INTERSECTION SPACE COHOMOLOGY SHEAVES

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Abstract. This talk is meant to give an overview on the sheaf theoretic approach to intersection space cohomology and discusses its advantages and disadvantages. It gives a hint on the construction of certain intersection space complexes on toric varieties with Poincaré duality for their hypercohomology groups.

1. Introduction

Poincaré duality is one of the most important topological properties of closed oriented manifolds and does in general not hold for singular spaces. The initial question of this talk is: Can we change the singular space (locally near the singular set) to produce a space that has Poincaré duality?

This is done by the theory of intersection spaces: It assigns to certain types of stratified pseudomanifolds by a local spatial procedure CW-complexes that have Poincaré duality for their reduced singular (co)homology groups with rational coefficients. The construction is explained for spaces with isolated singularities in [8, 2] and for general depth one spaces in [6]. For these types of spaces, the construction was modified in [12, 14] to yield actual rational Poincaré duality spaces in the sense of Browder. For spaces of greater stratification depth, intersection spaces were defined in [3, 1].

The resulting (co)homology theory (reduced singular cohomology of intersection spaces) was studied independently using differential forms on the nonsingular part or blowup in [4, 9, 13], using L^2-cohomology in [7] and a algebraic approach in [10].

In this talk, the focus is on another approach to the intersection space cohomology theory: Via sheaf theory. This was studied first in [5] for complex projective varieties with only isolated singularities. The authors proved the existence of a perverse sheaf that describes intersection space cohomology and carries a natural mixed Hodge structure. They give precise conditions on the self duality of this sheaf complex and the existence of a pure Hodge structure satisfying the Hard Lefschetz Theorem. For general pseudomanifolds

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there cannot exist a perverse sheaf describing intersection space cohomology because of the stalk vanishing conditions such a sheaf has to satisfy.

In the second part of [1], Agustín and Bobadilla explain the construction of a cohomologically bounded complex of constructible sheaves to describe intersection space cohomology. They give conditions on the existence of such a intersection space complex and characterize it by a set of properties. In general, there is not a unique intersection space complex. We provide an overview of these constructions and results in Section 2. It is not clear if or which intersection space complexes are self dual. The non-uniqueness makes it impossible to use the techniques for proving self duality one uses for the intersection homology sheaves. Agustín and Bobadilla give a partial solution to this problem for spaces with one singular stratum. They introduce the notion of general intersection spaces and show that general intersection spaces of complementary perversities are dual to each other.

In Section 3, we explain a construction of so called Künneth intersection space complexes for toric varieties (or other singular spaces with compatible triangulisations for their link bundles) that respect the trivializations on the link bundles of the singular strata. Those intersection space complexes are then the candidates to give a positive answer to the self duality question. This is work in progress and joint with M. Agustín and J. Bobadilla.

Conventions and Notation: Throughout the paper, the terms "singular space" and "pseudomanifold" denote a Thom-Mather stratified pseudomanifold, e.g. a Whitney-stratified complex projective variety.

Following [11], the cohomology sheaf of a complex of sheaves $\mathbf{A}^\bullet$ on a space X will be denotes by $\mathbf{H}^\bullet(\mathbf{A}^\bullet)$, while the hypercohomology groups will be $\mathcal{H}^i(X; \mathbf{A}^\bullet)$. We work in the derived category of cohomologically bounded constructible complexes of sheaves, which is denoted by $\mathcal{D}^b_{cc}$.

2. Intersection space sheaf complexes

Intersection homology by Goresky and MacPherson was one of the first approaches to re-establish Poincaré duality for singular spaces. Initially invented to define characteristic classes for singular spaces, it became famous for its impact on singular topology and singular alegbraic variety. A sheaf theoretic approach to intersection cohomology, motivated by Deligne and introduced by Goresky and MacPherson in [11] has been used to prove Poincaré duality and topological invariance of intersection cohomology for topological pseudomanifolds. It also led to a proof of the Kazhdan-Lusztig conjecture via $\mathcal{D}$-modules, relating representation theory and intersection cohomology. The axiomatic definition has another advantage: It makes it easy to check, whether a new approach computes intersection cohomology or not.

With these promising results in the back of one's mind, an analogous sheaf theoretical description for intersection space cohomology is desirable. Here, I want to highlight the approach of Agustín-Bobadilla in [1]. Based on their iterative construction of intersection space pairs, they derive a constructible complex of sheaves $\mathbf{IS}^{\bullet}_{\bar{p}}$ and show that its global hypercohomology is the cohomology of the intersection space pair, see Section 2.1. Moreover, in [1, Section 6], they introduce a set of properties, called the $\mathbf{IS}^{\bullet}_{\bar{p}}$-properties in the following, mimicking the axioms for intersection cohomology of [11, Section 3.3]. A sheaf complex satisfying these properties will be called an $\mathbf{IS}^{\bullet}_{\bar{p}}$-complex. There are three main differences from intersection cohomology:

(1) Except for the case of isolated singularities, an $\mathbf{IS}^{\bullet}_{\bar{p}}$-complex cannot be a perverse sheaf complex.
(2) The $\mathbf{IS}^{\bullet}_{\bar{p}}$-properties do not fix an $\mathbf{IS}^{\bullet}_{\bar{p}}$-complex of sheaves up to quasi-isomorphism.
(3) In unison with the other approaches to intersection space cohomology, there does not always have to be an $\mathbf{IS}^{\bullet}_{\bar{p}}$-complex.

In the following, we use the notation of Agustín-Bobadilla : Let X^d be a Thom-Mather stratified pseudomanifold (e.g. a complex projective variety) of dimension d with filtration

$$X = X_d \supset X_{d-2} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

Let $U_k := X \setminus X_{d-k}$ with open inclusion $i_k : U_k \hookrightarrow U_{k+1}$ and $j_k : X_{d-k} \setminus X_{d-k-1} \hookrightarrow U_{k+1}$. Let further $\bar{p}$ denote a perversity function in the sense of Goresky and MacPherson and $\bar{q}$ its dual perversity.

2.1. **From spaces to sheaf complexes.** Agustín-Bobadilla construct intersection space pairs $(I^{\bar{p}}X, I^{\bar{p}}X_{d-2})$, where X_{d-2} is the singular set, by iteratively replacing link bundles by their fiberwise homology cotruncations. The existence of such an intersection space pair is obstructed in general and each choice in the iterative construction of the pair might affect the following steps (in the spirit of obstruction theory). An intersection space pair always exists for toric varieties or other pseudomanifolds with compatibly trivializable link bundles, see [1, Theorem 3.30].

To assign to an intersection space pair a cohomologically bounded constructible complex of sheaves on X, with hypercohomology the cohomology of the pair, Agustín-Bobadilla define an increasing sequence $(I^{\bar{p},n}X, I^{\bar{p},n}X_{d-2})$ of modified intersection space pairs which are all contained in a certain homotopy model $\pi : X' \simeq X$ of X. Loosely speaking, for increasing n, the replacement of link bundles by their homology cotruncations is done on smaller tubular neighbourhoods of the singular strata. One then defines the sheaf complex $\mathcal{K}^{n,\bullet}$ as the kernel of the induced map of $\nu^n : I^{\bar{p},n}X_{d-2} \hookrightarrow I^{\bar{p},n}X$ on cubical singular cochains with rationals coefficients. Since the sequence of modified intersection spaces is increasing, there are canonical morphisms $i^{n_1,n_2} : \mathcal{K}^{n_2,\bullet} \to \mathcal{K}^{n_1,\bullet}$ for $n_1 < n_2$. This gives and inverse system and one sets

$$\mathbf{IS}^{\bullet}_{\bar{p}} := \pi_* \varprojlim_{n\in\mathbb{N}} \mathcal{K}^{n,\bullet}.$$

Agustín-Bobadilla prove in [1, Theorem 5.16], that the hypercohomology groups of this complex are the cohomology groups of the intersection space pair. For the whole constructing and proofs of the statements, [1, Section 5] can be consulted.

2.2. **An axiomatic approach to intersection space complexes.** The cohomologically bounded, constructible intersection space complex $\mathbf{IS}^\bullet_{\bar{p}}$ of the previous section satisfies the following set of properties:

(1) $\mathbf{IS}^\bullet_{\bar{p}}|_{U_2} \cong \mathbb{Q}_{U_2}$,
(2) $\mathbf{H}^i(\mathbf{IS}^\bullet_{\bar{p}}) = 0$ for $i \notin \{0, 1, \ldots, n\}$,
(3) $\mathbf{H}^i\left(j_k^*\mathbf{IS}^\bullet_{\bar{p}}|_{U_{k+1}}\right) = 0$ for $i \leq \bar{q}(k)$,
(4) the natural morphism $\mathbf{H}^i\left(j_k^*\mathbf{IS}^\bullet_{\bar{p}}|_{U_{k+1}}\right) \to \mathbf{H}^i\left(j_k^* i_{k*}\mathbf{IS}^\bullet_{\bar{p}}|_{U_k}\right)$ is an isomorphism for $i > \bar{q}(k)$.

From now on, any complex $\mathbf{B}^\bullet \in \mathcal{D}^b_{cc}$ that satisfies these properties will be called an $\mathbf{IS}^\bullet_{\bar{p}}$-complex. These properties look similar to the axioms for the intersection homology sheaf of Goresky and MacPherson, see [11, 3.3], but there are two major differences: The intersection homology sheaf always exists and is (up to quasi-isomorphism) uniquely determined by the axioms, while this is not true for $\mathbf{IS}^\bullet_{\bar{p}}$-complexes.

Agustín-Bobadilla also give a construction of $\mathbf{IS}^\bullet_{\bar{p}}$-complexes that is independent from intersection space pairs (and their existence). That approach is more general (see e.g. [1, Section 9.1]). Their construction is inductive, starting on the regular stratum. In the k-th step, an $\mathbf{IS}^\bullet_{\bar{p}}$-complex $(\mathbf{IS}^\bullet_{\bar{p}})_{k-1}$ on U_k can be extended to a complex satisfying the axioms on U_{k+1} if and only if the following distinguished triangle in the derived category splits.

$$\tau_{\leq \bar{q}(k)} j_k^* i_{k*}(\mathbf{IS}^\bullet_{\bar{p}})_{k-1} \longrightarrow j_k^* i_{k*}(\mathbf{IS}^\bullet_{\bar{p}})_{k-1} \longrightarrow \tau_{>\bar{q}(k)} j_k^* i_{k*}(\mathbf{IS}^\bullet_{\bar{p}})_{k-1} \xrightarrow{[+1]}$$

If this triangle splits, one has to choose such a splitting to proceed. The obstruction at each step might, as for the construction of the intersection space pairs, depend on all the previous choices. If X is an algebraic variety, Agustín-Bobadilla show that the construction can be lifted to the category of mixed Hodge modules on X: If an $\mathbf{IS}^\bullet_{\bar{p}}$-complex exists, it is a mixed Hodge module, i.e. its global hypercohomology groups have a mixed Hodge structure (see [1, Theorem 8.3]).

Agustín-Bobadilla give necessary and sufficient conditions on the existence and uniqueness of an $\mathbf{IS}^\bullet_{\bar{p}}$-sheaf complex, living in certain Ext^1 and Hom sheaves respectively, see [1, Corollary 7.6]. It follows from [1, Theorem 9.10], that spaces with isolated singularities have a unique intersection space complex.

Concerning Poincaré duality, Agustín-Bobadilla only have a partial answer. Although the Verdier dual of an $\mathbf{IS}^\bullet_{\bar{p}}$-complex is an $\mathbf{IS}^\bullet_{\bar{q}}$-complex ([1, Theorem 10.1]), this does not imply global Poincaré duality. If an intersection space of a two strata space X^d of dimension d with singular set

X_{d-k} exists for a given perversity $\bar{p}$, the intersection space complexes $\mathbf{IS}^\bullet_{\bar{p}}$ are parametrized by the vector space

$$E_{\bar{p}} := \hom\left(\tau_{>\bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}|_{U_k}, \tau_{\leq \bar{q}(k)} j_{k*} j_k^* i_{k*}\mathbb{Q}_{U_k}\right)$$

(see [1, Corollary 7.6]). An element $\mathbf{IS}^\bullet_{\bar{p}} \in E_{\bar{p}}$ is called a $GIS_{\bar{p}}$ or *general intersection space complex* of X with perversity $\bar{p}$, if the hypercohomology groups of $\mathbf{IS}^\bullet_{\bar{p}}$ are minimal compared to the hypercohomology groups of all complexes in $E_{\bar{p}}$, that is $\dim\left(\mathbb{H}^i(X, \mathbf{IS}^\bullet_{\bar{p}}\right) = \min_{\mathbf{S}^\bullet \in E_{\bar{p}}} \mathbb{H}^i(X, \mathbf{S}^\bullet)$. Provided the existence of such a $GIS_{\bar{p}}$ $\mathbf{IS}^\bullet_{\bar{p}}$, Agustín-Bobadilla prove that the Verdier dual $\mathbf{IS}^\bullet_{\bar{q}} := \mathcal{D}\mathbf{IS}^\bullet_{\bar{p}}[-d]$ is a $GIS_{\bar{q}}$, with $\bar{q}$ the dual perversity of $\bar{p}$, and that there is an isomorphism

$$\mathbb{H}^i(X, \mathbf{IS}^\bullet_{\bar{p}}) \cong \hom\left(\mathbb{H}^{d-i}(X, \mathbf{IS}^\bullet_{\bar{q}}), \mathbb{R}\right)$$

of $\mathbb{Q}$-vector spaces for all i, see [1, Theorem 10.6]. It is not clear, what the conditions are that determine the existence of such a $GIS_{\bar{p}}$-complex and how it can be constructed. It is interesting to know, whether the intersection space of Banagl-Chriestenson in the setting of [6] or the intersection form complex $\Omega I^\bullet_{\bar{p}}$ of [4] give rise to $GIS_{\bar{p}}$-complexes. In the following, the Poincaré duality question is discussed for toric varieties.

3. Künneth intersection space complexes for toric varieties

The purpose of this section is to introduce special $\mathbf{IS}^\bullet_{\bar{p}}$-complexes for toric varieties and other pseudomanifolds with compatible trivializations for the link bundles, that respect these trivializations.

3.1. **Künneth intersection space complexes in depth one.** We outline the idea of a Künneth intersection space complex for a two strata pseudomanifold first.

The setup in this section is the following: Let X^d be a d-dimensional stratified pseudomanifold with filtration $X = X_d \supset X_{d-r} = Y$, closed inclusion $j : Y \hookrightarrow X$, and let $U = X \setminus Y$ with open inclusion $i : U \hookrightarrow X$, such that there is an open (tubular) neighbourhood $Y \subset TY \subset X$ that is trivializable. That is there is a stratified isomorphism $\Phi : TY \xrightarrow{\cong} Y \times Z$, with $Z = \mathrm{cone}(L) = \mathring{Z} \cup \{v\}$ and inclusions $i_{\mathring{Z}} : \mathring{Z} \hookrightarrow Z$ and $j_v : \{v\} \hookrightarrow Z$. Let $Y \xleftarrow{\pi_1} Y \times Z \xrightarrow{\pi_2} Z$ be the factor projections. Let further $T\mathring{Y} = TY \setminus Y$ denote the tubular neigbourhood of Y with the singular part removed.

The idea is: By [1, Corollary 7.6 and Theorem 9.10], there is a unique intersection space complex $\mathbf{IS}^\bullet_{\bar{p}}(Z)$ on the cone Z for any perversity $\bar{p}$, that is there is only one splitting of the map $\tau_{\leq \bar{q}(r)} j_{v*} j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}} \to j_{v*} j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}}$. We use that fact to construct a unique intersection space complex $\mathbf{IS}^\bullet_{\bar{p}}(TY)$ on TY and hence for X (since it must be the constant sheaf on $X \setminus Y$) for perversity $\bar{p}$, which is compatible with the trivialization Φ. Its Verdier dual (which is an $\mathbf{IS}^\bullet_{\bar{q}}$ complex) will then also be compatible with the trivialization, implying the duality $\mathbf{IS}^\bullet_{\bar{p}} \cong \mathbf{IS}^\bullet_{\bar{q}}[-d]$. Compatibility with the trivialization means that the complex fits into a certain Künneth structure.

Definition 3.1.1 (Künneth property and structures)**.** *A sheaf complex* $\mathbf{K}^\bullet \in \mathcal{D}^b_{cc}(X)$ *satisfies the* Künneth property *with respect to the trivialization* $\Phi : TY \xrightarrow{\cong} Y \times Z$ *if there is exists a sheaf complex* $\mathbf{L}^\bullet \in \mathcal{D}^b_{cc}(Z)$ *and an isomorphism* $\beta : \mathbf{K}^\bullet|_{TY} \xrightarrow{\cong} \Phi^*\pi_2^*\mathbf{L}^\bullet$. *A triple* $(\mathbf{K}^\bullet, \mathbf{L}^\bullet, \beta)$ *is called a* Künneth structure *for* $\mathbf{K}^\bullet$.

For example, the constant sheaf $\mathbb{Q}_X$ on X has a Künneth structure. There is a unique complex that satisfies the properties of an $\mathbf{IS}^\bullet_{\bar{p}}$-complex and has a Künneth structure $(\mathbf{IS}^\bullet_{\bar{p}}, \mathbf{IS}^\bullet_{\bar{p}}(Z), \beta)$, where $\mathbf{IS}^\bullet_{\bar{p}}(Z)$ is the unique $\mathbf{IS}^\bullet_{\bar{p}}$-complex on the cone Z, which is compatible with the Künneth structure of the constant sheaf on $\mathring{TY}$. We want to elaborate the construction of such a complex: By [1, Theorem 9.10], the cone Z, which has only one isolated singularitiy v, has a unique splitting $\lambda : j_{v*}j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}} \to \tau_{\le \bar{q}(r)} j_{v*}j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}}$. Together with the canonical adjunction morphisms related to j and j_v and the natural Künneth structures of $i_*\mathbb{Q}_U$ and $j_*j^*i_*\mathbb{Q}_U$, this morphism fits into the following commutative diagram, where the dashed arrows are induced by the other morphisms.

$$\begin{array}{ccccc} i_*\mathbb{Q}_U & \xrightarrow{\cong} & \Phi^*\pi_2^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}} & \dashrightarrow & \\ \downarrow & & \downarrow & & \\ j_*j^*i_*\mathbb{Q}_U & \xrightarrow{\cong} & \Phi^*\pi_2^* j_{v*}j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}} & \xrightarrow{\Phi^*\pi_2^*\lambda} & \Phi^*\pi_2^*\tau_{\le \bar{q}(r)} j_{v*}j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}} \end{array} \qquad \varphi_\Phi : i_*\mathbb{Q}_U \dashrightarrow \Phi^*\pi_2^*\tau_{\le \bar{q}(r)} j_{v*}j_v^* i_{\mathring{Z}*}\mathbb{Q}_{\mathring{Z}} \tag{1}$$

We set $\mathbf{IS}^\bullet_{\bar{p}} := \operatorname{cone}(\varphi_\Phi)[-1]$. By the same arguments as in the proof of [1, Theorem 7.3] (see p.39 therein), $\mathbf{IS}^\bullet_{\bar{p}}$ satisfies the properties of an $\mathbf{IS}^\bullet_{\bar{p}}$-complex. By construction, it has a Künneth structure with

$$\mathbf{IS}^\bullet_{\bar{p}}|_{TY} = \Phi^*\pi_2^* \operatorname{cone}(\lambda) = \Phi^*\pi_2^*\mathbf{IS}^\bullet_{\bar{p}}(Z).$$

We do not provide the arguments that $\mathbf{IS}^\bullet_{\bar{p}}$ is unique up to quasi-isomorphism. To prove Poincaré duality for the rational hypercohomology groups of $\mathbf{IS}^\bullet_{\bar{p}}$, one shows that $\mathcal{D}\mathbf{IS}^\bullet_{\bar{p}}[-d]$, where $\mathcal{D}$ denotes the Verdier dual, is the Künneth intersection space complex of the dual perversity $\bar{q}$.

3.2. **Compatible systems of trivializations in depth two.** In generality, compatible systems of trivializations for Thom-Mather stratified pseudomanifolds are explained in [1, Section 3]. Since the notation in the general case is rather involved, we focus on the special case of a depth 2 pseuodmanifold X with filtration

$$X = X_d \supset X_{d-m} \supset X_{d-l}$$

and three strata $U = X \setminus X_{d-m}$, $S_{d-m} := X_{d-m} \setminus X_{d-l}$ and $S_{d-l} = X_{d-l}$. The Thom-Mather control data provide tubular neighbourhoods T_{d-m}, T_{d-l} of the singular strata that have a fiber bundle structure

$$\sigma_{d-m} : T_{d-m} \to S_{d-m}, \quad \sigma_{d-l} : T_{d-l} \to S_{d-l}$$

with fibers cone(L_{d-m}) and cone(L_{d-l}) (and suitable transition functions), such that on $T_{d-m} \cap T_{d-l}$ it holds that $\sigma_{d-l} \circ \sigma_{d-m} = \sigma_{d-l}$. A system of trivializations for such a space is a tuple of fiber bundle isomorphisms

$$\Psi_{d-m} : \partial T_{d-m} \xrightarrow{\cong} L_{d-m} \times S_{d-m},$$
$$\Psi_{d-l} : \partial T_{d-l} \xrightarrow{\cong} L_{d-l} \times S_{d-l},$$

where the bundle projections on the right hand sides are the second factor projections. Note that the link L_{d-l} of the bottom singular stratum is a singular space itself with singular stratum S^L and link L_{d-m}. The system of trivializations is then called compatible, if there is a tubular neigbourhood T^L_{d-l} of the singular set in L_{d-l} (defined by $\Psi_{d-l}|_{T_{d-m}}$) that has a trivialization

$$\Psi^L : \partial T^L_{d-l} \xrightarrow{\cong} L_{d-m} \times S^L$$

such that the following diagram commutes:

$$\begin{array}{ccc} \partial T_{d-l} \cap \partial T_{d-m} & \xrightarrow[\cong]{\Psi_{d-l}|} & \partial T^L_{d-l} \times S_{d-l} \\ \cong \Big\downarrow \Psi_{d-m}| & & \cong \Big\downarrow \Psi_L \times \mathrm{id} \\ L_{d-m} \times (S_{d-m} \cap \partial T_{d-l}) & \xrightarrow[\cong]{\mathrm{id} \times \Psi_{d-l}|} & L_{d-m} \times S^L \times S_{d-l} \end{array}$$

As an example, the torus action in toric varieties always induces a compatible trivialization for the link bundles.

3.3. **Künneth intersection space complexes in depth two.** We use the notation of the previous section. To generalize the construction of Künneth intersection space complexes of Section 3.1, one uses an induction on strata. Starting with the constant sheaf $\mathbb{Q}_U$ on U, one first shows that there is a unique Künneth intersection space complex ${}_{d-l}\mathbf{IS}^\bullet_{\bar{p}}$ on $X \setminus X_{d-l}$ with a Künneth structure as in the depth one setting. Then, one proves that there is a unique $\mathbf{IS}^\bullet_{\bar{p}}$-complex on X extendending ${}_{d-l}\mathbf{IS}^\bullet_{\bar{p}}$ that has a Künneth structure $(\mathbf{IS}^\bullet_{\bar{p}}, \mathbf{IS}^\bullet_{\bar{p}}(L_{d-l}), \beta)$, with $\mathbf{IS}^\bullet_{\bar{p}}(L_{d-l})$ the unique Künneth intersection space complex on the link L_{d-l}. Note, that the two Künneth structures can be chosen compatibly since the trivializations are compatible. Afterwards, one proves that the shifted Verdier dual $\mathcal{D}\mathbf{IS}^\bullet_{\bar{p}}[-n]$ is the unique $\mathbf{IS}^\bullet_{\bar{q}}$-complex with the above Künneth properties.

At the end, the uniqueness of the Künneth intersection space complexes and the Verdier duality theorem imply Poincaré duality for the hypercohomology groups of the Künneth intersection space complexes of complementary perversities. These ideas will be explained in more detail in an upcoming article.

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