to the normal operator

$$e^{\frac{2i\alpha}{3}}(h^2 D_u^2 + u) + e^{\frac{i\alpha}{2}}\left(h^2 D_s^2 + \frac{k_0 s^2}{2}\right)\,.$$

Let us now turn to the estimate of the resolvent in the disk $D(z_1 e^{\frac{2i\alpha}{3}} h^{\frac{2}{3}}, Rh)$. Consider

$$z = z_1 e^{\frac{2i\alpha}{3}} h^{\frac{2}{3}} + \zeta h\,, \quad \zeta \in D(0,R)\,,$$

with ζ avoiding the numbers $e^{\frac{i\alpha}{2}}(2n-1)\sqrt{\frac{k_0}{2}}$. We have

$$\mathscr{N}_{h,\alpha} - z = e^{\frac{2i\alpha}{3}}(h^2 D_u^2 + u - z_1 h^{\frac{2}{3}}) + h^2 D_s^2 + e^{i\alpha}\frac{k_0 s^2}{2} - \zeta h\,,$$

and also

$$e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha} - z) = e^{\frac{i\alpha}{6}}(h^2 D_u^2 + u - z_1 h^{\frac{2}{3}}) + e^{-\frac{i\alpha}{2}}\left(h^2 D_s^2 + e^{i\alpha}\frac{k_0 s^2}{2} - \zeta h\right)\,.$$

Let us denote g_h the (explicit) positive normalized groundstate of $h^2 D_u^2 + u$ and consider the orthogonal projection

$$\mathfrak{P}_h \psi(s,u) = \langle g_h, \psi(s,\cdot)\rangle_{L^2(\mathbb{R}_+)} g_h(u)\,.$$

We have

$$\operatorname{Re}\langle e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha}-z)(\mathrm{Id}-\mathfrak{P}_h)\psi,\psi\rangle \geqslant \left(h^{\frac{2}{3}}\cos(\alpha/6)(z_2-z_1)-Rh\right)\|(\mathrm{Id}-\mathfrak{P}_h)\psi\|^2 \tag{4.7}$$
$$\geqslant ch^{\frac{2}{3}}\|(\mathrm{Id}-\mathfrak{P}_h)\psi\|^2\,,$$

which implies that the restriction of $\mathscr{N}_{h,\alpha}-z$ to $\mathrm{Ker}(\mathfrak{P}_h)$ is injective and therefore bijective since $\mathscr{N}_{h,\alpha}$ is Fredholm of index 0. Moreover, we have the orthogonal decomposition

$$e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha}-z)=e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha}-z)(\mathrm{Id}-\mathfrak{P}_h)+e^{-\frac{i\alpha}{2}}\left(h^2D_s^2+e^{i\alpha}\frac{\kappa_0 s^2}{2}-\zeta h\right)\mathfrak{P}_h\,.$$

From (4.7), the fact that ζ avoids the numbers $e^{\frac{i\alpha}{2}}(2n-1)\sqrt{\frac{k_0}{2}}$, we get that $e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha}-z)$ is bijective and that its inverse is given by

$$\left(e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha}-z)\right)^{-1}=e^{\frac{i\alpha}{2}}\left((\mathscr{N}_{h,\alpha}-z)|_{\mathrm{Ker}(\mathfrak{P}_h)}\right)^{-1}(\mathrm{Id}-\mathfrak{P}_h)$$
$$+e^{\frac{i\alpha}{2}}\left(h^2D_s^2+e^{i\alpha}\frac{k_0s^2}{2}-\zeta h\right)^{-1}\mathfrak{P}_h\,.$$

Let us notice that there exists $C>0$ such that for all $\zeta\in D(0,R)$ avoiding the numbers $e^{\frac{i\alpha}{2}}(2n-1)\sqrt{\frac{k_0}{2}}$, we have

$$\left\|\left(h^2D_s^2+e^{i\alpha}\frac{k_0s^2}{2}-\zeta h\right)^{-1}\right\|\leqslant\frac{C}{\mathrm{dist}(\mathrm{sp}(h^2D_s^2+e^{i\alpha}\frac{k_0s^2}{2}),\zeta h)}\,. \tag{4.8}$$

Indeed, by rescaling, (4.8) is equivalent to

$$\left\|\left(D_s^2+e^{i\alpha}\frac{k_0s^2}{2}-\zeta\right)^{-1}\right\|\leqslant\frac{C}{\mathrm{dist}(\mathrm{sp}(D_s^2+e^{i\alpha}\frac{k_0s^2}{2}),\zeta)}\,,$$

and this last estimate can be checked by using a decomposition induced by the Riesz projections and by recalling that the eigenvalues of the complex harmonic oscillator have algebraic multiplicity one.

Thus, from the above orthogonal decomposition, we deduce that

$$\|[e^{-\frac{i\alpha}{2}}(\mathscr{N}_{h,\alpha}-z)]^{-1}\|\leqslant(ch^{\frac{2}{3}})^{-1}+\frac{C}{\mathrm{dist}(\mathrm{sp}(h^2D_s^2+e^{i\alpha}\frac{k_0s^2}{2}),\zeta h)}\,.$$

This concludes the proof. □

4.3. **Quasimodes and localization estimates.** If $\varphi=(\varphi_1,\phi_2)$ is a normalized eigenfunction of $\mathscr{M}_{h,\alpha}$ associated with $\lambda\in D(z_1e^{\frac{2i\alpha}{3}}h^{\frac{2}{3}},Rh)$, we have in particular

$$\mathcal{M}_{h,\alpha}\phi_2=\lambda\phi_2\,, \tag{4.9}$$

with $\mathcal{M}_{h,\alpha}$ denoting the formal operator

$$\mathcal{M}_{h,\alpha}=hD_s(m_\theta^{-2})hD_s+hD_u(J_\theta')^{-2}hD_u+e^{i\alpha}\Gamma_1(s,J_\theta(u))\,. \tag{4.10}$$

This section is devoted to the proof of the following proposition. We recall that the model operator $\mathscr{N}_{h,\alpha}$ is defined in (4.5) and that ϕ_2^{cut} is defined as a truncation of ϕ_2 in (4.4).

Proposition 4.6. *We have*

$$(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\mathrm{cut}}=\mathscr{O}(h^{\frac{3}{2}-3\eta})\|\phi_2^{\mathrm{cut}}\|\,.$$

This proposition essentially says that ϕ_2^{cut} is a good quasimode for our model operator $\mathscr{N}_{h,\alpha}$. The proof will be done in several steps including elliptic estimates and a refined localization in s.

4.3.1. *Preliminary estimates.* Recall that $\chi_h(s,u) = \chi(h^{-\frac{1}{3}+\eta}s, h^{-\frac{2}{3}+\eta}u)$ with χ a smooth cutoff function and η fixed and small, and that $\phi_2^{\mathrm{cut}} = \chi_h\phi_2$.

Lemma 4.7. *We have*

$$\mathcal{M}_{h,\alpha}\phi_2^{\mathrm{cut}} = \lambda\phi_2^{\mathrm{cut}} + r_h\,,$$

with

$$r_h = [\mathcal{M}_{h,\alpha}, \chi_h]\phi_2\,,$$

where the commutator is given by

$$\begin{aligned}(4.11)\quad [\mathcal{M}_{h,\alpha}, \chi_h]\phi = \big(hD_s(m_\theta^{-2}(hD_s\chi_h))\big)\phi + 2m_\theta^{-2}(hD_s\chi_h)(hD_s\phi)\\ + \big(hD_u((J_\theta')^{-2}(hD_u\chi_h))\big)\phi + 2(J_\theta')^{-2}(hD_u\chi_h)(hD_u\phi)\,.\end{aligned}$$

Moreover,

$$\|r_h\| = \mathscr{O}(h^\infty)\|\phi_2^{\mathrm{cut}}\|\,.$$

Proof. The expression (4.11) follows from a straightforward computation. The estimate $\|r_h\| = \mathscr{O}(h^\infty)$ is a consequence of Proposition 3.5 and support considerations. □

Remark 4.8. With a straightforward computation, we can check that, for all $k,\ell\in\mathbb{N}$, $\|D_s^kD_u^\ell r_h\| = \mathscr{O}(h^\infty)$.

Lemma 4.9. *We have*

$$\|hD_s\phi_2^{\mathrm{cut}}\|^2 + \|hD_u\phi_2^{\mathrm{cut}}\|^2 \leqslant Ch^{\frac{2}{3}}\|\phi_2^{\mathrm{cut}}\|^2\,.$$

Proof. Due to the Dirichlet boundary condition of ϕ_2 on the external part of ∂B_δ and the cutoff χ_h, the function ϕ_2^{cut} satisfies the Dirichlet condition on ∂B_δ.

Let us now choose β such that $(\alpha,\beta)\in\mathcal{T}$ as defined in Lemma 3.3. With an integration by parts using the expression of $e^{-i\beta}\mathcal{M}_{h,\alpha}$, we get that, for some $c>0$,

$$\mathrm{Re}\,\big(e^{-i\beta}\langle\mathcal{M}_{h,\alpha}\phi_2^{\mathrm{cut}}, \phi_2^{\mathrm{cut}}\rangle\big) \geqslant c(\|hD_s\phi_2^{\mathrm{cut}}\|^2 + \|hD_u\phi_2^{\mathrm{cut}}\|^2)\,.$$

Then, by Lemma 4.7, we have that

$$\big|\mathrm{Re}\,\big(e^{-i\beta}\langle\mathcal{M}_{h,\alpha}\phi_2^{\mathrm{cut}}, \phi_2^{\mathrm{cut}}\rangle\big)\big| \leqslant Ch^{\frac{2}{3}}\|\phi_2^{\mathrm{cut}}\|^2\,.$$

The conclusion follows. □

For further use, we check now that we even have a control of higher order derivatives.

Lemma 4.10. *We have*

$$\|(hD_s)^2\phi_2^{\mathrm{cut}}\|^2 + \|(hD_u)^2\phi_2^{\mathrm{cut}}\|^2 \leqslant Ch^{\frac{4}{3}}\|\phi_2^{\mathrm{cut}}\|^2\,.$$

Proof. By Lemma 4.7, we have

$$\langle hD_s(\mathcal{M}_{h,\alpha}\phi_2^{\mathrm{cut}}), hD_s\phi_2^{\mathrm{cut}}\rangle = \lambda\|hD_s\phi_2^{\mathrm{cut}}\|^2 + \langle r_h, (hD_s)^2\phi_2^{\mathrm{cut}}\rangle\,.$$

Thus, after computing a commutator, we get

$$\begin{aligned}&\langle\mathcal{M}_{h,\alpha}(hD_s\phi_2^{\mathrm{cut}}), hD_s\phi_2^{\mathrm{cut}}\rangle\\ &\quad= \lambda\|hD_s\phi_2^{\mathrm{cut}}\|^2 + \langle r_h, (hD_s)^2\phi_2^{\mathrm{cut}}\rangle + h\mathscr{O}(\|\phi_2^{\mathrm{cut}}\|\|hD_s\phi_2^{\mathrm{cut}}\|) + h^2\mathscr{O}(\|hD_s\phi_2^{\mathrm{cut}}\|^2)\\ &\qquad + h\mathscr{O}(\|(hD_s)^2\phi_2^{\mathrm{cut}}\|\|hD_s\phi_2^{\mathrm{cut}}\|)\,.\end{aligned}$$

Then, we use Lemma 4.9 and we proceed as in its the proof to get

$$\|(hD_s)^2\phi_2^{\rm cut}\|^2 + \|(hD_u)(hD_s\phi_2^{\rm cut})\|^2 \leqslant Ch^{\frac{4}{3}}\|\phi_2^{\rm cut}\|^2 + Ch^{\frac{4}{3}}\|(hD_s)^2\phi_2^{\rm cut}\|^2 .$$

We proceed in the same way to get the control of $(hD_u)^2\phi_2^{\rm cut}$. The conclusion follows. □

Let us consider the following intermediate operator,

$$\mathcal{N}_{h,\alpha} = e^{\frac{2i\alpha}{3}}(h^2D_u^2 + u) + h^2D_s^2 + e^{i\alpha}\gamma_1(s) ,$$

which differs from $\mathscr{N}_{h,\alpha}$ in (4.5) only through its potential part.

Proposition 4.11. *We have*

$$\mathcal{N}_{h,\alpha}\phi_2^{\rm cut} = \lambda\phi_2^{\rm cut} + R_h ,$$

with

$$R_h = r_h + ue^{2i\alpha/3}(\mathbf{n}_1(s)-1)\phi_2^{\rm cut} - (hD_s m_\theta^{-2})hD_s\phi_2^{\rm cut} + (1-m_\theta^{-2})(hD_s)^2\phi_2^{\rm cut}$$

and

$$\|R_h\| = \mathscr{O}(h^{\frac{4}{3}-\eta})\|\phi_2^{\rm cut}\| .$$

Proof. It is sufficient to use Lemma 4.7 and the explicit expression of $\mathcal{M}_{h,\alpha}$. The estimate of R_h follows from support considerations and Lemma 4.10 (note also that $\mathbf{n}_1(s) - 1 = \mathscr{O}(s^2)$). □

4.3.2. *Refined localization in s.* Thanks to the Agmon estimates in Proposition 3.5, we have proved so far a localization of order $h^{\frac{1}{3}}$ in the variable s. We improve this in the following proposition.

Proposition 4.12. *Consider $\chi_0 \in \mathscr{C}_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of 0 and $\eta > 0$. Then,*

$$\phi_2^{\rm cut} = \chi_0(h^{-\frac{1}{2}+\eta}s)\phi_2^{\rm cut} + \mathscr{O}(h^\infty) , \tag{4.12}$$

where the remainder is estimated in H^1-norm.

Moreover, for all $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$,

$$\|s^{\alpha_1}(hD_s)^{\alpha_2}(hD_u)^{\alpha_3}\phi_2^{\rm cut}\| \leqslant Ch^{\frac{\alpha_3}{3}+\frac{\alpha_1+\alpha_2}{2}}\|\phi_2^{\rm cut}\| . \tag{4.13}$$

Proof. Let us consider $\chi_{{\rm far},h}(s) = \chi_{\rm far}(h^{-\frac{1}{2}+\eta}s)$, where $\chi_{\rm far}$ is supported away from 0 and equal to 1 away form a compact set. Then, we use Proposition 4.11 to get that

$$\begin{aligned}\langle(\mathcal{N}_{h,\alpha}-\lambda)(\chi_{{\rm far},h}\phi_2^{\rm cut}), \chi_{{\rm far},h}\phi_2^{\rm cut}\rangle &= \langle\chi_{{\rm far},h}R_h, \chi_{{\rm far},h}\phi_2^{\rm cut}\rangle \\ &\quad + \langle[\mathcal{N}_{h,\alpha}, \chi_{{\rm far},h}]\phi_2^{\rm cut}, \chi_{{\rm far},h}\phi_2^{\rm cut}\rangle .\end{aligned} \tag{4.14}$$

Notice that $-\alpha/2 + 2\alpha/3 = \alpha/6$ and that, for all $\Psi \in H_0^1(\mathbb{R}_+)$,

$$\cos(\alpha/6)\langle(h^2D_u^2 + u - z_1h^{\frac{2}{3}})\Psi, \Psi\rangle \geqslant 0 . \tag{4.15}$$

This, combined with the fact that $\gamma_1(s)$ is bounded from below by $h^{1-2\eta}$ on the support of $\chi_{{\rm far},h}$, implies that, for some $c > 0$,

$$\begin{aligned}&\operatorname{Re}\langle(e^{-i\alpha/2}\big((\mathcal{N}_{h,\alpha}-\lambda)(\chi_{{\rm far},h}\phi_2^{\rm cut}), \chi_{{\rm far},h}\phi_2^{\rm cut}\big)\rangle \\ &\qquad\geqslant ch^{1-2\eta}\|\chi_{{\rm far},h}\phi_2^{\rm cut}\|^2 + c\|hD_s(\chi_{{\rm far},h}\phi_2^{\rm cut})\|^2 .\end{aligned}$$

From the support properties of $\phi_2^{\rm cut}$ (see before Lemma 4.7) and Proposition 4.11, we have

$$\begin{aligned}\left|\langle\chi_{{\rm far},h}R_h,\chi_{{\rm far},h}\phi_2^{\rm cut}\rangle\right| \leqslant & \mathscr{O}(h^\infty)\|\phi_2^{\rm cut}\|^2+Ch^{\frac43-2\eta}\|\chi_{{\rm far},h}\phi_2^{\rm cut}\|^2\\ &+Ch\|hD_s(\chi_{{\rm far},h}\phi_2^{\rm cut})\|\|\chi_{{\rm far},h}\phi_2^{\rm cut}\|\\ &+|\langle(1-m_\theta^{-2})\chi_{{\rm far},h}(hD_s)^2\phi_2^{\rm cut},\chi_{{\rm far},h}\phi_2^{\rm cut}\rangle|\,,\end{aligned}$$

where we used that $u(\mathbf{n}_1(s)-1)=\mathscr{O}(us^2)$.

Using a commutator in the last term of the previous expression, we also have

$$\begin{aligned}|\langle(1-m_\theta^{-2})\chi_{{\rm far},h}(hD_s)^2\phi_2^{\rm cut},\chi_{{\rm far},h}\phi_2^{\rm cut}\rangle| \leqslant\ & Ch^{\frac23-\eta}\|(hD_s)(\chi_{{\rm far},h}\phi_2^{\rm cut})\|^2\\ +Ch^{\frac43-3\eta}\|\underline{\chi}_{{\rm far},h}\phi_2^{\rm cut}\|^2+Ch\|hD_s(\chi_{{\rm far},h}\phi_2^{\rm cut})\|&\|\chi_{{\rm far},h}\phi_2^{\rm cut}\|+\mathscr{O}(h^\infty)\|\phi_2^{\rm cut}\|^2\,,\end{aligned}$$

where $\underline{\chi}_{\rm far}$ has the same properties as $\chi_{\rm far}$ and is such that $\underline{\chi}_{\rm far}\chi_{\rm far}=\chi_{\rm far}$.

In a similar way, we get that

$$\begin{aligned}\langle[\mathscr{N}_{h,\alpha},\chi_{{\rm far},h}]\phi_2^{\rm cut},\chi_{{\rm far},h}\phi_2^{\rm cut}\rangle| \leqslant & Ch^{\frac43-2\eta}\|\underline{\chi}_{{\rm far},h}\phi_2^{\rm cut}\|^2\\ &+Ch\|hD_s(\chi_{{\rm far},h}\phi_2^{\rm cut})\|\|\chi_{{\rm far},h}\phi_2^{\rm cut}\|+\mathscr{O}(h^\infty)\|\phi_2^{\rm cut}\|^2\,.\end{aligned}$$

It follows that

$$\frac{c}{2}\|hD_s(\chi_{{\rm far},h}\phi_2^{\rm cut})\|^2+ch^{1-2\eta}\|\chi_{{\rm far},h}\phi_2^{\rm cut}\|^2\leqslant Ch^{\frac43-3\eta}\|\underline{\chi}_{{\rm far},h}\phi_2^{\rm cut}\|^2+\mathscr{O}(h^\infty)\|\phi_2^{\rm cut}\|^2\,.$$

By choosing η small enough, and by using an induction argument, we get that, for all $N\in\mathbb{N}$,

$$\chi_{{\rm far},h}\phi_2^{\rm cut}=\mathscr{O}(h^N)\|\phi_2^{\rm cut}\|\,,$$

in H^1-norm with respect to s. Coming back to (4.14), we also get the control of hD_u. This gives (4.12).

Let us now turn to (4.13), which are better estimates than those in Lemmas 4.9 and 4.10. We again use Proposition 4.11 and see that

$${\rm Re}\,e^{-i\alpha/2}\langle(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\rm cut},\phi_2^{\rm cut}\rangle\leqslant|\langle R_h,\phi_2^{\rm cut}\rangle|\,.$$

By using (4.15), we get that

$$\|s\phi_2^{\rm cut}\|^2+\|hD_s\phi_2^{\rm cut}\|^2\leqslant Ch\|\phi_2^{\rm cut}\|^2+Ch^{\frac23-\eta}\|(hD_s)\phi_2^{\rm cut}\|^2+Ch\|(hD_s)\phi_2^{\rm cut}\|\|\phi_2^{\rm cut}\|\,,$$

and the estimate for $|(\alpha_1,\alpha_2,\alpha_3)|=1$ follows from the Young inequality. Let us now deal with $|(\alpha_1,\alpha_2,\alpha_3)|=2$. We have

$$\text{(4.16)}\qquad {\rm Re}\,e^{-i\alpha/2}\langle(hD_s)(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\rm cut},hD_s\phi_2^{\rm cut}\rangle\leqslant|\langle R_h,(hD_s)^2\phi_2^{\rm cut}\rangle|\,.$$

Thus,

$$\begin{aligned}c\|shD_s\phi_2^{\rm cut}\|^2+\|(hD_s)^2\phi_2^{\rm cut}\|^2\leqslant Ch\|hD_s\phi_2^{\rm cut}\|^2+Ch\|s\phi_2^{\rm cut}\|\|hD_s\phi_2^{\rm cut}\|&\\ +|\langle R_h,(hD_s)^2\phi_2^{\rm cut}\rangle|\,.&\end{aligned}$$

By using the Young inequality and Proposition 4.11 to deal with the last term, we get

$$\|shD_s\phi_2^{\rm cut}\|^2+\|(hD_s)^2\phi_2^{\rm cut}\|^2\leqslant Ch^2\|\phi_2^{\rm cut}\|^2\,.$$

Coming back to (4.16), we also get

$$\|hD_u(hD_s\phi_2^{\rm cut})\|^2\leqslant Ch^{\frac23}\|hD_s\phi_2^{\rm cut}\|^2+Ch^{2+\frac43-\eta}\|\phi_2^{\rm cut}\|^2\leqslant Ch^{\frac23+1}\|\phi_2^{\rm cut}\|^2\,.$$

To get the control of s^2, it is sufficient to notice that

$${\rm Re}\,e^{-i\alpha/2}\langle s(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\rm cut},s\phi_2^{\rm cut}\rangle\leqslant|\langle R_h,s^2\phi_2^{\rm cut}\rangle|\,,$$

and to estimate again a commutator. This concludes the case $|(\alpha_1,\alpha_2,\alpha_3)| = 2$. The proof of (4.13) for general $(\alpha_1,\alpha_2,\alpha_3)$ follows then by induction using the same method. □

4.3.3. *Proof of Proposition 4.6.* We are now in position to complete the proof of Proposition 4.6, namely that ϕ_2^{cut} is indeed a good quasimode for $\mathscr{N}_{h,\alpha}$. For this, we consider an operator P_h defined through:

$$\mathcal{M}_{h,\alpha} = \mathscr{N}_{h,\alpha} + P_h\,. \tag{4.17}$$

We check that P_h can be written in the following way:

$$P_h = hur_{h,1}(s,u)hD_s + ur_{h,2}(s,u)(hD_s)^2 + r_{h,3}(s,u)us^2 + r_{h,4}(s,u)s^3 + r_{h,5}(s,u)\,,$$

where the remainders $r_{h,j}$ all belong to $S_{\mathbb{R}^2}(1)$ and $r_{h,5}$ with support avoiding a fixed neighborhood of $(0,0)$. We recall that $S_{\mathbb{R}^2}(1) = \{a \in \mathscr{C}^\infty(\mathbb{R}^2) : \forall \alpha \in \mathbb{N}^2\,, \quad \exists C_\alpha > 0 : \forall x \in \mathbb{R}^2 : |\partial^\alpha a(x)| \leqslant C_\alpha\}$. For the other terms, we use the support property in the variable u and Proposition 4.12 to get

$$\|P_h\phi_2^{\mathrm{cut}}\| = \mathscr{O}(h^{\frac{3}{2}-3\eta})\|\phi_2^{\mathrm{cut}}\|\,.$$

The conclusion follows.

4.4. **Quasimodes for $(\mathscr{N}_{h,\alpha}-\lambda)^2$.** Since we are in a non-selfadjoint context, the algebraic and geometric dimension associated with a given eigenvalue λ may differ. For further use, we now deal with localization estimates similar to the ones in Proposition 4.6, but in the case of generalized eigenfunctions. For this, let us consider such a $\lambda \in D(z_1 e^{\frac{2i\alpha}{3}} h^{\frac{2}{3}}, Rh)$ associated with $\varphi = (\varphi_1,\phi_2) \in \ker(\mathcal{M}_{h,\alpha}-\lambda)^2$ with $\|\varphi\| = 1$ and such that $\varphi \notin \ker(\mathcal{M}_{h,\alpha}-\lambda)$ (if it exists). We still denote

$$\phi_2^{\mathrm{cut}} = \chi_h\phi_2\,.$$

The following proposition states that ϕ_2^{cut} is a generalized quasimode of $\mathscr{N}_{h,\alpha}$. Its proof is the object of the following two sections.

Proposition 4.13. *We have*

$$(\mathscr{N}_{h,\alpha}-\lambda)^2\phi_2^{\mathrm{cut}} = \mathscr{O}(h^{\frac{13}{6}})\|\phi_2^{\mathrm{cut}}\|\,.$$

4.4.1. *Localization estimates.* The function ϕ_2^{cut} satisfies the same localization estimates as in the previous section. Let us explain this. We have

$$(\mathscr{M}_{h,\alpha}-\lambda)^2\varphi = 0\,,$$

for φ normalized and $(\mathscr{M}_{h,\alpha}-\lambda)\varphi = f \neq 0$. We have $(\mathscr{M}_{h,\alpha}-\lambda)f = 0$ and thus f satisfies the estimates of the previous section. For instance, we have

$$f_2^{\mathrm{cut}} = \chi(h^{-\frac{1}{2}+\eta}s)f_2^{\mathrm{cut}} + \mathscr{O}(h^\infty)\|f\|\,,$$

where f_2^{cut} is defined without ambiguity and satifies, from Proposition 4.12,

$$\|s^{\alpha_1}(hD_s)^{\alpha_2}(hD_u)^{\alpha_3}f_2^{\mathrm{cut}}\| \leqslant Ch^{\frac{\alpha_3}{3}+\frac{\alpha_1+\alpha_2}{2}}\|f_2^{\mathrm{cut}}\| \tag{4.18}$$

for all $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$. Coming back to the eigenvalue equation, this implies that

$$(h^2D_u^2 + u - z_1h^{\frac{2}{3}})f_2^{\mathrm{cut}} = \mathscr{O}(h)\|f_2^{\mathrm{cut}}\|\,. \tag{4.19}$$

We can easily adapt the proof of the Agmon estimates given in Proposition 3.5 with the right-hand side f to get

$$\int_{\Omega\setminus T_\delta} e^{2|x|/h^{\frac{2}{3}}}|\varphi_1|^2\mathrm{d}x + \int_{B_\delta} e^{2|\Gamma(s,u)|/h^{\frac{2}{3}}}|\phi_2|^2\mathrm{d}s\mathrm{d}u \leqslant C\|\varphi\|_{E_0}^2 + Ch^{-\frac{2}{3}}\|f\|_{E_0}^2\,, \tag{4.20}$$

and

$$(4.21)\quad \int_{\Omega\setminus T_\delta} e^{2|x|/h^{\frac{2}{3}}}|h\nabla\varphi_1|^2\mathrm{d}x + \int_{B_\delta} e^{2|\Gamma(s,u)|/h^{\frac{2}{3}}}|h\nabla_{s,u}\phi_2|^2\mathrm{d}s\mathrm{d}u \leqslant Ch^{\frac{2}{3}}\|\varphi\|^2_{E_0} + C\|f\|^2_{E_0}\,.$$

These estimates imply that

$$(4.22)\qquad (\mathcal{M}_{h,\alpha}-\lambda)\phi_2^{\mathrm{cut}} = f_2^{\mathrm{cut}} + r_h\,,$$

where r_h has the same expression as in Lemma 4.7 and satisfies

$$r_h = \mathscr{O}(h^\infty)(\|f\| + \|\varphi\|)\,.$$

In the following proposition we prove that, in fact, f is small compared to ϕ. This estimate is reminiscent of the famous Caccioppoli estimates (see the original article [10], and, for instance, the article [15] or the book [8, Section 5.4.1]), since it allows us to control the derivatives of ϕ with ϕ.

Proposition 4.14. *We have*

$$(4.23)\qquad \|f\| \leqslant Ch\|\varphi\| \leqslant \tilde{C}h\|\phi_2^{\mathrm{cut}}\|\,,$$

and, for all $(\alpha_1,\alpha_2)\in\mathbb{N}^2$,

$$(4.24)\qquad \|(hD_u)^{\alpha_1}(hD_s)^{\alpha_2}\phi_2^{\mathrm{cut}}\| \leqslant Ch^{\frac{\alpha_1}{3}}h^{\frac{\alpha_2}{2}}\|\varphi\|\,.$$

Proof. Let us start by noticing that, from (4.22), we have

$$\mathrm{Re}\, e^{-i\alpha/2}\langle(\mathcal{M}_{h,\alpha}-\lambda)\phi_2^{\mathrm{cut}},\phi_2^{\mathrm{cut}}\rangle \leqslant \|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\| + \mathscr{O}(h^\infty)(\|f\|+\|\varphi\|)\,,$$

and thus, with (4.17) and localization estimates, we get

$$(4.25)\quad \cos\left(\frac{\alpha}{6}\right)q_{\mathrm{Ai},h}(\phi_2^{\mathrm{cut}}) + c\|hD_s\phi_2^{\mathrm{cut}}\|^2 + c\|s\phi_2^{\mathrm{cut}}\|^2 \leqslant Ch\|\phi_2^{\mathrm{cut}}\|^2 + \|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\| + \mathscr{O}(h^\infty)(\|f\|+\|\varphi\|)\,,$$

where, for all $\Psi\in B^1_0(\mathbb{R}_+) := \{\Psi\in H^1_0(\mathbb{R}_+) : \sqrt{u}\Psi\in L^2(\mathbb{R}_+)\}$,

$$q_{\mathrm{Ai},h}(\Psi) = \|hD_u\Psi\|^2 + \int_{\mathbb{R}_+\times\mathbb{R}}(u - z_1h^{\frac{2}{3}})|\Psi|^2\mathrm{d}u\mathrm{d}s \geqslant 0\,.$$

We will denote the corresponding operator by $\mathscr{A}_h$ and we observe that ϕ_2^{cut} and f_2^{cut} belong to its domain.

At this stage, we still have to control f. By using (4.22), an integration by parts and (4.18), we have

$$(4.26)\quad \begin{aligned}\langle hD_s(\mathcal{M}_{h,\alpha}-\lambda)\phi_2^{\mathrm{cut}}, hD_s\phi_2^{\mathrm{cut}}\rangle &= \langle(hD_s)^2f_2^{\mathrm{cut}},\phi_2^{\mathrm{cut}}\rangle + \langle r_h,(hD_s)^2\phi_2^{\mathrm{cut}}\rangle\\ &\leqslant Ch\|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\| + \mathscr{O}(h^\infty)\|(hD_s)^2\phi_2^{\mathrm{cut}}\|(\|f\|+\|\varphi\|)\,.\end{aligned}$$

Multiplying by $e^{-i\alpha/2}$, taking the real part and estimating commutators give

$$(4.27)\quad \begin{aligned}&\|(hD_s)^2\phi_2^{\mathrm{cut}}\|^2 + q_{\mathrm{Ai},h}(hD_s\phi_2^{\mathrm{cut}})\\ &\leqslant Ch\|hD_s\phi_2^{\mathrm{cut}}\|^2 + Ch\|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\| + Ch\|(hD_s)^2\phi_2^{\mathrm{cut}}\|\|hD_s\phi_2^{\mathrm{cut}}\|\\ &\quad + \mathscr{O}(h^\infty)\|(hD_s)^2\phi_2^{\mathrm{cut}}\|(\|f\|+\|\varphi\|)\,,\end{aligned}$$

where we note that $hD_s\phi_2^{\mathrm{cut}}$ is also in $B^1_0(\mathbb{R}_+)$.

Proceeding in the same way, we find

$$\mathrm{Re}\,\left(e^{-i\alpha/2}\langle(\mathcal{M}_{h,\alpha}-\lambda)\phi_2^{\mathrm{cut}},\mathscr{A}_h\phi_2^{\mathrm{cut}}\rangle\right) \leqslant \mathscr{O}(h^\infty)\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|(\|f\|+\|\varphi\|) + C\|\mathscr{A}_hf_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\|\,,$$

and thus, with (4.19),

$$\operatorname{Re}\left(e^{-i\alpha/2}\langle(\mathcal{M}_{h,\alpha}-\lambda)\phi_2^{\mathrm{cut}},\mathscr{A}_h\phi_2^{\mathrm{cut}}\rangle\right)\leqslant\mathscr{O}(h^\infty)\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|(\|f\|+\|\varphi\|)+Ch\|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\|\,.$$

Therefore, estimating similarly commutators,

$$\begin{aligned}q_{\mathrm{Ai},h}(hD_s\phi_2^{\mathrm{cut}})+\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|^2&\leqslant Chq_{\mathrm{Ai},h}(\phi_2^{\mathrm{cut}})+\mathscr{O}(h^\infty)\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|(\|f\|+\|\varphi\|)\\&+Ch\|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\|+Ch^{\frac{4}{3}-2\eta}\|hD_s\phi_2^{\mathrm{cut}}\|^2+Ch\|hD_s\phi_2^{\mathrm{cut}}\|\|hD_u(hD_s\phi_2^{\mathrm{cut}})\|\,.\end{aligned}$$

Note that

$$\|hD_u(hD_s\phi_2^{\mathrm{cut}})^2\|\leqslant q_{\mathrm{Ai},h}(hD_s\phi_2^{\mathrm{cut}})+Ch^{\frac{2}{3}}\|hD_s\phi_2^{\mathrm{cut}}\|^2\,,$$

so that, using the definition of $q_{\mathrm{Ai},h}$, we get

$$\begin{aligned}q_{\mathrm{Ai},h}(hD_s\phi_2^{\mathrm{cut}})+\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|^2\leqslant Chq_{\mathrm{Ai},h}(\phi_2^{\mathrm{cut}})+\mathscr{O}(h^\infty)\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|(\|f\|+\|\varphi\|)\\+Ch\|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\|+Ch^{\frac{4}{3}-2\eta}\|hD_s\phi_2^{\mathrm{cut}}\|^2\,.\end{aligned}\tag{4.28}$$

Thus, with (4.27), (4.28) and the Young inequality, we get, for some constant $c>0$,

$$\begin{aligned}&c\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|^2+c\|(hD_s)^2\phi_2^{\mathrm{cut}}\|^2+q_{\mathrm{Ai},h}(hD_s\phi_2^{\mathrm{cut}})\\&\qquad\leqslant Ch^2\|\phi_2^{\mathrm{cut}}\|^2+\mathscr{O}(h^\infty)(\|f\|^2+\|\varphi\|^2)+Ch\|f_2^{\mathrm{cut}}\|\|\phi_2^{\mathrm{cut}}\|\,.\end{aligned}$$

Recalling (4.22) to bound the right-hand side, we deduce that

$$c\|\mathscr{A}_h\phi_2^{\mathrm{cut}}\|^2+c\|(hD_s)^2\phi_2^{\mathrm{cut}}\|^2+q_{\mathrm{Ai},h}(hD_s\phi_2^{\mathrm{cut}})\leqslant Ch^2\|\phi_2^{\mathrm{cut}}\|^2\,.\tag{4.29}$$

Again with (4.22), this implies that

$$\|f\|\leqslant C\|f_2^{\mathrm{cut}}\|\leqslant Ch\|\phi_2^{\mathrm{cut}}\|\,,$$

which gives (4.23).

With (4.25), (4.27), and (4.29), we get (4.24) for $|(\alpha_1,\alpha_2)|\in\{0,1,2\}$. The control of the higher powers can be obtained by induction and similar estimates. □

Proposition 4.15 (Localisation in s)**.** *Consider $\chi_0\in\mathscr{C}_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of 0 and $\eta>0$. Then, in H^1-norm, we have*

$$\phi_2^{\mathrm{cut}}=\chi_0(h^{-\frac{1}{2}+\eta}s)\phi_2^{\mathrm{cut}}+\mathscr{O}(h^\infty)\,.\tag{4.30}$$

Moreover, for all $(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3$,

$$\|s^{\alpha_1}(hD_s)^{\alpha_2}(hD_u)^{\alpha_3}\phi_2^{\mathrm{cut}}\|\leqslant Ch^{\frac{\alpha_3}{3}+\frac{\alpha_1+\alpha_2}{2}}\|\phi_2^{\mathrm{cut}}\|\,.\tag{4.31}$$

Proof. The proof is similar to that of Proposition 4.12 since we can write

$$\begin{aligned}\langle(\mathcal{N}_{h,\alpha}-\lambda)(\chi_{\mathrm{far},h}\phi_2^{\mathrm{cut}}),\chi_{\mathrm{far},h}\phi_2^{\mathrm{cut}}\rangle&=\langle\chi_{\mathrm{far},h}f_2^{\mathrm{cut}},\chi_{\mathrm{far},h}\phi_2^{\mathrm{cut}}\rangle\\&+\langle R_h,\chi_{\mathrm{far},h}\phi_2^{\mathrm{cut}}\rangle+\langle[\mathcal{N}_{h,\alpha},\chi_{\mathrm{far},h}]\phi_2^{\mathrm{cut}},\chi_{\mathrm{far},h}\phi_2^{\mathrm{cut}}\rangle\,.\end{aligned}$$

The only new term satisfies

$$\langle\chi_{\mathrm{far},h}f_2^{\mathrm{cut}},\chi_{\mathrm{far},h}\phi_2^{\mathrm{cut}}\rangle=\mathscr{O}(h^\infty)\|\phi_2^{\mathrm{cut}}\|^2\,,$$

since we can apply Proposition 4.12 to the eigenfunction f and use (4.23). Following the same lines as in the proof of Proposition 4.12, we get (4.30).

Let us explain (4.31). Let us consider the case $|(\alpha_1,\alpha_2,\alpha_3)|=1$. When $\alpha_3=1$ or $\alpha_2=1$, the estimate comes from Proposition 4.14. Then, we recall (4.25) and (4.23) and we get (4.31) with $\alpha=(1,0,0)$. For $|(\alpha_1,\alpha_2,\alpha_3)|\geqslant 2$, the result follows by induction. □

4.4.2. *Proof of Proposition 4.13.* We have

$$(\mathcal{M}_{h,\alpha}-\lambda)\varphi = f\,,\qquad \text{with}\qquad (\mathcal{M}_{h,\alpha}-\lambda)f=0\,.$$

Then, from Proposition 4.14 and (4.22), we get that

$$(\mathscr{M}_{h,\alpha}-\lambda)f_2^{\rm cut}=\mathscr{O}(h^\infty)\,,\quad (\mathscr{M}_{h,\alpha}-\lambda)\phi_2^{\rm cut}=f_2^{\rm cut}+\mathscr{O}(h^\infty)\,.$$

Thus,

$$(\mathscr{M}_{h,\alpha}-\lambda)^2\phi_2^{\rm cut}=\mathscr{O}(h^\infty)\,.$$

Recalling (4.17) we get

$$(\mathscr{N}_{h,\alpha}-\lambda)^2\phi_2^{\rm cut}=-P_h(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\rm cut}-(\mathscr{N}_{h,\alpha}-\lambda)P_h\phi_2^{\rm cut}-P_h^2\phi_2^{\rm cut}+\mathscr{O}(h^\infty)\,.$$

By means of Proposition 4.15, we deduce Proposition 4.13. Let us explain this. Among the terms on the right-hand side (coming from the definition of P_h), we have to estimate the following

$$\begin{aligned}\|r_{h,4}s^3(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\rm cut}\| \leqslant & C|\lambda|\|s^3\phi_2^{\rm cut}\|+C\|s^3(hD_u)^2\phi_2^{\rm cut}\|+C\|s^3(hD_s)^2\phi_2^{\rm cut}\|\\ &+C\|hs^3uhD_s\phi_2^{\rm cut}\|+C\|s^3u\phi_2^{\rm cut}\|+C\|s^5\phi_2^{\rm cut}\|\\ \leqslant & Ch^{\frac{13}{6}}\,.\end{aligned}$$

All the other terms can be analyzed in the same way. It appears that the order of magnitude $h^{\frac{13}{6}}$ is the biggest one among all powers of h appearing in the remainders. This completes the proof of Proposition 4.13.

4.5. Proof of Propositons 4.2 & 4.3.

4.5.1. *Proof of Proposition 4.2.* Let us first consider $\varphi=(\varphi_1,\phi_2)$ an eigenfunction of $\mathscr{M}_{h,\alpha}$. From Proposition 4.6, we have

$$\|(\mathscr{N}_{h,\alpha}-\lambda)\phi_2^{\rm cut}\|\leqslant \tilde{C}h^{\frac{3}{2}-\eta}\|\phi_2^{\rm cut}\|\,.$$

Then, we use the resolvent estimate of Proposition 4.5. We write $\lambda=z_1e^{\frac{2i\alpha}{3}}h^{\frac{2}{3}}+\zeta h$, with $\zeta\in D(0,R)$. If ζ does not belong to the spectrum of our complex harmonic oscillator, then, we have

$$1\leqslant \tilde{C}h^{\frac{3}{2}-\eta}\left(Ch^{-\frac{2}{3}}+\frac{C}{\mathrm{dist}(\mathrm{sp}(h^2D_s^2+e^{i\alpha}\frac{k_0s^2}{2}),\zeta h)}\right)\,.$$

We deduce that

$$\mathrm{dist}\left(\zeta,\{(2n-1)e^{\frac{i\alpha}{2}}\sqrt{\frac{k_0}{2}}\,,1\leqslant n\leqslant N\}\right)\leqslant Ch^{\frac{1}{2}-\eta}\,,$$

which implies (4.2).

Let us now discuss the rank of the Riesz projector $\Pi_{n,h}$. From the estimate (4.2), we can draw the circle $\mathscr{C}_{n,h}$ with center $\mu_n(h,\alpha)$ and radius $h^{\frac{3}{2}-3\eta}$ in the resolvent set of $\mathscr{M}_{h,\alpha}$. Let us assume that the rank of the projector is at least two. There are two possibilities. Either there are two distinct (possibly not simple) eigenvalues (which coincide with $\mu_n(h,\alpha)$ modulo $\mathscr{O}(h^{\frac{3}{2}-\eta})$), or there is an eigenvalue with algebraic multiplicity at least two. The strategy is to evaluate the Riesz projector on the corresponding (possibly generalized) eigenfunctions

$$\hat{\Pi}_{n,h}=\frac{1}{2i\pi}\int_{\mathscr{C}_{n,h}}(z-\mathscr{N}_{h,\alpha})^{-1}\mathrm{d}z\,,$$

whose rank is one by Proposition 4.5.

Consider the first case and ψ and $\tilde{\psi}$ corresponding normalized eigenfunctions. Let us denote $F_h = \mathrm{span}(\psi, \tilde{\psi})$, which is of dimension two. Then, the map $Q_h : F_h \ni f \mapsto \chi_h f_2$ is injective. Indeed, from the Agmon estimates satisfied by the eigenfunctions, we see that

$$\|\chi_h f_2\| = \|f\| + \mathscr{O}(h^\infty)\|f\|\,,$$

and, in particular, for h small enough,

$$\|f\| \leqslant 2\|Q_h f\|\,.$$

Moreover, we have, for all $f \in F_h$,

$$\|(\mathscr{N}_{\alpha,h} - \lambda)Q_h f\| \leqslant \tilde{C}h^{\frac{3}{2}-\eta}\|Q_h f\|\,, \quad \lambda = \mu_n(h,\alpha)\,.$$

We notice that

$$\hat{\Pi}_{n,h}Q_h f = Q_h f + \frac{1}{2i\pi}\int_{\mathscr{C}_{n,h}} (z-\lambda)^{-1}(z-\mathscr{N}_{h,\alpha})^{-1}\,(\mathscr{N}_{h,\alpha}-\lambda)\,Q_h f \mathrm{d}z\,.$$

Thus,

$$\|\hat{\Pi}_{n,h}Q_h f - Q_h f\| \leqslant Ch^\eta\|Q_h f\| \leqslant \frac{1}{2}\|Q_h f\|\,.$$

This shows that $\mathrm{rank}\,\hat{\Pi}_{n,h} = 2$, which is a contradiction.

Let us now consider the second case of an eigenvalue with algebraic multiplicity at least two. This implies the existence of $\varphi = (\varphi_1, \phi_2) \in \ker(\mathscr{M}_{h,\alpha} - \lambda)^2$ such that $\varphi \notin \ker(\mathscr{M}_{h,\alpha} - \lambda)$.

Then, we write

$$\hat{\Pi}_{n,h}\phi_2^{\mathrm{cut}} = \phi_2^{\mathrm{cut}} + \frac{1}{2i\pi}\int_{\mathscr{C}_{n,h}} (z-\lambda)^{-2}(z-\mathscr{N}_{h,\alpha})^{-1}(\mathscr{N}_{h,\alpha}-\lambda)^2\phi_2^{\mathrm{cut}}\mathrm{d}z\,.$$

Combining Propositions 4.13 and 4.5, we get

$$\|\hat{\Pi}_{n,h}\phi_2^{\mathrm{cut}} - \phi_2^{\mathrm{cut}}\| = o(1)\|\phi_2^{\mathrm{cut}}\|\,,$$

where we used the fact that $13/6 > 2$. We conclude that the range of $\hat{\Pi}_{n,h}$ has dimension at least two. This is a contradiction with Proposition 4.5.

4.5.2. *Proof of Proposition 4.3.* Considering the result of Proposition 4.2, it is sufficient to show that for each fixed $n \in \{1, \cdots N\}$, the Riesz projector $\Pi_{n,h}$ is not zero. For this consider the function

$$\psi_h(x) = (0, \chi(s,u)\Psi_{1,n,h}(s,u))\,,$$

where $\Psi_{1,n,h}$ is defined in (4.6) and χ is a smooth function with compact support equal to 1 near 0 and equal to 0 outside a small neighborhood of $(0,0)$.

Then, we consider

$$\Pi_{n,h}\psi_h = \frac{1}{2i\pi}\int_{\mathscr{C}_{n,h}} (z-\mathscr{M}_{h,\alpha})^{-1}\psi_h\mathrm{d}z = \frac{1}{2i\pi}\int_{\widetilde{\mathscr{C}}_{n,h}} (z-\mathscr{M}_{h,\alpha})^{-1}\psi_h\mathrm{d}z\,,$$

where $\widetilde{\mathscr{C}}_{n,h}$ is a circle with the same center as $\mathscr{C}_{n,h}$, but with radius of order ϵh for ϵ small enough. Given $z \in \widetilde{\mathscr{C}}_{n,h}$, we consider $\varphi_{h,z} = (\varphi_{h,z,1}, \phi_{h,z,2})$ the unique solution of

$$(z - \mathscr{M}_{h,\alpha})\varphi_{h,z} = \psi_h\,.$$

Then, $\varphi_{h,z}$ satisfies the Agmon estimates with a right-hand side (4.20) and (4.21). In particular, we have, in H^1-norm,

$$\varphi_{h,z,1} = \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|) \tag{4.32}$$

and, with similar notations as in (4.4),

$$\phi_{h,z,2} = \phi^{\rm cut}_{h,z,2} + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|)\,. \tag{4.33}$$

One needs to estimate $\|\varphi_{h,z}\|$. To do so, let us consider $\phi^{\rm cut}_{h,z,2}$, which satisfies

$$(\mathscr{M}_{h,\alpha} - z)\phi^{\rm cut}_{h,z,2} = -\psi^{\rm cut}_{h,2} + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|)\,.$$

As in (the beginning of) the proof of Proposition 4.14, we get the following.

Lemma 4.16. *We have*

$$\|hD_u\phi^{\rm cut}_{h,z,2}\|^2 \leqslant Ch^{\frac{2}{3}}\|\phi^{\rm cut}_{h,z,2}\|^2 + \|\psi_h\|\|\phi^{\rm cut}_{h,z,2}\| + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\|^2 + \|\psi_h\|^2)\,,$$

and

$$\|hD_s\phi^{\rm cut}_{h,z,2}\|^2 + \|s\phi^{\rm cut}_{h,z,2}\|^2 \leqslant Ch\|\phi^{\rm cut}_{h,z,2}\|^2 + C\|\psi_h\|\|\phi^{\rm cut}_{h,z,2}\| + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\|^2 + \|\psi_h\|^2)\,.$$

Similarly, we get the control of the second order derivative with respect to s.

Lemma 4.17. *We have*

$$\|(hD_s)^2\phi^{\rm cut}_{h,z,2}\| \leqslant Ch\|\phi^{\rm cut}_{h,z,2}\| + Ch^{\frac{1}{2}}\|\psi^{\rm cut}_{h,2}\|^{\frac{1}{2}}\|\phi^{\rm cut}_{h,z,2}\|^{\frac{1}{2}} + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|)\,.$$

Proof. Adapting (4.27) with our notations gives

$$\begin{aligned}\|(hD_s)^2\phi^{\rm cut}_{h,z,2}\|^2 &\leqslant Ch\|hD_s\phi^{\rm cut}_{h,z,2}\|^2 + Ch\|\psi^{\rm cut}_{h,2}\|\|\phi^{\rm cut}_{h,z,2}\| \\ &\quad + Ch\|(hD_s)^2\phi^{\rm cut}_{h,z,2}\|\|hD_s\phi^{\rm cut}_{h,z,2}\| + \mathscr{O}(h^\infty)\|(hD_s)^2\phi^{\rm cut}_{h,z,2}\|(\|\psi_h\| + \|\varphi_{h,z}\|)\,.\end{aligned}$$

With the Young inequality, this gives

$$\|(hD_s)^2\phi^{\rm cut}_{h,z,2}\|^2 \leqslant Ch^2\|\phi^{\rm cut}_{h,z,2}\|^2 + Ch\|\psi^{\rm cut}_{h,2}\|\|\phi^{\rm cut}_{h,z,2}\| + \mathscr{O}(h^\infty)(\|\psi_h\|^2 + \|\varphi_{h,z}\|^2)\,.$$

The proof is complete. ☐

We first write

$$(\mathscr{N}_{h,\alpha} - z)\phi^{\rm cut}_{h,z,2} = -\psi^{\rm cut}_{h,2} + R_{h,z}\,,$$

with

$$\|R_{h,z}\| \leqslant Ch^{\frac{2}{3}-\eta}\|(hD_s)^2\phi^{\rm cut}_{h,z,2}\| + Ch\|(hD_s)\phi^{\rm cut}_{h,z,2}\| + C\|s^3\phi^{\rm cut}_{h,z,2}\| + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|)\,.$$

From Lemmas 4.16 and 4.17, we get

$$\|R_{h,z}\| \leqslant Ch^{\frac{2}{3}-2\eta+\frac{1}{2}}\|\phi^{\rm cut}_{h,z,2}\| + Ch^{\frac{2}{3}-2\eta}\|\psi_h\|^{\frac{1}{2}}\|\phi^{\rm cut}_{h,z,2}\|^{\frac{1}{2}} + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|)\,.$$

By using Proposition 4.5 and the fact that $z \in \widetilde{\mathscr{C}}_{n,h}$, we infer that

$$\|\phi^{\rm cut}_{h,z,2}\| \leqslant Ch^{-1}\|\psi_h\| + \mathscr{O}(h^\infty)(\|\varphi_{h,z}\| + \|\psi_h\|)\,.$$

With (4.32) and (4.33), this gives

$$\|\varphi_{h,z}\| \leqslant Ch^{-1}\|\psi_h\|\,.$$

Then, we also deduce that

$$\left\|\int_{\widetilde{\mathscr{C}}_{n,h}} (\mathscr{N}_{h,\alpha} - z)^{-1}R_{h,z}\mathrm{d}z\right\| = o(1)\|\psi_h\|\,.$$

This shows that

$$\Pi_{n,h}\psi_h = \frac{1}{2i\pi}\int_{\mathscr{C}_{n,h}} (z - \mathscr{N}_{h,\alpha})^{-1}\psi^{\rm cut}_{h,2}\mathrm{d}z + o(1)\|\psi_h\|\,.$$

Recalling the resolvent formula

$$(z - \mathscr{N}_{h,\alpha})^{-1} - (z - \mu_n(h,\alpha))^{-1} = (z - \mathscr{N}_{h,\alpha})^{-1}(z - \mu_n(h,\alpha))^{-1}(\mathscr{N}_{h,\alpha} - \mu_n(h,\alpha))\,,$$

and that

$$(\mathscr{N}_{h,\alpha} - \mu_n(h,\alpha))\psi_{h,2} = \mathscr{O}(h^\infty)\,,$$

we get

$$\Pi_{n,h}\psi_h = \psi_h + o(1)\|\psi_h\|\,.$$

Therefore, $\Pi_{n,h}$ is not zero for h small enough. Recalling the discussion at the beginning of this section, this completes the proof of Proposition 4.3.

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