EQUIVARIANT MOTIVIC INTEGRATION AND ITS APPLICATION TO THE INTEGRAL IDENTITY CONJECTURE

LÊ QUY THUONG

Abstract. It is well known that the classical motivic integartion invented by Kontsevich and developed by Denef-Loeser is among important objects concerning several fields of mathematics such as algebraic geometry, non-archimedean geometry, singularity theory and mathematical physics. One of its important influence to the motivic Donaldson-Thomas invariants is the integral identity conjecture. To obtain a complete proof of this conjecture for regular functions we develop an equivariant version of Denef-Loeser's theory where actions of algebraic groups are mentioned. In this note, we review important aspects in our theory and the proof of the conjecture.

1. Introduction

Let k be a field of characteristic zero. Let X be a k-variety, that is, a k-scheme of finite type. For semi-algebraic subsets of the arc space $\mathcal{L}(X)$ of X one constructs an additive measure μ taking value in a reasonable completion of the Grothendieck ring of k-varieties. Let A be a semi-algebraic subset of $\mathcal{L}(X)$, and let $\alpha : A \to \mathbb{Z}$ be a simple function. The motivic integral of α can be defined whenever all the fibers of α are semi-algebraic subsets of $\mathcal{L}(X)$, namely,

$$\int_A \mathbb{L}^{-\alpha} := \sum_{n\in\mathbb{Z}} \mu(\alpha^{-1}(n))\mathbb{L}^{-n},$$

with $\mathbb{L}$ the isomorphism class of $\mathbb{A}_k^1$ (see [DL99]). The integral admits a change of variables formula with respect to a proper birational morphism onto X. Kontseich in his Orsay seminar talk used this formula and resolution of singularity to prove Batyrev's conjecture on the Betti numbers of birationally equivalent complex Calabi-Yau varieties (stated in [Ba99]).

In [LN20], Nguyen and the author developed an equivariant version of the classical motivic integration. The work is motivated from the studies of Denef-Loeser on motivic Milnor fibers where the canonical action of the group of roots of unity on a motivic Milnor fiber reflects the monodromy action on the classical Milnor fiber of a hypersurface complex singularity. The most important contribution of [LN20] is to introduce a criterion of being equal in the G-equivariant Grothendieck ring of k-varieties endowed with an action of a group scheme G. This result allows to construct a G-equivariant motivic measure and motivic integration as well as to prove a G-equivariant change of variables formula. It together with a rationality result of ours also allows to prove the integral identity conjture for regular functions. The most important contributions of the paper [LN20] will be reviewed in the note.

Sebag [Se04], Loeser-Sebag [LS03] and Nicaise-Sebag [NS07] develop a theory of geometric motivic integration for formal schemes topologically of finite type over a discrete valuation ring with perfect residue field. When applying their theory to formal completion of k-varieties one recovers Denef-Loeser's classical motivic integration [DL99]. Recently, Hartmann [Ha15] has provided a G-equivariant version of the genmetric motivic integration for formal schemes topologically of finite type, where G is an abelian finite group. However, one can not view

Lê-Nguyen's equivariant motivic integration [LN20] as a special case of Hartmann's equivariant motivic integration because Hartmann's equivariant Grothendieck ring defined in [Ha15, Definition 4.1] is quite different to Denef-Loeser's definition in [DL98, DL99] that Lê-Nguyen use. Furthermore, the foundation [Ha15, Lemma 4.7] of Hartmann's work is also different to Theorems 2.1.

This note mentions in Section 3.2 the version for regular functions of the integral identity conjecture which was already proved in [LN20]. The integral identity conjecture, which was proposed by Kontsevich-Soibelman, is one of the most important tools in studying motivic Donaldson-Thomas invariants for noncommutative Calabi-Yau threefolds (cf. [KS08], [Le17, Section 1]). The author proved the conjecture for some particular forms of f in [Le12], proved a weak form of the conjecture for ground fields algebraically closed in [Le15], and had some improvement in [Le17]. More recently, Nicaise-Payne [NP17] have proved the conjecture for regular functions over ground fields containing all roots of unity. The proof of Lê-Nguyen in [LN20] finishes attempts in proving the conjecture for regular functions.

2. Equivariant motivic integration, after [LN20]

2.1. **Equivariant Grothendieck rings of varieties.** Let k be a field of characteristic zero, S an algebraic k-variety and Var_S the category of S-varieties. The *Grothendieck group* $K_0(\mathrm{Var}_S)$ of S-varieties is an abelian group generated by symbols $[X \to S]$ for objects $X \to S$ in Var_S modulo the relation $[X \to S] = [Y \to S]$ whenever X and Y are S-isomorphic, and modulo the relation $[X \to S] = [Y \to S] + [X \setminus Y \to S]$ if Y is a Zariski closed subvariety of X. The group $K_0(\mathrm{Var}_S)$ becomes a commutative ring by means of the fiber product of S-varieties, and it has the class of the identity morphism $S \to S$ as the unity. Define $\mathcal{M}_S$ to be the localization $K_0(\mathrm{Var}_S)[\mathbb{L}^{-1}]$; when $S = \mathrm{Spec}(k)$, we simply write this ring as $\mathcal{M}_k$.

Assume that X, Y and F are k-varieties, and that A and B are respectively constructible subsets of X and Y. Let $f : X \to Y$ be a morphism with $f(A) \subseteq B$. The restriction map $f : A \to B$ is a *piecewise trivial fibration with fiber* F if there exists a stratification of B into finitely many locally closed subsets B_i in Y such that, for every i, $f^{-1}(B_i) := A \times_Y B_i$ is locally closed in X and isomorphic as a k-variety to $B_i \times_k F$, and that, over B, f corresponds to the projection $B_i \times_k F \to B_i$. More generally, f is a *piecewise trivial fibration over* $C \subseteq B$ if $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a piecewise trivial fibration. Sebag in [Se04, Théorème 4.2.3] proves that the map $f : A \to B$ is a piecewise trivial fibration with fiber F if and only if for every y in B, the fiber $f^{-1}(y)$ is isomorphic as a $\kappa(y)$-variety to $F \times_k \kappa(y)$.

Assume that there is an action of an algebraic group G on a k-variety X. This action is *good* if every G-orbit is contained in an affine open subset of X. If both k-varieties X and Y are endowed with good G-actions, then a morphism $f : X \to Y$ is called *G-equivariant* if it is compatible with the G-actions, i.e., $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and $g \in G$. Let S be a k-variety endowed with the trivial G-action. The *G-equivariant Grothendieck group* $K_0^G(\mathrm{Var}_S)$ of all the G-equivariant morphisms of k-varieties $X \to S$, where X is endowed with a good G-action, is the quotient of the free abelian group generated by the G-equivariant isomorphism classes $[X \to S, \sigma]$, where X is a k-variety endowed with a good G-action σ and $X \to S$ is a G-equivariant morphism of k-varieties, modulo the relations

$$[X \to S, \sigma] = [Y \to S, \sigma|_Y] + [X \setminus Y \to S, \sigma|_{X \setminus Y}]$$

for Y being G-invariant Zariski closed in X under σ, and

$$[X \times_k \mathbb{A}_k^n \to S, \sigma] = [X \times_k \mathbb{A}_k^n \to S, \sigma']$$

if σ and σ' lift the same good G-action on X. Similarly as above, we have a natural commutative ring with unity structure on $K_0^G(\mathrm{Var}_S)$ by fiber product. We define the localization $\mathcal{M}_S^G$ of the ring $K_0^G(\mathrm{Var}_S)$ inverting the class of $\mathbb{A}_k^1 \times_k S \to S$ endowed with the trivial G-action.

Let $\mu_n = \mathrm{Spec}(k[t]/(t^n - 1))$ be the group scheme of nth roots of unity in k. The family of all μ_n forms a projective system with respect to morphisms $\mu_{nm} \to \mu_n$ given by $\xi \mapsto \xi^m$, whose limit is denoted by $\hat{\mu}$. We define $K_0^{\hat{\mu}}(\mathrm{Var}_S) = \varinjlim_{n\in\mathbb{N}} K_0^{\mu_n}(\mathrm{Var}_S)$ and $\mathcal{M}_S^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_S)[\mathbb{L}^{-1}]$, both are called in a same name the *monodromic Grothendieck ring of k-varieties*. Clearly, there is a canonical isomorphism of rings $\mathcal{M}_S^{\hat{\mu}} = \varinjlim_{n\in\mathbb{N}^*} \mathcal{M}_S^{\mu_n}$.

For a morphism of k-varieties $f : S \to S'$ and an algebraic group G, one defines group morphisms $f_! : \mathcal{M}_S^G \to \mathcal{M}_{S'}^G$ and $f_! : \mathcal{M}_S^{\hat{\mu}} \to \mathcal{M}_{S'}^{\hat{\mu}}$ by composition, and ring morphisms $f^* : \mathcal{M}_{S'}^G \to \mathcal{M}_S^G$ and $f^* : \mathcal{M}_{S'}^{\hat{\mu}} \to \mathcal{M}_S^{\hat{\mu}}$ by fiber product. If $S' = \mathrm{Spec}(k)$, $f_!$ is denoted by $\int_S$.

Let X, Y and F be k-varieties endowed with good action of an algebraic group G. Let A and B be G-invariant constructible subsets of X and Y, respectively. Let $f : X \to Y$ be a G-equivariant morphism such that $f(A) \subseteq B$. The restriction $f : A \to B$ is called a *G-equivariant piecewise trivial fibration with fiber F* if there exists a stratification of B into finitely many G-invariant locally closed subsets B_i such that $f^{-1}(B_i)$ is a G-invariant constructible subset of A and $f^{-1}(B_i)$ is G-equivariant isomorphic to $B_i \times_k F$, with the action of G on $B_i \times_k F$ being the diagonal one, and over B_i, f equals the projection $B_i \times_k F \to B_i$. Clearly, if there is such a map, one gets the identity $[A] = [B] \cdot [F]$ in $K_0^G(\mathrm{Var}_k)$.

For a morphism of k-varieties $X \to Y$ and any immersion $S \to Y$, we write X_S for the fiber product $X \times_Y S$. If Y is endowed with a good G-action, then for y in Y, the *stabilizer subgroup G_y* of G with respect to y is the subgroup of elements in G fixing y.

Theorem 2.1 (Lê-Nguyen [LN20], Theorem 2.2)**.** *Let X and Y be k-varieties endowed with good G-action such that the categorical quotient $Y \to Y//G$ exists and is quasi-finite. Let $f\colon X \to Y$ be a G-equivariant morphism. Then f is a G-equivariant piecewise trivial fibration if and only if there exists a k-variety F endowed with good G-action such that for every y in Y, there exists a G_y-equivariant isomorphism of $\kappa(y)$-varieties $X_y \xrightarrow{\cong} F \times_k \kappa(y)$.*

This theorem has an interesting consequence, which is shown in [LN20, Theorem 2.3], that with the assumption as in Theorem 2.1 augmented by the condition that there exists for every y in Y an isomorphism of $\kappa(y)$-varieties $X_y \cong \mathbb{A}^n_{\kappa(y)}$, for a given natural number n we have the identity $[X] = [Y]\mathbb{L}^n$ in $K_0^G(\mathrm{Var}_k)$.The condition that $Y \to Y//G$ exists and is quasi-finite is always satisfied if G is finite and the G-action is good.

2.2. **Arc spaces.** Let X be a k-variety. Consider for each $n \in \mathbb{N}$ the k-scheme $\mathcal{L}_n(X)$ representing the functor which sends k-algebras A to the sets $\mathrm{Hom}_k(\mathrm{Spec}(A[t]/A(t^{n+1})), X)$. For $m \geq n$, the truncation $k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$ gives rise to a morphism of k-schemes $\pi^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. If X is a smooth k-variety of dimension d, the morphism π^m_n is a locally trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$. The n-jet schemes and the morphisms π^m_n form a projective system of k-schemes, thus one defines the *arc space of* X to be the k-scheme

$$\mathcal{L}(X) := \varprojlim \mathcal{L}_n(X).$$

the *arc space of* X. While $\mathcal{L}_n(X)$ is of finite type for every n, $\mathcal{L}(X)$ is not. For any field extension $K \supseteq k$, the K-points of $\mathcal{L}(X)$ correspond one-to-one to the $K[[t]]$-points of X. Denote by π_n the natural morphism $\mathcal{L}(X) \to \mathcal{L}_n(X)$.

In the Denef-Pas language, a formula is a conjunction of clauses of the following forms $\mathrm{ord}_t f_1(x) \geq \mathrm{ord}_t f_2(x) + m_1$, $\mathrm{ord}_t f_3(x) \equiv m_2 \mod n$, and $\Phi(\overline{\mathrm{ac}}(g_1(x)), \ldots, \overline{\mathrm{ac}}(g_n(x))) = 0$, where f_i, g_j and Φ are polynomials over k, ℓ_1 and ℓ_2 are $\mathbb{Z}$-polynomials of degree ≤ 1, n is in $\mathbb{N}$, and $\overline{\mathrm{ac}}(g_j(x))$ is the angular component of $g_j(x)$. A *semi-algebraic* subset of $\mathcal{L}(X)$ is a subset defined by a formula in the Denef-Pas language. More precisely, $A \subseteq \mathcal{L}(X)$ is semi-algebraic if there exists a covering of X by affine Zariski open sets U such that

$$A \cap \mathcal{L}(U) = \{\gamma \in \mathcal{L}(U) \mid \theta(h_1(\tilde{\gamma}), \ldots, h_m(\tilde{\gamma}))\},$$

where h_i are regular functions on U, θ is a Denef-Pas formula, and $\tilde{\gamma}$ is the element in $\mathcal{L}(U)(\kappa(\gamma))$ corresponding to a point γ in $\mathcal{L}(U)$ of residue field $\kappa(\gamma)$ (cf. [DL99, Section 2.2]). A *simple* function α on a semi-subset $A \subseteq \mathcal{L}(X)$ is a function $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ such that the set $\{\gamma \in A \mid \alpha(\gamma) = n\}$, with $n \in \mathbb{N}$, is semi-algebraic. If f is a regular function on a k-variety X and A is a semi-algebraic subset of $\mathcal{L}(X)$, then $\mathrm{ord}_t f$ is a simple function on A.

2.3. **Equivariant motivic integration.** Let X be a k-variety of pure dimension d. We consider semi-algebraic subsets A of $\mathcal{L}(X)$ having the following

Property 2.2.
(i) *A is a union of fibers of the canonical morphism $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ for some $n \in \mathbb{N}$;*
(ii) *the map $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a piecewise trivial fibration over $\pi_m(A)$ for every m bigger than some $n \in \mathbb{N}$.*

We observe that if A satisfies Property 2.2 (i) and if $A \cap \mathcal{L}(X_{\mathrm{sing}}) = \emptyset$, then A also satisfies Property 2.2 (ii). Here, X_{Sing} is the locus of singular points of X. If a semi-algebraic subset $A \subseteq \mathcal{L}(X)$ has the above-mentioned properties, then the morphism $\pi_m(A) \to \pi_n(A)$ is a piecewise trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$ for every $m \geq n$ big enough, hence the quantity

$$\mu(A) := [\pi_m(A)]\mathbb{L}^{-(m+1)d} = [\pi_n(A)]\mathbb{L}^{-(n+1)d} \in \mathcal{M}_k$$

is independent of the choice of m, n big enough.

Let $\hat{G} = \varprojlim G_i$ be the limit of a projective system of finite algebraic groups over a directed ordered set $(I, \leq)$. Assume that every G_i acts well on $\mathcal{L}_n(X)$, that for $m \geq n$, the morphisms $\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$ are G_i-equivariant, and that $\pi_m(\mathcal{L}(X))$ is G_i-invariant. We consider semi-algebraic subsets $A \subseteq \mathcal{L}(X)$ satisfying the following

Property 2.3.
(i) *A is a union of fibers of the canonical morphism $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ for some $n \in \mathbb{N}$;*
(ii) *for m big enough, $\pi_m(A)$ is G_i-invariant for every $i \in I$,*
(iii) *the map $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a G_i-equivariant piecewise trivial fibration over $\pi_m(A)$ for every m bigger than some $n \in \mathbb{N}$ and for every $i \in I$.*

Then the morphism $\pi_m(A) \to \pi_n(A)$ is a G_i-equivariant piecewise trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$ for every $i \in I$ and $m \geq n$ big enough. A consequence of Theorem 2.1 shows that $[\pi_m(A)]\mathbb{L}^{-(m+1)d} = [\pi_n(A)]\mathbb{L}^{-(n+1)d}$ holds in $\mathcal{M}_k^{G_i}$ for every $m \geq n$ big enough, hence the quantity $\mu^{G_i}(A) := [\pi_m(A)]\mathbb{L}^{-(m+1)d}$ in $\mathcal{M}_k^{G_i}$ is independent of the choice of m big enough.

Assume that $A \subseteq \mathcal{L}(X)$ satisfies Property 2.3. Let $\alpha : A \to \mathbb{N}$ be a simple function such that all the fibers $\alpha^{-1}(n)$ also satisfy Property 2.3. It implies from [DL99, Lemma 2.4] that

A is the disjoint union of finitely many subsets $\alpha^{-1}(n)$. Moreover, these sets $\alpha^{-1}(n)$ are all G_i-invariant. By [LN20], the *$\hat{G}$-equivariant motivic integral* of α is defined to be

$$\int_A \mathbb{L}^{-\alpha} d\tilde{\mu}^{\hat{G}} := \varinjlim_{i\in I} \sum_{n\in\mathbb{N}} \tilde{\mu}^{G_i}\left(\alpha^{-1}(n)\right)\mathbb{L}^{-n},$$

which takes value in $\mathcal{M}_k^{\hat{G}} := \varinjlim_{i\in I} \mathcal{M}_k^{G_i}$.

Theorem 2.4 (Lê-Nguyen [LN20])**.** *Let X, Y be k-varieties of pure dimension d, with Y smooth. Let $h : Y \to X$ be a proper birational morphism. Let $\hat{G} = \varprojlim G_i$. Assume that every G_i acts well on $\mathcal{L}_n(X)$ and $\mathcal{L}_n(Y)$ such that all the morphisms $h_n : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$ are G_i-equivariant. Let $A \subseteq \mathcal{L}(X)$ satisfy Property 2.3 and $A \cap \mathcal{L}(h(E)) = \emptyset$, where E is the exceptional locus of h. Let $\alpha : A \to \mathbb{N}$ a simple function whose fibers satisfy Property 2.3. Then $h^{-1}(A)$ and the fibers of $\alpha\circ h + \mathrm{ord}_t \det \mathrm{Jac}_h$ satisfy Property 2.3 for Y, and the identity*

$$\int_A \mathbb{L}^{-\alpha} d\tilde{\mu}^{\hat{G}} = \int_{h^{-1}(A)} \mathbb{L}^{-\alpha\circ h - \mathrm{ord}_t \det \mathrm{Jac}_h} d\tilde{\mu}^{\hat{G}}$$

holds in $\mathcal{M}_k^{\hat{G}}$.

3. Rationality of motivic zeta functions and the integral identity

3.1. **Rationality of motivic zeta functions.** Let X be a smooth algebraic k-variety of pure dimension d, and $f : X \to \mathbb{A}_k^1$ a regular function with the zero locus X_0 nonempty. For $n \geq 1$, we put

$$\mathcal{X}_n(f) := \left\{\gamma \in \mathcal{L}_n(X) \mid f(\gamma) = t^n \mod t^{n+1}\right\}.$$

Then $\mathcal{X}_n(f)$ is naturally an X_0-variety and invariant under the natural action σ of μ_n on $\mathcal{L}_n(X)$ given by $\xi \cdot \gamma(t) := \gamma(\xi t)$, which defines an element of $\mathcal{M}_{X_0}^{\hat{\mu}}$ denoted by $[\mathcal{X}_n(f)]$. Then the *motivic zeta function* of f is the following formal series

$$Z_f(T) := \sum_{n\geq 1} [\mathcal{X}_n(f)]\mathbb{L}^{-nd}T^n.$$

If x is a closed point in X_0, by putting $\mathcal{X}_{n,x}(f) = \{\gamma \in \mathcal{X}_n(f) \mid \gamma(0) = x\}$ we obtain in the same way the *motivic zeta function of f at x*

$$Z_{f,x}(T) := \sum_{n\geq 1} [\mathcal{X}_{n,x}(f)]\mathbb{L}^{-nd}T^n.$$

For a k-variety S, we consider the set $\mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}}$, which is the $\mathcal{M}_S^{\hat{\mu}}$-submodule of $\mathcal{M}_S^{\hat{\mu}}[[T]]$ generated by 1 and by finite products of elements of the form $\frac{\mathbb{L}^a T^b}{1-\mathbb{L}^a T^b}$ with (a, b) in $\mathbb{Z} \times \mathbb{N}_{>0}$. An element of $\mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ is called a *rational series*. By [DL98], there is a unique $\mathcal{M}_S^{\hat{\mu}}$-linear morphism $\lim_{T\to\infty} : \mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}} \to \mathcal{M}_S^{\hat{\mu}}$ such that $\lim_{T\to\infty} \frac{\mathbb{L}^a T^b}{1-\mathbb{L}^a T^b} = -1$.

Theorem 3.1 (Denef-Loeser [DL98])**.** *For any regular function on a k-variety X and for any closed point x of the zero locus X_0 of f, the series $Z_f(T)$ and $Z_{f,x}(T)$ are rational.*

The element $\mathcal{S}_f := -\lim_{T\to\infty} Z_f(T)$ of $\mathcal{M}_{X_0}^{\hat{\mu}}$ is called the *motivic nearby cycles* of f. If $x \in X_0$ is a closed point, $\mathcal{S}_{f,x} := -\lim_{T\to\infty} Z_{f,x}(T)$ of $\mathcal{M}_k^{\hat{\mu}}$ is called the *motivic Milnor fiber of f at x*.

We can also define $\mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ for multivariate $T = (T_1, \ldots, T_r)$ in the same way, namely, it is the $\mathcal{M}_S^{\hat{\mu}}$-submodule of $\mathcal{M}_S^{\hat{\mu}}[[T]]$ generated by 1 and by finite products of elements of the form $\frac{\mathbb{L}^a T^b}{1-\mathbb{L}^a T^b}$ with (a, b) in $\mathbb{Z} \times \mathbb{N}^r_{>0}$, where T^b stands for $T_1^{b_1} \cdots T_r^{b_r}$ when $b = (b_1, \ldots, b_r)$. An element of $\mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ is a *rational series* in multivariate T.

Theorem 3.2 (Lê-Nguyen [LN20])**.** *Let X be an affine k-variety, and let f be a regular function on X. Let $\varphi(x, \alpha_1, \ldots, \alpha_r)$ be a formula which is a conjunction of clauses of the forms $\mathrm{ord}_t f_1(x) \geq \mathrm{ord}_t f_2(x) + \ell_1(b_1, \ldots, b_r)$ and $\mathrm{ord}_t f_3(x) \equiv \ell_2(b_1, \ldots, b_r) \mod n$, where f_i are polynomials over k and ℓ_j are $\mathbb{Z}$-polynomials of degree ≤ 1. For each $b \in \mathbb{N}^r$, put*

$$A_b := \left\{\gamma \in \mathcal{L}(X) \mid K((t)) \models \varphi(\gamma(t), b) \ \forall \text{ field extension } k \subseteq K = \overline{K}\right\}.$$

Assume that, for any $\alpha \in \mathbb{N}^r$, A_α satisfies Property 2.2 and disjoint with $\mathcal{L}(X_{\mathrm{Sing}})$. For $n \geq 1$, we put

$$A_{n,\alpha} := \left\{\gamma \in A_\alpha \mid f(\gamma) = t^n \mod t^{n+1}\right\},$$

which admits the natural μ_n-action for every α in $\mathbb{N}^r$. Then the formal power series

$$Z(T_0, T_1, \ldots, T_r) := \sum_{(n,\alpha) \in \mathbb{N}^{r+1}} \mu\left(A_{n,\alpha}\right) T_0^n T_1^{\alpha_1} \cdots T_r^{\alpha_r}$$

is rational in $T_0, T_1, \ldots, T_r$.

Theorems 3.1 and 3.2 are both proved using resolution of singularities together with some principles in model theory. The latter has an important consequence as follows. Assume that X, f, A_α and $A_{n,\alpha}$ are as in Theorem 3.2. Let Δ be a rational polyhedral convex cone in $\mathbb{R}^{r+1}_{\geq 0}$ and $\bar{\Delta}$ its closure. Let ℓ and ε be integral linear forms on $\mathbb{Z}^{r+1}$ with $\ell(n, \alpha) > 0$ and $\varepsilon(n, \alpha) \geq 0$ for all (n, α) in $\bar{\Delta} \setminus \{0\}$. Then, as shown in [LN20, Proposition 4.6], the series

$$Z(T) := \sum_{(n,\alpha) \in \Delta \cap \mathbb{N}^{r+1}} \mu\left(A_{n,\alpha}\right) \mathbb{L}^{-\varepsilon(n,\alpha)} T^{\ell(n,\alpha)}$$

is rational in univariate T and its limit $\lim_{T \to \infty} Z(T)$ is independent of such an ℓ and ε.

3.2. **The integral identity concerning regular functions.** One of important applications of the $\hat{\mu}$-equivariant motivic integration is to prove the integral identity conjecture for regular functions. We now sketch of the proof emphasising the role of Theorems 2.1 and 3.2.

Theorem 3.3 (Lê-Nguyen [LN20])**.** *Let (x, y, z) be the standard coordinates of the vector space $k^d = k^{d_1} \times k^{d_2} \times k^{d_3}$. Let f be in $k[x, y, z]$ such that $f(0, 0, 0) = 0$ and $f(\lambda x, \lambda^{-1} y, z) = f(x, y, z)$ for all λ in $\mathbb{G}_{m,k}$. Then the integral identity $\int_{\mathbb{A}_k^{d_1}} i^* \mathcal{S}_f = \mathbb{L}^{d_1} \mathcal{S}_{\tilde{f},0}$ holds in $\mathcal{M}_k^{\hat{\mu}}$, where $\tilde{f}$ is the restriction of f to $\mathbb{A}_k^{d_3}$, and i is the inclusion of $\mathbb{A}_k^{d_1}$ in $f^{-1}(0)$.*

Sketch of proof. By the hypothesis on f we have $f(x, 0, z) = f(0, 0, z) = \tilde{f}(z)$. This implies that $i^*[\mathcal{X}_n(f)]$ equals $\left[\left\{\gamma \in \mathcal{X}_n(f) \mid \gamma(0) \in \mathbb{A}_k^{d_1}\right\}\right]$ as elements of $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{A}_k^{d_1}})$. Write $\gamma(t) = (x(t), y(t), z(t))$ with $x(t) = (x_j(t))_{1 \leq j \leq d_1}$, $y(t) = (y_j(t))_{1 \leq j \leq d_2}$. Define

$$\mathrm{ord}_t x(t) := \min_{1 \leq j \leq d_1} \mathrm{ord}_t x_j(t), \quad \mathrm{ord}_t y(t) := \min_{1 \leq j \leq d_2} \mathrm{ord}_t y_j(t.$$

Put

$$U_n := \left\{\gamma \in \mathcal{X}_n(f) \mid \gamma(0) \in \mathbb{A}_k^{d_1}, \mathrm{ord}_t x(t) + \mathrm{ord}_t y(t) > n\right\},$$

$$W_n := \left\{\gamma \in \mathcal{X}_n(f) \mid \gamma(0) \in \mathbb{A}_k^{d_1}, \mathrm{ord}_t x(t) + \mathrm{ord}_t y(t) \leq n\right\},$$

which are respectively closed and open k-subvarieties of $\mathcal{X}_n(f)\times_{X_0}\mathbb{A}_k^{d_1}$ and invariant under the natural μ_n-action $\xi\gamma(t):=\gamma(\xi t)$. Thus we have

$$\int_{\mathbb{A}_k^{d_1}} i^*\left[\mathcal{X}_n(f)\right]\mathbb{L}^{-nd}=\left[U_n\right]\mathbb{L}^{-nd}+\left[W_n\right]\mathbb{L}^{-nd}.$$

By Theorem 3.2, the series $U(T):=\sum_{n\geq 1}\left[U_n\right]\mathbb{L}^{-nd}T^n$ and $W(T):=\sum_{n\geq 1}\left[W_n\right]\mathbb{L}^{-nd}T^n$ are rational. Since (it is not difficult to prove) the minus limit of the former is the quantity on the right hand side of the integral identity, it suffices to prove that $\lim_{T\to\infty}W(T)=0$.

For $n\geq m\geq 1$, putting $W_{n,m}:=\{\gamma\in W_n\mid \mathrm{ord}_t x+\mathrm{ord}_t y=m\}$, which is invariant under the natural $\hat{\mu}$-action, we get $W(T)=\sum_{1\leq m\leq n}\left[W_{n,m}\right]\mathbb{L}^{-nd}T^n$. Since the reductive group $\mathbb{G}_{m,k}$ acts on the affine k-variety $X:=\mathbb{A}_k^{d_1}\times_k\mathbb{A}_k^{d_2}\times_k\mathbb{A}_k^{d_3}$ with $\xi\cdot(x,y,z):=(\xi x,\xi^{-1}y,z)$, for ξ in $\mathbb{G}_{m,k}$ and (x,y,z) in X, it admits a categorical quotient $\phi: X\to Y$ such that Y is a k-variety (cf. [MFK94, Chapter 1, Section 2]).

We can prove that the morphism $\phi_n\colon W_{n,m}\to V_{n,m}:=\phi_n(W_{n,m})$ which is the restriction to $W_{n,m}$ of the morphism $\phi_n\colon\mathcal{L}_n(X)\to\mathcal{L}_n(Y)$ is a μ_n-equivariant piecewise trivial fibration with fiber $F:=\{\tau\in\mathcal{L}_n(\mathbb{A}_k^1)\mid \mathrm{ord}_t\tau<m\}$, where the action of μ_n on F is induced from the natural action of μ_n on $\mathcal{L}_n(\mathbb{A}_k^1)$. Indeed, for every field extension $k\subseteq K$, we take an arbitrary K-arc ψ in $V_{n,m}$ and consider the fiber $W_{n,m,\psi}$ of ϕ_n over ψ. For all $1\leq i\leq d_1$, $1\leq j\leq d_2$ and $1\leq l\leq d_3$, the morphisms f_{ij} and z_l from X to $\mathbb{A}_k^1$ defined respectively by x_iy_j and z_l are $\mathbb{G}_{m,k}$-equivariant with respect to the trivial action of $\mathbb{G}_{m,k}$ on $\mathbb{A}_k^1$. By the universality of ϕ, these morphisms are constant on every fiber of ϕ, thus the induced morphisms $(f_{ij})_n,(z_l)_n\colon\mathcal{L}_n(X)\to\mathcal{L}_n(\mathbb{A}_k^1)$ are constant on $W_{n,m,\psi}$. Fix an element $\gamma^\circ=(x^\circ,y^\circ,z^\circ)$ in $W_{n,m,\psi}$. We may assume that $\mathrm{ord}_t x^\circ=\mathrm{ord}_t x_1^\circ$, and under this assumption, may prove that $\mathrm{ord}_t x=\mathrm{ord}_t x_1$ for all $\gamma=(x,y,z)$ in $W_{n,m,\psi}$. Consider the morphism $\chi_\psi\colon W_{n,m,\psi}\to F\times_k K$ sending a K-arc $\gamma=(x,y,z)$ to its first component x_1. Clearly, χ_ψ is a $(\mu_n)_\psi$-equivariant morphism. It is injective because $(f_{ij})_n$ and $(z_l)_n$ are constant on $W_{n,m,\psi}$. For any v in $F\times_k K$, put $\tau=v(x_1^\circ)^{-1}\in K((t))$ and consider $(x,y,z):=(\tau x^\circ \mod t^{n+1},\tau^{-1}y^\circ \mod t^{n+1},z^\circ)$. Then (x,y,z) is in $W_{n,m,\psi}$ and $\chi_\psi(x,y,z)=v$, hence χ_ψ is surjective. So χ_ψ is isomorphic, from which χ_ψ is a $(\mu_n)_\psi$-equivariant isomorphism. It follows from Theorem 2.1 that ϕ_n is a μ_n-equivariant piecewise trivial fibration with fiber F. As a consequence, $[W_{n,m}]=[V_{n,m}]\cdot[F]=[V_{n,m}]\cdot(\mathbb{L}^{n+1}-\mathbb{L}^{n-m+1})$ as elements in $\mathcal{M}_k^{\hat{\mu}}$.

Consider the morphism $\phi_\infty\colon\mathcal{L}(X)\to\mathcal{L}(Y)$ induced by ϕ. Fro every $m\in\mathbb{N}_{>0}$, let $A_m\subseteq\mathcal{L}(Y)$ be the image of $\{(x,y,z)\in\mathcal{L}(X)\mid \mathrm{ord}_t x+\mathrm{ord}_t y=m\}$ under ϕ_∞. By the hypothesis, f is $\mathbb{G}_{m,k}$-equivariant, hence by the universal property of the quotient ϕ, f induces a regular function $g\colon Y\to\mathbb{A}_k^1$ such that $f=g\circ\phi$. Putting $A_{n,m}:=\left\{\gamma\in A_m\mid g(\gamma)=t^n \mod t^{n+1}\right\}$ we have $\mu(A_{n,m})=\left[\pi_n\left(A_{n,m}\right)\right]\mathbb{L}^{-(n+1)(d-1)}=\left[V_{n,m}\right]\mathbb{L}^{-(n+1)(d-1)}$, since $\dim_k Y=d-1$. Therefore

$$W(T)=\mathbb{L}^d\sum_{1\leq m\leq n}\mu\left(A_{n,m}\right)T^n-\mathbb{L}^d\sum_{1\leq m\leq n}\mu\left(A_{n,m}\right)\mathbb{L}^{-m}T^n,$$

from which $\lim_{T\to\infty}W(T)=0$, thanks to Theorem 3.2 (cf. [LN20, Proposition 4.6]). □

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Faculty of Mathematics, Mechanics and Informatics, VNU University of Science, Vietnam National University, Hanoi, Hanoi, Vietnam

334 Nguyen Trai Street, Thanh Xuan District, Hanoi, Vietnam

Email address: leqthuong@gmail.com