COBORDISM THEORY OF MORSE FUNCTIONS AND APPLICATIONS

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Abstract. This is a survey talk on recent developments in the cobordism theory of Morse functions. We present results on the computation of several cobordism groups of Morse functions of compact manifolds possibly with boundary.

1. Introduction

We are concerned with differentiable[1] maps between differentiable manifolds. Cobordism groups of differentiable maps with prescribed singularities are generally studied by means of stable homotopy theory (see e.g. the works of Rimányi and Szűcs [13], Ando [1], Kalmár [10], Sadykov [14], and Szűcs [19]). Historically, the topic was pioneered in the middle of the 20th century by René Thom [20], who applied the Pontryagin-Thom construction to study embeddings of manifolds into Euclidean spaces up to cobordism. In doing so, Thom reduced the study of cobordism groups of closed differentiable manifolds to the computation of homotopy groups of certain spaces. In the sequel, the structures of the n-dimensional oriented cobordism group Ω_n^{SO} and its unoriented version Ω_n^{O} have been completely determined by several authors. It remains an interesting problem to study cobordism theory of differentiable maps with concrete prescribed types of singularities.

In this short note, we shall focus on cobordism theory of Morse functions. Recall that Morse functions of closed differentiable manifolds are real valued differentiable functions whose critical points are all nondegenerate. We point out that Morse theory is a fundamental tool in the study of differentiable manifolds, for example by virtue of Smale's h-cobordism theorem. Thus, when studying Morse functions up to suitable notions of cobordism, we expect that we can still detect important information about algebraic topology and differential topology of manifolds.

Cobordism groups of various types of Morse functions have been studied by several authors by applying explicit methods of global singularity theory of differentiable maps. For instance, Ikegami [4] used Levine's cusp elimination technique to compute cobordism groups of Morse functions on closed manifolds (this generalized results of Ikegami-Saeki [5] and Kalmàr [9]). An application of Ikegami's techniques to the construction of topological invariants of generic differentiable map germs was found by Ikegami and Saeki [6]. Saeki and Yamamoto [17, 18] studied Morse functions on compact surfaces with boundary up

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[1]In this note, "differentiable" always means differentiable of class C^∞.

to so-called admissible cobordism by using the cohomology of the universal complex of singular fibers in [17], as well as a combinatorial argument based on labeled Reeb graphs in [18]. By using similar techniques, Yamamoto [26] studied versions of these cobordism groups without cusps. Saeki [15] applied the technique of Stein factorization and Cerf's pseudoisotopy theorem to study cobordism groups of so-called special generic functions, i.e., Morse functions with only maxima and minima as their critical points. In [22], the author has imposed more general index constraints on the Morse functions, and studied the resulting cobordism relations for such "constrained" Morse functions by means of the two-index theorem of Hatcher and Wagoner, as well as handle extension techniques for fold maps due to Gay and Kirby. As an application to high-dimensional topological field theory (compare [21]), the author has shown how exotic Kervaire spheres can be distinguished in infinitely many dimensions from other exotic spheres as elements of the cobordism group of constrained Morse functions.

Let $n \geq 2$ be an integer. In this note, we consider several variants of cobordism relations for Morse functions of compact n-dimensional manifolds possibly with boundary. In principle, one defines cobordisms of such Morse functions to be certain differentiable maps of $(n+1)$-dimensional cobordisms (with corners) into the plane. Following Saeki and Yamamoto [17, 18], we impose the natural requirement that these maps are locally modeled on C^∞ stable map germs into the plane. At interior points, it is well-known that the possible C^∞ stable map germs $(\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^2, 0)$ are

$$(1.1) \qquad (x_0, \ldots, x_n) \mapsto \begin{cases} (x_0, x_1), & \text{regular point,} \\ (x_0, \pm x_1^2 \pm \cdots \pm x_n^2), & \text{fold point,} \\ (x_0, x_0x_1 + x_1^3 \pm x_2^2 \pm \cdots \pm x_n^2), & \text{cusp.} \end{cases}$$

At boundary points, we point out that the possible C^∞ stable map germs $(\mathbb{R}^n \times [0, \infty), 0) \to (\mathbb{R}^2, 0)$ are given by

$$(1.2) \qquad (x_0, \ldots, x_n) \mapsto \begin{cases} (x_0, x_1), & \partial\text{-regular point,} \\ (x_0, \pm x_1^2 \pm \cdots \pm x_{n-1}^2 + x_n), & \partial\text{-fold point,} \\ (x_0, x_0x_1 + x_1^3 \pm x_2^2 \pm \cdots \pm x_{n-1}^2 + x_n), & \partial\text{-cusp,} \\ (x_0, \pm x_1^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^2 + x_0x_n), & B_2 \text{ point,} \end{cases}$$

where the first three types are regular map germs that are named after their restrictions to the boundary $(\mathbb{R}^n \times \{0\}, 0) \subset (\mathbb{R}^n \times [0, \infty), 0)$, while the so-called B_2 point[2] is a singular map germ. In Definition 1.2 below, we introduce various cobordism relations of Morse functions by requiring that cobordisms are locally modeled on prescribed subsets of the possible C^∞ stable map germs into the plane. On the technical side, we note that our definition of cobordism relations differs from that of Saeki and Yamamoto [17, 18]

[2] According to [26], the terminology "B_2 point" has its origin in the case of dimension $n = 3$, where the map germ is a versal unfolding of the function germ $B_2 = \pm x^2 \pm y^2$ (see Arnold [2]).

in that we do *not* assume Morse functions and their cobordisms to be proper C^∞ stable maps. Nevertheless, by means of slight perturbations we see that both definitions result in isomorphic cobordism groups.

Next, we introduce the cobordism relations that will be discussed in this note. Let M be an n-dimensional compact differentiable manifold possibly with boundary. By a *Morse function of* M we mean a real valued differentiable function $f\colon M \to \mathbb{R}$ which is a submersion near the boundary ∂M, and such that the critical points of both f and $f|_{\partial M}$ are all nondegenerate. We consider the following notion of oriented generic cobordisms between two Morse functions $f_0\colon M_0 \to \mathbb{R}$ and $f_1\colon M_1 \to \mathbb{R}$ of oriented compact n-dimensional manifolds possibly with boundary.

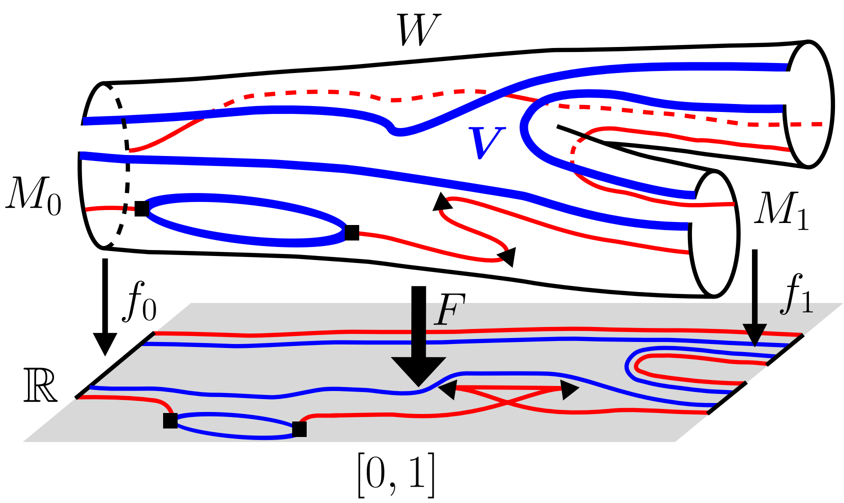

FIGURE 1. Illustration of an oriented generic cobordism (W, V, F) from f_0 to f_1. The singular point set of F and its image in the plane are indicated as follows. Fold lines are red, cusps are triangles, and B_2 points are squares.

Definition 1.1. An *oriented generic cobordism from* $f_0\colon M_0 \to \mathbb{R}$ *to* $f_1\colon M_1 \to \mathbb{R}$ is a triple (W, V, F) (see Figure 1), where

- the pair (W, V) is an oriented cobordism (with corners) from M_0 to M_1, that is, W is a compact oriented $(n+1)$-dimensional manifold with corners such that[3] $\partial W = M_0 \cup_{\partial M_0} V \cup_{-\partial M_1} -M_1$, where M_0, $-M_1$ and V are oriented codimension 0 submanifolds of ∂W such that $M_0 \cap M_1 = \emptyset$, $V \cap M_0 = \partial M_0$ and $V \cap M_1 = \partial M_1$, V is an oriented cobordism from ∂M_0 to ∂M_1 (that is, V is a compact oriented n-dimensional manifold with boundary $\partial V = \partial M_0 \cup -\partial M_1$), and W has corners precisely along ∂V, and

- $F\colon W \to [0, 1] \times \mathbb{R}$ is a differentiable map such that

[3] For an oriented manifold X, the manifold with opposite orientation is denoted by $-X$.

– there exist collars (with corners) $[0,\varepsilon) \times M_0 \subset W$ of $M_0 \subset W$ and $(1-\varepsilon, 1] \times M_1 \subset W$ of $M_1 \subset W$ such that $F|_{[0,\varepsilon)\times M_0} = \mathrm{id}_{[0,\varepsilon)} \times f_0$ and $F|_{(1-\varepsilon,1]\times M_1} = \mathrm{id}_{(1-\varepsilon,1]} \times f_1$, and
– at every point $x \in W \setminus (M_0 \sqcup M_1)$, the map germ of F at x is C^∞ right-left equivalent[4] to a C^∞ stable map germ into $\mathbb{R}^2$ (see (1.1) and (1.2)).

Similarly, we can define an unoriented version of the notion of oriented generic cobordism by ignoring orientations of manifolds in the above definition.

As it turns out, oriented generic cobordism (or its unoriented version) is not an interesting cobordism relation to study because any Morse function $f\colon M \to \mathbb{R}$ is *nullcobordant*, i.e., there exists an oriented generic cobordism from f to the unique function on the empty set. (In fact, the double $M \cup_{\partial M} -M$ of M is oriented nullcobordant, and any oriented nullcobordism W can be considered as an oriented cobordism (W, M) (with corners) from M to the empty set. Then, the desired differentiable map $W \to [0,1] \times \mathbb{R}$ is obtained by a generic extension of the $\{0\} \times M$-germ of the map $\mathrm{id}_{[0,\varepsilon)} \times f$ defined on a collar (with corners) $[0,\varepsilon) \times M \subset W$.) Nevertheless, we can use the above notion of (oriented) generic cobordism to define the following more interesting cobordism relations[5].

Definition 1.2. An (oriented) generic cobordism (W, V, F) is called

(i) an *(oriented) admissible cobordism* if F has no B_2 points.

(ii) an *(oriented) fold cobordism* if F has no cusps and no ∂-cusps.

(iii) an *(oriented) admissible fold cobordism* if F has no cusps, no ∂-cusps, and no B_2 points.

The oriented cobordism relations of the previous definition clearly define equivalence relations on the set $b\mathcal{M}_n$ of Morse functions of oriented compact n-dimensional manifolds possibly with boundary. Let $b\mathfrak{C}_n$, $b\mathfrak{F}_n$, and $b\mathfrak{A}_n$ denote the sets of equivalence classes $[f\colon M \to \mathbb{R}]$ of Morse functions in $b\mathcal{M}_n$ up to oriented admissible cobordism, oriented fold cobordism, and admissible fold cobordism, respectively. Disjoint union "$\sqcup$" induces an additive group law on each of the sets $b\mathfrak{C}_n$, $b\mathfrak{F}_n$, and $b\mathfrak{A}_n$ as follows. The identity element is represented by the unique map $\emptyset \to \mathbb{R}$, and the inverse of a class $[f\colon M \to \mathbb{R}]$ is represented by $-f\colon -M \to \mathbb{R}$, $x \mapsto -f(x)$, where $-M$ denotes the manifold M equipped with the opposite orientation. We call $b\mathfrak{C}_n$ (resp. $b\mathfrak{F}_n$, $b\mathfrak{A}_n$) the *n-dimensional oriented admissible (resp. fold, admissible fold) cobordism group of Morse functions.* Similarly, we can define $b\mathcal{M}_n^O$ by ignoring orientations of manifolds, and the unoriented versions

[4]Given differentiable manifolds N possibly with boundary and P without boundary, two differentiable maps $f, g\colon N \to P$ are called C^∞ right-left equivalent if there exist diffeomorphisms $\Phi\colon N \to N$ and $\Psi\colon P \to P$ such that $\Psi \circ f = g \circ \Phi$.

[5]More generally, it seems interesting to study cobordism relations for Morse functions on compact differentiable manifolds *with corners*. As this problem is beyond the scope of the methods presented in this note, we mention it as an interesting direction of future research.

$b\mathfrak{C}_n^O$, $b\mathfrak{F}_n^O$, and $b\mathfrak{A}_n^O$ of the oriented cobordism groups $b\mathfrak{C}_n$, $b\mathfrak{F}_n$, and $b\mathfrak{A}_n$ by using the unoriented versions of the cobordism relations of Definition 1.2.

Saeki and Yamamoto [17, 18] introduced the cobordism groups $b\mathfrak{C}_n$, $b\mathfrak{A}_n$ and their unoriented versions $b\mathfrak{C}_n^O$, $b\mathfrak{A}_n^O$. In [18], they showed that $b\mathfrak{C}_2^O \cong \mathbb{Z}_2$ by means of a geometric method using Reeb graphs which is based on [5, 9, 16]. Moreover, they posed the problem to study the group structures of $b\mathfrak{C}_n$, $b\mathfrak{A}_n$, and $b\mathfrak{C}_n^O$, $b\mathfrak{A}_n^O$ for arbitrary $n \geq 2$ (see Section 6 in [18]). Based on similar techniques, Yamamoto [26] showed that $b\mathfrak{F}_2^O \cong \mathbb{Z}_2$ and $b\mathfrak{A}_2^O \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

In the following sections, we outline our results on the computation of the oriented cobordism groups $b\mathfrak{C}_n$, $b\mathfrak{F}_n$, and $b\mathfrak{A}_n$, and their unoriented versions $b\mathfrak{C}_n^O$, $b\mathfrak{F}_n^O$, and $b\mathfrak{A}_n^O$ for arbitrary $n \geq 2$.

2. Admissible cobordism group of Morse functions

Our Theorem 2.1 below answers the problem of Saeki and Yamamoto [18] to determine the group structures of the (oriented) admissible cobordism groups of Morse functions $b\mathfrak{C}_n$ and $b\mathfrak{C}_n^O$ for all $n \geq 2$. Our proof is based on a geometric method that combines Levine's cusp elimination technique [12] with the complementary process of creating pairs of cusps along fold lines.

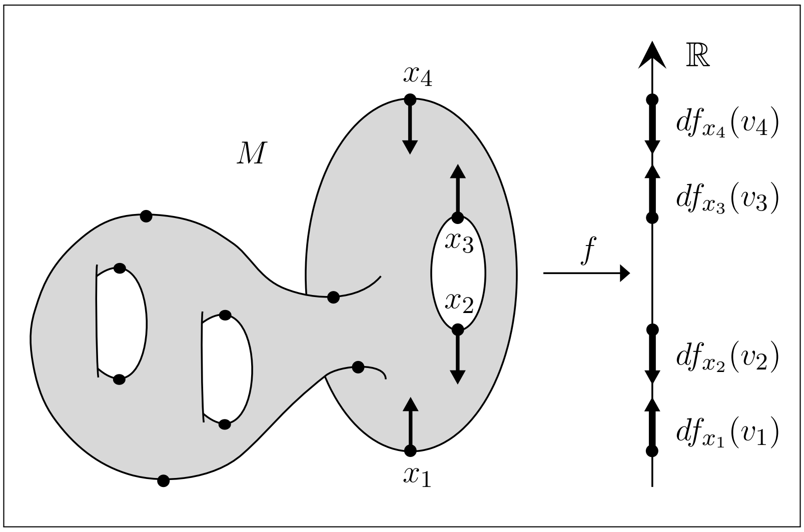

Figure 2. Illustration of a Morse function $f\colon M \to \mathbb{R}$ of a compact surface with boundary induced by the height function in $\mathbb{R}^3$. The surface M is the connected sum of two tori with two small open 2-disks removed. The critical points of $f|_{\partial M}$ are x_1, x_2, x_3, and x_4. Using the indicated inward pointing tangent vectors $v_i \in T_{x_i}M$, we see that $\sigma_f(x_i) = +1$ if and only if $i \in \{1, 3\}$. Hence, we have $S_0^+[f] = \{x_1\}$, $S_1^+[f] = \{x_3\}$, and thus $\chi^+[f] = 1 - 1 = 0$.

In order to present our result, we need to introduce some more notation for Morse functions $g\colon N \to \mathbb{R}$ defined on p-dimensional manifolds possibly with boundary, $p \geq 1$.

Following Curley [3], we assign to every critical point $x \in \partial N$ of the Morse function $g|_{\partial N}$ a sign $\sigma_g(x) \in \{\pm 1\}$ (see Figure 2) that is uniquely determined by requiring that for an *inward* pointing tangent vector $v \in T_x N$ the tangent vector

$$\sigma_g(x) \cdot dg_x(v) \in T_{g(x)}\mathbb{R} = \mathbb{R}$$

points into the *positive* direction of the real axis. Let $S(g|_{\partial N})$ denote the set of critical points of the Morse function $g|_{\partial N}$. We note that the resulting assignment $\sigma_g \colon S(g|_{\partial N}) \to \{\pm 1\}$ depends only on the map germ $[g]$ of g near ∂N. Let $S_i^+[g] \subset S(g|_{\partial N})$ denote the subset of those critical points x of the Morse function $g|_{\partial N}$ of index i for which $\sigma_g(x) = +1$. If ∂N is compact, then $S(g|_{\partial N})$ is finite, and we define in analogy with a well-known Euler characteristic formula[6] the integer

$$\chi^+[g] = \sum_{i=0}^{p-1} (-1)^i \cdot \nu_i^+[g],$$

where $\nu_i^+[g]$ denotes the cardinality of the finite set $S_i^+[g]$ (for example, see Figure 2).

Theorem 2.1 (W. [23], 2019). *Let $n \geq 2$ be an integer. The assignment $b\mathcal{M}_n^{(O)} \to \mathbb{Z}$, $(f \colon M \to \mathbb{R}) \mapsto \chi(M) - \chi^+[f]$, induces group isomorphisms*

$$b\mathfrak{C}_n^{(O)} \xrightarrow{\cong} \begin{cases} \mathbb{Z}_2, & n \text{ even}, \\ \mathbb{Z}, & n \text{ odd}. \end{cases}$$

In particular, $b\mathfrak{C}_n^O \cong b\mathfrak{C}_n$ for all $n \geq 2$.

3. Admissible fold cobordism group of Morse functions

In this section, we discuss a structural relationship between admissible fold cobordism groups of Morse functions and SKK-groups of compact differentiable manifolds possibly with boundary (see Theorem 3.2 below). The concept of SKK-groups of manifolds goes back to Jänich [7, 8], who observed that the index of elliptic operators is invariant under natural cutting and pasting operations on manifolds. This operation cuts a closed n-dimensional manifold along a submanifold Q of codimension 1 with trivial normal bundle, and pastes back together the two resulting copies of Q in the boundary by means of some gluing automorphism $Q \to Q$. The resulting notion of SK-invariants[7] of closed n-manifolds was studied systematically in [11] by viewing SK-invariants as homomorphisms on a universal SK-group SK_n with values in some abelian group. As a generalization,

[6] If $h \colon P \to \mathbb{R}$ is a Morse function on a closed $(p-1)$-dimensional manifold, then the Euler characteristic of P can be computed as $\chi(P) = \sum_{i=0}^{p-1}(-1)^i \cdot \nu_i(h)$, where $\nu_i(h)$ denotes the number of critical points of h of index i.

[7] from German "Schneiden und Kleben" = "cutting and pasting"

the notion of SKK-invariants[8] and the corresponding universal SKK-group SKK_n incorporate a correction term that may depend on the gluing automorphism $Q \to Q$.

In Theorem 3.2 below, we compute the (oriented) admissible fold cobordism groups of Morse functions, $b\mathfrak{A}_n$ and $b\mathfrak{A}_n^O$, in terms of the (oriented) SKK-groups of compact manifolds possibly with boundary, which we will denote by $bSKK_n$ and $bSKK_n^O$. The underlying $bSKK$-relations are a version with boundary of the SKK-relations which are studied systematically in the manuscript [11] by Karras, Kreck, Neumann, and Ossa.

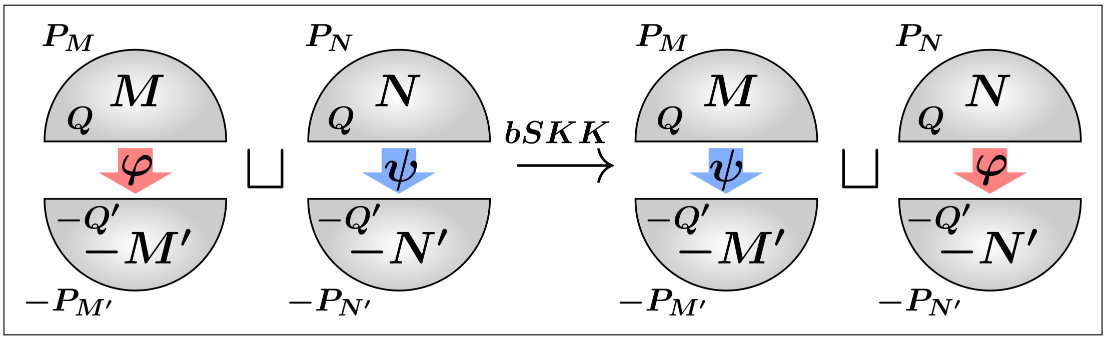

FIGURE 3. $bSKK$-related oriented n-dimensional manifolds X and Y.

Definition 3.1. Two compact oriented n-dimensional differentiable manifolds possibly with boundary X, Y are called *$bSKK$-related*, $X \xrightarrow{bSKK} Y$ (see Figure 3), if there exist

- compact oriented $(n-1)$-dimensional differentiable manifolds possibly with boundary $P_M, P_{M'}, P_N, P_{N'}, Q, Q'$ such that $-\partial P_M = \partial Q = -\partial P_N$ and $-\partial P_{M'} = \partial Q' = -\partial P_{N'}$,
- compact oriented n-dimensional differentiable manifolds M, M', N, N' with boundaries $\partial M = P_M \cup_{\partial Q} Q$, $\partial M' = P_{M'} \cup_{\partial Q'} Q'$, $\partial N = P_N \cup_{\partial Q} Q$, $\partial N' = P_{N'} \cup_{\partial Q'} Q'$, and corners along ∂P_M, $\partial P_{M'}$, ∂P_N, $\partial P_{N'}$, respectively, and
- orientation preserving diffeomorphisms $\varphi, \psi \colon Q \to Q'$ such that

$$X = \left(M \cup_\varphi -M'\right) \sqcup \left(N \cup_\psi -N'\right),$$
$$Y = \left(M \cup_\psi -M'\right) \sqcup \left(N \cup_\varphi -N'\right).$$

Let $b\mathfrak{M}_n$ denote the set of oriented diffeomorphism classes of oriented compact n-dimensional differentiable manifolds possibly with boundary. We regard $b\mathfrak{M}_n$ as an abelian semigroup with addition $[M] + [N] = [M \sqcup N]$ and identity element $0 = [\emptyset]$.

While the $bSKK$-relation on $b\mathfrak{M}_n$ given by Definition 3.1 is obviously symmetric, it might not be an equivalence relation. Nevertheless, we can use the $bSKK$-relation to define an equivalence relation $\sim_{bSKK}$ via stabilization as follows. Given two manifolds

[8]from German "SK-Kontrollierbar" = "SK-controllable"

M and N in $b\mathfrak{M}_n$, we define $[M] \sim_{bSKK} [N]$ if there exist manifolds X and Y in $b\mathfrak{M}_n$ such that $X \xrightarrow{bSKK} Y$ and $[M]+[X]=[N]+[Y]$ in $b\mathfrak{M}_n$. Then, it is straightforward to check that "$\sim_{bSKK}$" is an equivalence relation on $b\mathfrak{M}_n$. The quotient $b\mathfrak{M}_n/\sim_{bSKK}$ inherits an abelian semigroup structure from $b\mathfrak{M}_n$. We define the additive group $bSKK_n$ to be the Grothendieck group of $b\mathfrak{M}_n/\sim_{bSKK}$. In particular, note that an element of $bSKK_n$ is not always represented by a manifold, but can in general be written as a difference $[M]-[N]$. The group $bSKK_n$ is called the *n-dimensional oriented SKK-group of manifolds possibly with boundary.* Similarly, we can define an unoriented version $bSKK_n^O$ by ignoring orientations of manifolds.

By taking boundaries of manifolds, we obtain natural maps $bSKK_n^{(O)} \to SKK_{n-1}^{(O)}$ to the usual SKK-groups of closed manifolds. The groups $bSKK_n$ and $bSKK_n^O$ do not depend on singularity theory of differentiable maps, and will be computed in [24].

To relate admissible fold cobordism groups to $bSKK$-groups, we define an assignment

$$\Sigma_n^{(O)}\colon b\mathcal{M}_n^{(O)} \to \mathbb{Z}, \qquad (f\colon M\to\mathbb{R}) \mapsto \begin{cases} \nu_0(f)+\cdots+\nu_k(f), & n=2k+1,\\ \mu_{k-1}(f)+\frac{\sigma(M)-\chi(M)}{2}, & n=2k, \end{cases} \tag{3.1}$$

where $\nu_i(f)$ denotes the number of critical points of f of index i, and $\mu_i(f)=\nu_{n-i}(f)-\nu_i(f)$. We also recall the definition of the assignment $b\mathcal{M}_n^{(O)} \to \mathbb{Z}$, $f \mapsto \nu_i^+[f]$ from Section 2, and set $\mu_i^+[f]=\nu_{n-i}^+[f]-\nu_i^+[f]$.

Theorem 3.2 (W. [24], 2020)**.** *Let $n\geq 2$ be an integer. The assignment*

$$\omega_n^{(O)}\colon b\mathcal{M}_n^{(O)} \to bSKK_n, \qquad (f\colon M\to\mathbb{R}) \mapsto [M]+\Sigma_n^{(O)}(f)\cdot[S^n]+\Sigma_{n-1}^{(O)}(f|_{\partial M})\cdot[D^n],$$

induces a group isomorphism

$$b\mathfrak{A}_n^{(O)} \xrightarrow{\cong} bSKK_n^{(O)} \oplus \mathbb{Z}^{\lfloor (n-1)/2\rfloor} \oplus \mathbb{Z}^{\lfloor (n-2)/2\rfloor} \oplus \mathbb{Z}^{\lceil n/2\rceil},$$
$$[f\colon M\to\mathbb{R}] \mapsto (\omega_n^{(O)}[f], \boldsymbol{\mu}_{\lfloor (n-1)/2\rfloor}(f), \boldsymbol{\mu}_{\lfloor (n-2)/2\rfloor}(f|_{\partial M}), \boldsymbol{\mu}^+_{\lceil n/2\rceil}[f]),$$

where we make use of the vector notation $\boldsymbol{\mu}_{\boldsymbol{N}}^{(+)}=(\mu_0^{(+)},\ldots,\mu_{N-1}^{(+)})$.

4. Fold cobordism group of Morse functions

In Theorem 4.2 below, we determine the group structure of the (oriented) fold cobordism groups of Morse functions, $b\mathfrak{F}_n$ and $b\mathfrak{F}_n^O$ (except for $b\mathfrak{F}_n$ in the case $n\equiv 1 \bmod 4$).

First, we need to review the group structure of the (oriented) fold cobordism group of Morse functions of *closed* manifolds. For this purpose, let $\mathcal{M}_{n-1}$ denote the set of Morse functions of oriented closed $(n-1)$-dimensional manifolds. An oriented generic cobordism between two Morse functions $f_0\colon M_0\to\mathbb{R}$ and $f_1\colon M_1\to\mathbb{R}$ in $\mathcal{M}_{n-1}$ is an oriented generic cobordism (W,V,F) from f_0 to f_1 (seen as elements of $b\mathcal{M}_{n-1}$) in the sense of Definition 1.1 such that $V=\emptyset$. We call $(W,\emptyset,F)$ an *oriented fold cobordism* from f_0 to f_1 in $\mathcal{M}_{n-1}$ if F is an oriented fold cobordism from f_0 to f_1 (seen as elements of

$b\mathcal{M}_{n-1}$) in the sense of Definition 1.2. This is obviously equivalent to requiring that all singular points of F are fold points. Oriented fold cobordism clearly defines an equivalence relation on $\mathcal{M}_{n-1}$, and we denote the set of equivalence classes by $\mathfrak{F}_{n-1}$. Disjoint union defines an additive group law on $\mathfrak{F}_{n-1}$ in a similar way as for $b\mathfrak{F}_n$. We call $\mathfrak{F}_{n-1}$ the *oriented fold cobordism group of Morse functions (on closed manifolds)*. We can also define the unoriented version $\mathfrak{F}^O_{n-1}$ by ignoring orientations of manifolds.

Theorem 4.1 (Ikegami [4], 2004)**.** *For $n \geq 2$, there are group isomorphisms of the form*

$$\mathfrak{F}^O_{n-1} \xrightarrow[\cong]{(\beta^O,\Phi^O)} \Omega^O_{n-1} \oplus \mathbb{Z}^{\lfloor n/2 \rfloor},$$

and

$$\begin{cases} \mathfrak{F}_{n-1} \xrightarrow[\cong]{(\beta,\Phi)} \Omega^{SO}_{n-1} \oplus \mathbb{Z}^{\lfloor n/2 \rfloor}, & n \not\equiv 1 \mod 4, \\ \mathfrak{F}_{n-1} \xrightarrow[\cong]{(\beta,\Phi,\Lambda)} \Omega^{SO}_{n-1} \oplus \mathbb{Z}^{\lfloor n/2 \rfloor} \oplus \mathbb{Z}_2, & n \equiv 1 \mod 4, \end{cases}$$

where $\beta^O \colon \mathfrak{F}^O_{n-1} \to \Omega^O_{n-1}$ (resp. $\beta \colon \mathfrak{F}_{n-1} \to \Omega^{SO}_{n-1}$) is the natural map $[f \colon M \to \mathbb{R}] \mapsto [M]$.

By construction, there is a natural map $\alpha^{(O)} \colon b\mathfrak{F}^{(O)}_n \to \mathfrak{F}^{(O)}_{n-1}$ induced by restriction to the boundary, $[f \colon M \to \mathbb{R}] \mapsto [f|_{\partial M}]$.

Theorem 4.2 (W. [25], 2020)**.** *For $n \geq 2$, there are short exact sequences*

$$\begin{cases} 0 \to b\mathfrak{F}^O_n \xrightarrow{\alpha^O} \mathfrak{F}^O_{n-1} \xrightarrow{\beta^O} \Omega^O_{n-1} \to 0, & n \equiv 1 \mod 2, \\ 0 \to b\mathfrak{F}^O_n \xrightarrow{(\gamma^O,\alpha^O)} \mathbb{Z}_2 \oplus \mathfrak{F}^O_{n-1} \xrightarrow{\beta^O \circ \mathrm{pr}_2} \Omega^O_{n-1} \to 0, & n \equiv 0 \mod 2, \end{cases}$$

and

$$\begin{cases} 0 \to b\mathfrak{F}_n \xrightarrow{\alpha} \mathfrak{F}_{n-1} \xrightarrow{\beta} \Omega^{SO}_{n-1} \to 0, & n \equiv 2, 3 \mod 4, \\ 0 \to b\mathfrak{F}_n \xrightarrow{(\gamma,\alpha)} \mathbb{Z}_2 \oplus \mathfrak{F}_{n-1} \xrightarrow{\beta \circ \mathrm{pr}_2} \Omega^{SO}_{n-1} \to 0, & n \equiv 0 \mod 4, \end{cases}$$

where the map $\gamma^O \colon b\mathfrak{F}^O_{2k} \to \mathbb{Z}_2$ (resp. $\gamma \colon b\mathfrak{F}_{4k} \to \mathbb{Z}_2$) is given by[9]

$$[f \colon M \to \mathbb{R}] \mapsto \chi(M) + \frac{1}{2} \# S(f|_{\partial M}) \mod 2.$$

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[9] Here, $\# S(f|_{\partial M})$ is the cardinality of the set of critical points of the Morse function $f|_{\partial M}$.

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